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Topology I

Problem Sheet 2

Exercise 1.

(a) Verify that the quotient topology really defines a topology.

(b) The quotient topology is the finest topology (i.e. the topology with the most open sets) for which the canonical projection $\pi\colon X \to X/_\sim$ is continuous.

(c) Let Y be another topological space. A map $f\colon X/_\sim \to Y$ is continuous if and only if $f \circ \pi\colon X \to Y$ is continuous.

(d) Show that the quotient space D^n/S^{n-1} is homeomorphic to S^n. (Here, X/A denotes the quotient space under the equivalence relation $x \sim y :\Leftrightarrow (x = y \text{ or } x, y \in A)$.)

(e) Show that the suspension ΣS^n of the n-sphere S^n is homeomorphic to S^{n+1}.

Exercise 2.

A map $f\colon X \to Y$ is called an **embedding** if it is injective and $f\colon X \to f(X)$ is a homeomorphism when $f(X)$ is equipped with the subspace topology from Y.

(a) Closed subsets of compact spaces are compact.

(b) Compact subsets of Hausdorff spaces are closed. In particular, singletons are closed sets.

(c) Use parts (a) and (b) to show that every continuous and injective map from a compact space to a Hausdorff space is an embedding.

(d) The map

$$f\colon [0, 2\pi] \times [0, \pi] \longrightarrow \mathbb{R}^5$$
$$(x, y) \longmapsto (\cos x, \cos 2y, \sin 2y, \sin x \cos y, \sin x \sin y)$$

induces an embedding of the Klein bottle into $\mathbb{R}^5$.
Hint: Use part (c) and Exercise 1(c).
Bonus Exercise: Can the Klein bottle be embedded into $\mathbb{R}^4$?

(e) Describe continuous bijective maps that are not homeomorphisms.

Exercise 3.

The boundary of a Möbius band is homeomorphic to a circle S^1. Describe an embedding of a Möbius band in $\mathbb{R}^3$ such that its boundary is a standard circle. Create a 3D model of this embedding (e.g. out of paper).

to be continued...

Exercise 4.
The n-dimensional real projective space $\mathbb{R}P^n$ is the quotient of S^n under the identification of antipodal points, i.e. $\mathbb{R}P^n := S^n/_\sim$ with $x \sim y$ for $x, y \in S^n$ if and only if $y = x$ or $y = -x$.

(a) Show that the following definitions are equivalent to this one, i.e., they describe spaces homeomorphic to $\mathbb{R}P^n$:

  (i) Start with $\mathbb{R}^{n+1} \setminus \{0\}$ and identify points lying on the same line through the origin, i.e., consider the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/_\sim$ with $x \sim y$ iff there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. (This describes $\mathbb{R}P^n$ as the space of lines through the origin in $\mathbb{R}^{n+1}$.)

  (ii) Start with the n-dimensional closed disk D^n and identify antipodal points on the boundary $\partial D^n = S^{n-1}$, i.e. define $D^n/_\sim$ with $x \sim y$ if $y = x$ or $y \in S^{n-1}$ with $y = -x$.

(b) Let M be a Möbius band. Its boundary is $\partial M = S^1$. Attach M to a 2-disk D^2 along their boundaries via the identity map, i.e., form $D^2 \cup_\varphi M$ with $\varphi = \mathrm{id}_{S^1}$. Show that this space is homeomorphic to $\mathbb{R}P^2$.

(c) Gluing two Möbius bands along their boundaries yields a Klein bottle.

Bonus Exercise.

(a) Revisit the proof of the Heine–Borel theorem.

(b) Every compact, locally Euclidean space has a countable basis for its topology. A space is called **locally Euclidean** if every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

(c) A topological space X is called **sequentially compact** if every sequence $\{x_i\}_{i \in I}$ has a convergent subsequence. Define what a convergent sequence in a topological space should mean, and show that subsets of $\mathbb{R}^n$ are compact if and only if they are sequentially compact.

(d) Thus, we have seen three notions of compactness (which coincide in $\mathbb{R}^n$): closed and bounded subsets in metric spaces, sequential compactness, and (open cover) compactness. Describe topological spaces in which these notions of compactness do not coincide.

These exercises will be discussed in the session on Thursday, May 8.