SS 2025

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Topology I

Exercise Sheet 13

During the last exercise session we will have the exam. So this sheet will not be discussed in the exercise session. Nevertheless, you may use these exercises to prepare for the exam.

Exercise 1.

(a) Compute the homology groups of D^n and S^n with coefficients in $\mathbb{Z}_2$ and $\mathbb{Q}$.

(b) Compute the homology groups of all surfaces with coefficients in $\mathbb{Z}_2$ and $\mathbb{Q}$ and compare them with the integral homology groups.

Exercise 2.
Construct for every $n \in \mathbb{N}$ and every $k \in \mathbb{Z}$ a continuous map $f\colon S^n \to S^n$ with $\deg(f) = k$.
Hint: First compute the degree of the map $S^1 \ni z \mapsto z^k \in S^1 \subset \mathbb{C}$. Then construct from a map $f\colon S^n \to S^n$ of degree k a map $S^{n+1} \to S^{n+1}$ of the same degree by viewing S^{n+1} as the suspension of S^n.

Exercise 3.

(a) Compute the Euler characteristic of a connected sum and of a Cartesian product from those of their summands and factors.

(b) The mod 2 Betti numbers of an n-dimensional simplicial complex K are defined as

$$\bar{b}_q = \dim_{\mathbb{Z}_2} H_q(K;\mathbb{Z}_2), \quad q = 0, \dots, n.$$

Show that

$$\chi(K) = \sum_{q=0}^{n} (-1)^q \bar{b}_q.$$

Exercise 4.

(a) Let $f\colon S^n \to S^n$ be a continuous map which extends to a continuous map $F\colon D^{n+1} \to S^n$. Then there exists a point $x \in S^n$ with $f(x) = f(-x)$.

(b) Let $f\colon S^n \to S^n$ be a continuous map with even degree. Then there exists a point $x \in S^n$ with $f(x) = f(-x)$.

(c) Let $f\colon |K| \to |K|$ be a simplicial map of a polyhedron to itself. Then the Lefschetz number $L(f)$ equals the Euler characteristic of the fixed point set of f.

Exercise 5.

(a) Show that the following statements are equivalent to the Borsuk-Ulam theorem:

   (i) Every antipodal continuous map $S^n \to \mathbb{R}^n$ has a zero.

   (ii) There exists no antipodal continuous map $S^n \to S^{n-1}$.

   (iii) There exists no continuous map $D^n \to S^{n-1}$ which is antipodal on the boundary.

(b) Deduce Brouwer's fixed point theorem from the Borsuk-Ulam theorem.
Hint: Both the Borsuk-Ulam theorem and Brouwer's fixed point theorem are true, so trivially one follows from the other. Here a direct elementary proof is sought, which uses only the Borsuk-Ulam theorem without deeper theory to verify Brouwer's fixed point theorem.