SS 2025

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Topology I

Exercise Sheet 0

This sheet will be discussed in the first exercise session on Thursday, April 17.

Exercise 1.

(a) Let X be a topological space. Show:

   (i) $\emptyset$ and X are closed,

   (ii) the union of any two closed sets is closed,

   (iii) the intersection of arbitrarily many closed sets is closed.

(b) A map $f\colon X \to Y$ between topological spaces X and Y is continuous if and only if the preimage $f^{-1}(A) \subset X$ of every closed set $A \subset Y$ is closed.

Exercise 2.
Let (X, d) be a metric space. A subset $U \subset X$ is called **open** if for every point $x \in U$ there exists $\varepsilon > 0$ such that the open ε-ball around x,

$$B_\varepsilon(x) := \big\{p \in X \big| d(x,p) < \varepsilon\big\},$$

is entirely contained in U.

(a) This defines a topology on X, called the **metric** topology.

(b) An open ball is open.

(c) From introductory courses you know the ε-δ definition of continuity for maps between metric spaces. Show that a map between metric spaces is continuous in the ε-δ sense if and only if it is continuous with respect to the metric topologies.

(d) A subset $U \subset X$ is called a **neighborhood** of a point $x \in X$ if there exists an open set V with $x \in V \subset U$. A topology is called **Hausdorff** if any two distinct points have disjoint neighborhoods. Show that every metric topology is Hausdorff.

(e) Describe topological spaces that are **not** Hausdorff.

Exercise 3.
Show that the surfaces in Figure 1 are homeomorphic.

to be continued

Exercise 4.

(a) The relative topology on a subset A of a topological space X is indeed a topology on A.

(b) Let O be the standard topology on $\mathbb{R}^2$, i.e., the topology defined by the Euclidean metric. Then the induced topology on $\mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is the standard topology on $\mathbb{R}$.

(c) Let $f\colon X \to Y$, where $X = A_1 \cup A_2$ is the union of two closed sets $A_1, A_2 \subset X$. If the restrictions $f|_{A_i}\colon A_i \to Y$ are continuous for $i = 1, 2$, then $f\colon X \to Y$ is also continuous.

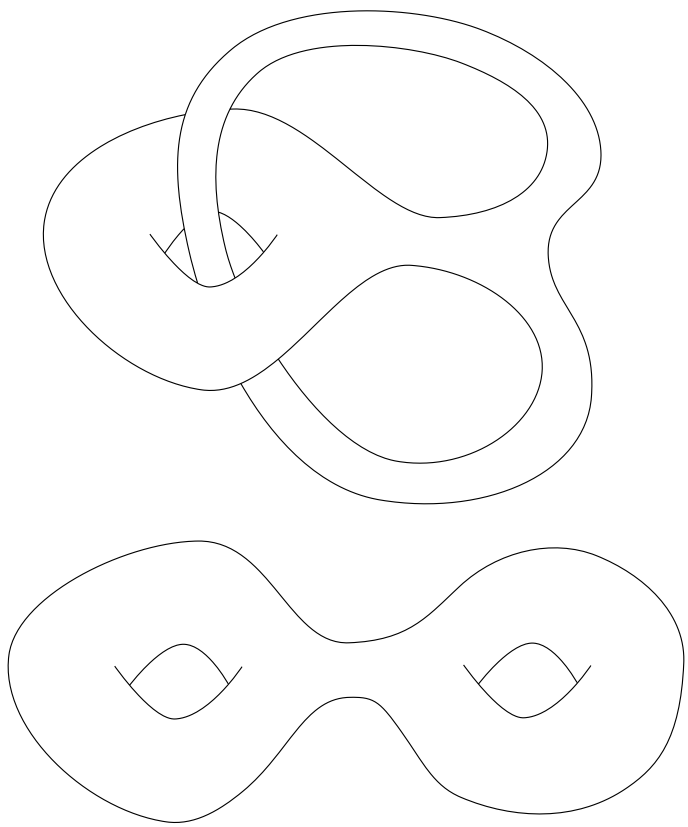

Figure 1: Homeomorphic surfaces

Exercise 5.
Let X be a topological space. A set B of open sets is called a **basis** of the topology if every open set is a union of sets from B.

(a) Convince yourself that every topology has a basis.

(b) $\mathbb{R}^n$ with the standard topology (with respect to the Euclidean metric) has a countable basis.

(c) Describe topological spaces that do not have countable bases.