SS 2025

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Topology I

Problem Sheet 11

Exercise 1.
An **exact sequence** is a sequence of groups and homomorphisms

$$\cdots \longrightarrow G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \longrightarrow \cdots$$

such that $\operatorname{Im}(\varphi_i) = \ker(\varphi_{i-1})$ for all i.

(a) Show that any exact sequence can be seen as a chain complex. Compute the homology. Use this to give another (algebraic) interpretation of what homology is measuring.

(b) Describe examples of exact sequences.

(c) Let

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$$

be an exact sequence of abelian groups. Then α is an isomorphism and G is isomorphic to $\mathbb{Z}_6$.

Exercise 2.
A **short exact sequence** of abelian groups is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

(a) The following statements are equivalent:

- There exists a homomorphism $\lambda\colon C \to B$ such that $\beta \circ \lambda = \mathrm{id}_C$.
- There exists a homomorphism $\mu\colon B \to A$ such that $\mu \circ \alpha = \mathrm{id}_A$.

In this case, the short exact sequence is called **split**. Further show that then $B \cong A \oplus C$ holds.

(b) Any short exact sequence of vector spaces splits.

(c) If C is a free abelian group, then every short exact sequence of the above form splits. (If the general case is difficult for you, discuss at least the case $C \cong \mathbb{Z}$.)

(d) Every exact sequence of the form

$$0 \to \mathbb{Z}_m \to G \to \mathbb{Z}_n \to 0,$$

with m and n coprime, splits.

(e) For every natural number $n \geq 2$, there is an exact sequence

$$0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0.$$

Does this sequence split?

(f) For $n \geq 2$, give two abelian groups G which are not isomorphic to each other for which there exists an exact sequence

$$0 \to \mathbb{Z} \to G \to \mathbb{Z}_n \to 0.$$

(g) Show that any long exact sequence can be decomposed into a collection of short exact sequences.

Exercise 3.
Let the simplicial complex K be the union of two simplicial complexes K_1 and K_2 such that their intersection $K_1 \cap K_2$ consists of a single vertex. (We call K the **simplicial wedge sum** of K_1 and K_2.) Show that for $q > 0$

$$H_q(K) \cong H_q(K_1) \oplus H_q(K_2).$$

What holds in the case $q = 0$?

Exercise 4.

(a) Show that the relation 'chain homotopic' is an equivalence relation on the set of chain maps from a chain complex $C = (C_q)$ to a chain complex $D = (D_q)$.

(b) A chain map $f\colon C \to D$ is called a **chain equivalence**, and the two chain complexes are then called **chain equivalent**, if there exists a chain map $g\colon D \to C$ such that $g \circ f$ and $f \circ g$ are chain homotopic to id_C and id_D, respectively. Show that 'chain equivalent' is an equivalence relation on any set of chain complexes.

Bonus Exercise.
Use Lemma 7.10 from the lecture to show that $i_q j_q\colon C_q(K^1) \to C_q(K^1)$ is chain homotopic to the identity. For this, consider that the assumptions of this lemma are satisfied if one defines for a q-simplex $\sigma \in K^1$ the subcomplex $L(\sigma) \subset K^1$ as the barycentric subdivision of the q-simplex of K that contains σ.

These exercises will be discussed in the session on Thursday, July 3.