Contact Geometry

Exercise sheet 3

Let ξ be a tangential 2-plane field on a 3-manifold M. An embedding $c\colon \mathbb{R} \to M$ is called **Legendrian** if it is tangent to ξ, i.e. $Tc \subset \xi$. Similarly, an embedding $c\colon \mathbb{R} \to M$ is called **transverse** if it is transverse to ξ, i.e. $Tc \oplus \xi = TM$.

Exercise 1.
Consider the following two curves in $(\mathbb{R}^3, \xi_{st})$

$$c_1\colon t \longmapsto \Big(3\sin(t)\cos(t), \cos(t), \sin^3(t)\Big),$$
$$c_2\colon t \longmapsto \Big(\cos(t), \sin(2t), \frac{2}{3}\sin(t)\cos(2t) - \frac{4}{3}\cos(t)\sin(2t)\Big).$$

(a) Show that c_1 and c_2 define Legendrian knots.

(b) Draw sketches of these curves and determine the smooth knot types.

(c) Draw diagrams of c_1 and c_2 in the (x, y)-plane and the (y, z)-plane.

Exercise 2.

(a) Any two points in $(\mathbb{R}^3, \xi_{st})$ can be connected by a Legendrian curve.

(b) Any two points in an arbitrary connected contact 3-manifold can be connected by a Legendrian curve.

(c) Any smooth knot in a contact 3-manifold is isotopic to a Legendrian knot.

(d) Are (a), (b) and (c) also true for transverse curves?

Exercise 3.
Describe a Legendrian knot L in $(\mathbb{R}^3, \xi_{ot})$ that is the boundary of an embedded disk D such that along L the contact planes agree with the tangent spaces of D. Deduce that one can find such a Legendrian knot in any contact manifold that is contactomorphic to $(\mathbb{R}^3, \xi_{ot})$.
Remark: We will later show that in $(\mathbb{R}^3, \xi_{st})$ such a Legendrian knot cannot exists. This will prove that these two contact structures are not contactomorphic on $\mathbb{R}^3$.

Exercise 4.

(a) Let ξ be a tangential 2-plane field on a 3-manifold M, that is induced by a foliation. Show that two points in M can be connected by a Legendrian curve if and only if the two points lie in the same leaf.

(b) What can be said about transverse curves and transverse knots in 2-planes fields induced by foliations?

(c) Use (a) to deduce that a contact structure cannot be induced from a foliation.

(d) Describe tangential 2-plane fields on 3-manifolds that are not contact structures but in which any two points can be connected by Legendrians.

This sheet will be discussed on Wednesday 15.11. and should be solved by then.