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Topology I

Exercise Sheet 1

Exercise 1.
We denote by $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ the **closed unit ball** in $\mathbb{R}^n$ and by $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ the **unit sphere**.

(a) $\mathbb{R}^n$ is homeomorphic to $D^n \setminus S^{n-1}$.

(b) The line through the **north pole** $N := (0, ..., 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ and another point $(x_1, \cdots, x_{n+1}) \in S^n$ intersects the equatorial plane $\{x_{n+1} = 0\}$ in exactly one point. This defines a map $S^n \setminus \{N\} \to \mathbb{R}^n$, the so-called **stereographic projection**. Give an explicit formula for this map and use it to show that $\mathbb{R}^n$ is homeomorphic to $S^n \setminus \{p\}$ for any point p in S^n.

Exercise 2.

(a) Describe a space-filling curve, i.e. a continuous and surjective map

$$[0, 1] \to [0, 1] \times [0, 1].$$

(b) Show that $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ if and only if $n = 1$.

(c) Later in the lecture, we will prove invariance of dimension, i.e. we will show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n \neq m$. Convince yourself that this is a non-trivial statement.

Bonus Exercise 1.
Let X and Y be topological spaces. We denote by $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the canonical projections. The **product topology** on $X \times Y$ is the **coarsest** topology (i.e., the topology with the fewest open sets) on $X \times Y$ for which π_X and π_Y are continuous. Describe the open sets in this topology and explicitly show that this defines a topology.

to be continued

Bonus Exercise 2.
Show that the following statements about a topological space X are equivalent:

(i) X is connected.

(ii) The only subsets of X that are both open and closed are $\emptyset$ and X.

(iii) If $X = A \cup B$ with $A, B \neq \emptyset$ subsets of X, then $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. Here $\overline{A}$ denotes the closure of A.

Bonus Exercise 3.
A topological space is called **discrete** if every subset is open, or equivalently: if all singletons are open. This is the **finest** possible topology, i.e. the one with the most open sets. One imagines the points of such a space as lying discretely, as opposed to being continuously distributed. A space is called **totally disconnected** if every connected component consists of exactly one point. Show:

(a) Every discrete space is totally disconnected.

(b) The set $\mathbb{Q}$ of rational numbers, with the topology induced from the real numbers $\mathbb{R}$, is totally disconnected but not discrete.

Bonus Exercise 4.
Work out the details of the sketch of the proof of Theorem 4 (the ham sandwich theorem).

These exercises will be discussed in the session on Thursday, April 24.