WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 10

Exercise 1.

(a) Construct symplectic fillings of $(S^1 \times S^2, \xi_{st})$ and (T^3, ξ_1).

(b) Construct infinitely many pairwise non-diffeomorphic symplectic caps of (S^3, ξ_{st}).
Hint: Use Darboux's theorem.

(c) Show that S^4 carries no symplectic structure.
Hint: Use de Rham's theorem and show more generally that any 4-manifold with vanishing second homology does not carry a symplectic structure.

(d) Construct a symplectic form on $\mathbb{C}P^2$.

Bonus: Can $-\mathbb{C}P^2$ carry a symplectic structure?
Hint: Why is the symplectization $\mathbb{R} \times M$ and not $M \times \mathbb{R}$? Use the intersection form.

Exercise 2.
Prove Lemma 6.7 from the lecture.

Exercise 3.
An n-dimensional 1-handle is a copy of $D^1 \times D^{n-1}$ attached to an n-manifold M via an embedding $\varphi\colon \partial D^1 \times D^{n-1}$.

(a) Draw sketches of 1-handle attachments in dimension 2, 3, and 4. And analyze what is happening to the boundaries of the manifolds.

(b) Express the connected sum of two manifolds as a certain 1-handle attachement.

(c) Construct a smooth compact manifold W whose boundary is $M \# M$. Which surfaces are boundaries of compact 3-manifolds?

(d) Let W be a 4-dimensional symplectic cobordism. And let W' be the result of attaching a 1-handle to the positive boundary of W. Show that W' carries a symplectic structure making W' into a symplectic cobordism.
Hint: Use the same approach as in Theorem 6.6 from the lecture.

(e) Construct a symplectic filling of $(S^1 \times S^2, \xi_{st})$ by attaching a single 1-handle to D^4.

Exercise 4.
Prove Theorem 6.1 from the lecture.

Bonus exercise.
Prove Lemma 6.2 from the lecture.

This sheet will be discussed on Wednesday 17.1. and should be solved by then.