SS 2025

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Topology I

Exercise Sheet 4

Exercise 1.

(a) For $n \geq 3$, the space $\mathbb{R}^n \setminus \{0\}$ is simply connected.

(b) Every path-connected space is connected.

(c) Deduce once again that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$.

(d) Describe a space that is connected but not path-connected.

Exercise 2.

(a) Every continuous map $f\colon X \to S^n$ that is not surjective is **null-homotopic**, i.e. homotopic to a constant map.

(b) Any two continuous maps $f, g\colon X \to CY$ are homotopic, where CY denotes the cone over Y.

(c) A continuous map $f\colon X \to Y$ is null-homotopic if and only if it extends to a continuous map $CX \to Y$.

Exercise 3.
Let u and v be loops in a topological group $(G, \circ)$ based at the identity element e. Define the loop $u \circ v$ by $(u \circ v)(s) := u(s) \circ v(s)$. Show that

$$uv \simeq u \circ v \simeq vu \quad \mathrm{rel}\,\{0, 1\}$$

and deduce that $\pi_1(G, e)$ is abelian.

Exercise 4.

(a) Let $f\colon (X, x_0) \to (X, x_0)$ be a continuous map that is homotopic to the identity (not necessarily $\mathrm{rel}\,\{x_0\}$). Show that the induced map $f_\star\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ is an inner automorphism, i.e. it has the form

$$f_\star[u] = [v]^{-1}[u][v]$$

for a suitable loop v based at x_0.

(b) Let $f, g\colon X \to Y$ be homotopic maps via a homotopy F. Let x_0 be a base point in X and define the path $u(t) = F(x_0, t)$ from $f(x_0)$ to $g(x_0)$. Then

$$f_\star = u_{\#}^{-1} g_\star\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$

Bonus Exercise 1.
Fill in the missing details in the proofs of Theorem 4.4 and Theorem 4.6 from the lecture.

Bonus Exercise 2.
The **complex projective space** $\mathbb{C}P^n$ is defined as the quotient of $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ (or of $S^{2n+1} \subset \mathbb{C}^{n+1}$) under the equivalence relation

$$(z_0, \dots, z_n) \sim (w_0, \dots, w_n) \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{C} \setminus \{0\} : (z_0, \dots, z_n) = (\lambda w_0, \dots, \lambda w_n).$$

The equivalence class of a point $(z_0, \dots, z_n)$ is denoted by its so-called **homogeneous coordinates** $[z_0 : \dots : z_n]$. One can also view $\mathbb{C}P^n$ as the space of complex lines through the origin in $\mathbb{C}^{n+1}$.

The **one-point compactification** $\widehat{\mathbb{C}}^n$ of $\mathbb{C}^n$ is defined as the set $\widehat{\mathbb{C}}^n = \mathbb{C}^n \cup \{\infty\}$ (i.e. the disjoint union of $\mathbb{C}^n$ and a point denoted ∞), equipped with the following topology: The open sets of $\widehat{\mathbb{C}}^n$ are precisely the open subsets of $\mathbb{C}^n \subset \widehat{\mathbb{C}}^n$ together with the sets of the form $\widehat{\mathbb{C}}^n \setminus K$ where $K \subset \mathbb{C}^n$ is compact. Show that:

(a) $\widehat{\mathbb{C}}^n$ is indeed a compact topological space.

(b) $\widehat{\mathbb{C}}^n$ is homeomorphic to S^{2n}.
Hint: Use the stereographic projection from Exercise 1(b) of Sheet 1.

(c) The map

$$\mathbb{C}P^1 \longrightarrow \widehat{\mathbb{C}}, \quad [z_0 : z_1] \longmapsto \begin{cases} z_1/z_0, & \text{if } z_0 \neq 0, \\ \infty, & \text{if } z_0 = 0 \end{cases}$$

is a homeomorphism.

These exercises will be discussed in the session on Thursday, May 15.