WS 2023/24 Marc Kegel

Contact Geometry

Exercise sheet 9

Exercise 1.
Let K be an isolating simple closed curve on a convex surface S then there is no isotopy of S through convex surfaces such that K lies in the characteristic foliation.

Exercise 2.
Let S be a closed oriented surface. We write S^z for $S \times z$ in $S \times [-1, 1]$.

(a) Let ξ_0 and ξ_1 be two contact structures on $S \times [-1, 1]$ such that their characteristic foliations $S^z_{\xi_0}$ and $S^z_{\xi_1}$, coincide for all $z \in [-1, 1]$. Then ξ_0 and ξ_1 are isotopic rel boundary.

(b) Let ξ_0 and ξ_1 be two contact structures on $S \times [-1, 1]$ with the following properties:

- The characteristic foliations $S^{\pm 1}_{\xi_i}$ coincide on the boundary of $S \times [-1, 1]$ for $i = 0, 1$.
- Each surface S^z, $z \in [-1, 1]$, is convex for both contact structures, and there is a smoothly varying family of multi-curves Γ_z dividing both$S^z_{\xi_0}$ and $S^z_{\xi_1}$.

Then ξ_0 and ξ_1 are isotopic rel boundary.

Exercise 3.
For this and the next exercise we assume Bennequin's theorem, i.e. that the standard contact structure on S^3 is tight.

(a) The standard contact structure on $\mathbb{R}^3$ is tight.

(b) Let $f\colon (M', \xi') \to (M, \xi)$ be a contact covering (i.e. a covering that is a local contactomorphism). If ξ' is tight then ξ is tight.

(c) (T^3, ξ_n) and $(S^1 \times \mathbb{R}^2, \xi_n)$ are tight for all n.

Exercise 4.

Let (M, ξ) be a contact manifold. We write $\widetilde{M}$ for the universal covering of M and $\widetilde{\xi}$ for the lift of ξ to $\widetilde{M}$. We call $(\widetilde{M}, \widetilde{\xi} \to (M, \xi)$ the *universal contact covering* of (M, ξ). We say that a tight contact structure ξ on M is *universally tight* if its universal contact covering is tight. A tight contact structure ξ on M is *virtually overtwisted* if there exists a finite contact covering $(M', \xi') \to (M, \xi)$ such that ξ' is overtwisted.

(a) A virtually overtwisted contact structure is not universally tight.

(b) A tight contact structure is either universally tight or virtually overtwisted.
Hint: The geometrization theorem for 3-manifolds implies that the fundamental group of every compact 3-manifold M is *residually finite*, i.e. for every non-trivial element $g \in \pi_1(M)$ there exist a normal subgroup N in $\pi_1(M)$ of finite index that does not contain g. Use this result without proof.

(c) Show that the standard contact structures on $\mathbb{R}^3$, S^3, T^3, and $S^1 \times \mathbb{R}^2$ are universally tight.

(d) Construct a virtually overtwisted contact structure on $S^1 \times D^2$.
Hint: See [V. Colin, *Recollement de variétés de contact tendues*, Bull. Soc. Math. France **127**] or Example 2.27(2) [M. Kegel, Symplektisches Füllen von Torusbündeln, `https://www.mathematik.hu-berlin.de/~kegemarc/Publications/Masterarbeit.pdf`] for a exposition of that result in German.

This sheet will be discussed on Wednesday 10.1. and should be solved by then.