SS 2025

Marc Kegel
Theo Müller

Topology I

Problem Sheet 10

Exercise 1.
Prove using the existence of the simplicial approximation:

(a) The set of homotopy classes of continuous maps $|K| \to |L|$ between the underlying topological spaces of two simplicial complexes is countable.

(b) $\pi_1(S^n) = 1$ for $n \geq 2$.

(c) Every continuous map $S^m \to S^n$ with $0 \leq m < n$ is homotopic to a constant map.

Exercise 2.
Compute the homology groups of the following simplicial complexes directly using the definition of homology groups:

(a) Three copies of the boundary of a 2-simplex glued together at one vertex.

(b) Two copies of the boundary of a tetrahedra glued together along an edge.

(c) A complex whose underlying topological space is homeomorphic to a Möbius strip.

Exercise 3.
An oriented q-simplex $\sigma = (x_0, \ldots, x_q)$ **induces** an orientation on each of its $(q-1)$-dimensional faces τ by

$$\tau = (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_q).$$

We call a triangulated n-manifold **orientable** if it is possible to orient its n-simplices so that any two adjacent n-simplices induce opposite orientations on their common $(n-1)$-dimensional face.

(a) Make sketches in dimensions 2 and 3.

(b) Show that this notion of orientability agrees with the definitions from Exercise 3 on Sheet 7.

(c) Show that a triangulable closed n-manifold M is orientable if and only if $H_n(K) \cong \mathbb{Z}$ for every simplicial complex K whose underlying topological space is homeomorphic to M.

Hint: Start with an explicit triangulation of the 2-torus and identify a 2-cycle generating the second homology group. Next consider the Klein bottle. Can there be a 2-cycle on the Klein bottle? Then try to consider the general case, perhaps first only in dimension 2.
Remark: A generator of $H_n(M)$ is also called a **fundamental class** $[M]$ of M. Choosing such a generator therefore fixes an orientation on M.

Exercise 4.

Construct for every finite sequence $G_1, \ldots, G_n$ of finitely generated abelian groups a simplicial complex K with $H_0(K) \cong \mathbb{Z}$ and $H_q(K) \cong G_q$ for $1 \leq q \leq n$, as well as $H_q(K) = 0$ for $q > n$.
Hint: Use the classification theorem for finitely generated abelian groups.

Exercise 5.

(a) Verify explicitly that $\partial\sigma$ (for an oriented q-simplex σ) is well-defined, i.e. independent of the ordering of the vertices within the given orientation class. Further verify that $\partial(-\sigma) = -\partial\sigma$.

(b) A simple closed oriented polygonal curve in a simplicial complex K defines a 1-cycle when one thinks of the curve as the formal sum of the edges it traverses. Show that $Z_1(K)$ is generated by such 'elementary' cycles.

Hint: One way to prove this is via the following steps:

(o) It suffices to prove the statement for simplicial complexes K with $\dim(K) = 1$ and $|K|$ connected.

(i) For the augmentation homomorphism $\varepsilon : C_0(K) \to \mathbb{Z}$ we have $\varepsilon \circ \partial_1 = 0$.

(ii) Let z be a 1-cycle. We may orient the 1-simplices of K so that all coefficients in z are non-negative. Let (x, y) be an edge appearing in z with multiplicity $\lambda > 0$. Let K' be the complex obtained from K by removing the edge (x, y) (but not the vertices x, y). Then $|K'|$ is still connected, because otherwise $z - \lambda(x, y)$ would split in K' into two 1-chains c_1, c_2 with $\varepsilon \circ \partial_1(c_i) \neq 0$.

(iii) There is therefore a path in K' from y to x. This path together with (x, y) defines an elementary cycle z_1 in K. Then $z - \lambda z_1$ is a cycle in K'. Iterate this argument to write z as a sum of elementary cycles.

These exercises will be discussed in the session on Thursday, June 26.