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Topology I

Exercise Sheet 12

Exercise 1.
Calculate the homology using the Mayer–Vietoris sequence

(a) of the projective plane $\mathbb{R}P^2$, considered as the space obtained by gluing a Möbius band to a 2-disk;

(b) of the 2-torus, obtained by gluing two cylinders;

(c) of the Klein bottle, also obtained by gluing two cylinders;

(d) Further describe the homology of a connected sum $M_1 \# M_2$ of two n-manifolds M_1 and M_2 in terms of the homology of the summands.

Exercise 2.

(a) Show that
$$S^{m+n+1} \cong (S^m \times D^{n+1}) \cup (D^{m+1} \times S^n)$$
with
$$(S^m \times D^{n+1}) \cap (D^{m+1} \times S^n) = S^m \times S^n.$$
Hint: Consider the boundary of $D^{m+n+2} \cong D^{m+1} \times D^{n+1}$.

(b) Use the Mayer–Vietoris sequence to show that for $m \neq n$:
$$H_q(S^m \times S^n) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, m, n, m+n \\ 0 & \text{otherwise.} \end{cases}$$

(c) What is $H_q(S^m \times S^m)$?

Exercise 3.

(a) Complete the proof of Theorem 7.18 from the lecture by verifying that $\mathrm{Im}(j_*) = \ker(\Delta)$ and $\mathrm{Im}(\Delta) = \ker(i_*)$.

(b) Provide a geometric description of the homomorphism Δ in the Mayer–Vietoris sequence (Theorem 7.19) for the decomposition of a complex K into two subcomplexes L and M. In other words: Given a q-cycle x in $K = L \cup M$, how can one geometrically find a $(q-1)$-cycle z in $L \cap M$ such that $\Delta[x] = [z]$?

Exercise 4.
Given a commutative diagram of groups and homomorphisms with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "i_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "i_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "i_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "j_1"] & B_2 \arrow[r, "j_2"] & B_3 \arrow[r, "j_3"] & B_4 \arrow[r, "j_4"] & B_5
\end{tikzcd}$$

Determine the minimal assumptions on f_1, f_2, f_4, and f_5 (regarding injectivity and surjectivity) that guarantee that f_3 is

(i) injective,

(ii) surjective,

(iii) bijective.

Show by examples that these assumptions cannot be further weakened.

Bonus Exercise.
Let L, M be subcomplexes of the simplicial complex $K = L \cup M$ with $|K|$, $|L|$, and $|M|$ path-connected. Derive from the Mayer–Vietoris sequence a description of the first homology group $H_1(L \cup M)$ as a quotient group of $H_1(L) \oplus H_1(M)$ under certain additional relations, analogous to the Seifert–van Kampen theorem.

These exercises will be discussed in the session on Thursday, July 10.