SS 2025

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Topology I

Problem Sheet 3

Exercise 1.

The **quaternions** are defined as the 4-dimensional real vector space

$$\mathbb{H} := \{a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

equipped with an associative multiplication given by the rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

and distributivity. The topology on $\mathbb{H}$ is given via the identification with $\mathbb{R}^4$. The Euclidean norm is $|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$, and the **conjugate** of $a \in \mathbb{H}$ is defined as

$$\overline{a} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}.$$

Show:

(a) $\mathbf{ij} = \mathbf{k}$ and $\mathbf{ji} = -\mathbf{k}$.

(b) $\overline{ab} = \overline{b}\,\overline{a}$.

(c) For $a \neq 0$, the element $\overline{a}/|a|^2$ is the inverse of a with respect to multiplication in $\mathbb{H}$.

(d) $|ab| = |a||b|$.

(e) The unit sphere $S^3 \subset \mathbb{H}$, with the topology and multiplication induced from $\mathbb{H}$, is a topological group.

Exercise 2.

Identify $\mathbb{R}^3$ with the space of purely imaginary quaternions, i.e., quaternions of the form $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ with $a_1, a_2, a_3 \in \mathbb{R}$, and S^3 with the space of quaternions of length 1. Show:

(a) Conjugation of $\mathbb{R}^3 \subset \mathbb{H}$ by an element of $S^3 \subset \mathbb{H}$ defines an element of $SO(3)$, i.e. the map

$$\begin{aligned} \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ a &\longmapsto uau^{-1} \end{aligned}$$

is, for each $u \in S^3$, a special orthogonal transformation (thus a rotation of $\mathbb{R}^3$ about a suitable axis).

(b) The map

$$\begin{aligned} S^3 &\longrightarrow SO(3) \\ u &\longmapsto \{a \mapsto uau^{-1}\} \end{aligned}$$

defined in this way is a continuous, surjective homomorphism of topological groups with $\mathrm{Ker} = \{\pm 1\}$.

(c) Deduce that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

p.t.o.

Exercise 3.

(a) The set of real $(n \times n)$ matrices $\mathcal{M}(n \times n, \mathbb{R})$ can canonically be identified with $\mathbb{R}^{n^2}$. This induces a topology on the group of invertible $(n \times n)$ matrices $GL_n(\mathbb{R}) \subset \mathcal{M}(n \times n, \mathbb{R})$. Show that $GL_n(\mathbb{R})$ with this topology is a topological group.

(b) Show that the group of orthogonal $(n \times n)$ matrices $O(n)$, equipped with the topology induced by the inclusion $O(n) \subset GL(n)$, is a compact topological group.

An **isomorphism** between topological groups G_1 and G_2 is a homeomorphism $G_1 \to G_2$ which is also a group isomorphism. Let $SO(n)$ denote the special orthogonal group, i.e. the group of orthogonal $(n \times n)$ matrices with determinant 1. Show:

(c) The multiplicative group $S^1 \subset \mathbb{C}$ is isomorphic to $SO(2)$ (as topological groups).

(d) $O(n)$ is homeomorphic to $SO(n) \times \mathbb{Z}_2$. Are these two topological groups isomorphic?

Exercise 4.

(a) Describe an action of $\mathbb{Z}$ on $\mathbb{R} \times [0, 1]$ that has the Möbius strip as its orbit space.

(b) Describe three different actions of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on the torus with orbit spaces the cylinder, the 2-sphere, and the Klein bottle, respectively.