Contact Geometry

Exercise sheet 12

Exercise 1.

(a) We denote by U the Legendrian unknot with $\mathrm{tb} = -1$. In the lecture we have seen that there exist a contactomorphism $f\colon U(+2) \to (S^3, \xi_{st})$. Now let L be a Legendrian link in the complement of U. Then $f(L)$ is a Legendrian link in (S^3, ξ_{st}). Describe a front project of $f(L)$ depending on a front projection of L. How do the contact framings of L and $f(L)$ differ?
Hint: Draw a front projection of U and L and perform handle slides of L over U (after applying the transformation lemma to U).

(b) Use (a) to show that any contact manifold admits a contact (± 1)-surgery diagram in which every component is a Legendrian unknot.

(c) Show that handle slides and cancellations are not enough to relate any two contact (± 1)-surgery diagrams of the same contact manifold.
Hint: Describe a contact (± 1)-surgery diagram of (S^3, ξ_{st}) consisting of three Legendrian knots and observe that handle slides and cancellations preserve the parity of the number of components in a surgery diagram.

Exercise 2.
Classify the number of tight contact structures on the lens spaces $L(p, 1)$.
Hint: Try to generalize the strategy on $\mathbb{R}P^3 = L(2, 1)$ from the lecture:

1. Get an upper bound by gluing together two tight contact solid tori and using the classifications of the tight contact structures on solid tori.

2. Describe as many contact surgery diagrams of tight contact structures on $L(p, 1)$ as possible.

3. Get a lower bound by distinguishing as many of the contact structures from (2) by computing their homotopical invariants.

4. Hope that the lower and upper bounds agree.

Exercise 3.
Finish the computations in Part (b) of the proof of Theorem 7.21 from the lecture, i.e. show that the contact surgery diagrams presented in that proof yields smoothly S^3 and compute its homotopical invariants.

p.t.o.

Exercise 4.

(a) Construct for every natural number n a manifold with surgery number n.
Hint: Show that the number of generators of the first homology is a lower bound on the surgery number.

(b) Construct for every natural number n a manifold with contact surgery number n.

Bonus exercise.
Prove Lemma 7.14 and Lemma 7.15 from the lecture. Deduce that negative contact surgery preserves fillability (and tightness if you assume Wand's theorem).

This sheet will be discussed on Wednesday 31.1. and should be solved by then.