SS 2025

Marc Kegel
Theo Müller

Topology I

Problem Sheet 5

Exercise 1.
The goal of this exercise is to prove Theorem 4.14. Let X be simply connected and let G be a topological group acting discretely on X. Show:

(a) $\pi\colon X \to X/G$ is a covering map.

(b) The fundamental group $\pi_1(X/G)$ is isomorphic to G.
Hint: Proceed similarly to the proof of Theorem 4.13.

Exercise 2.
Determine the universal cover of the Klein bottle and describe the fundamental group of the Klein bottle. Can the Klein bottle carry the structure of a topological group?

Exercise 3.
We view S^1 as the unit circle in $\mathbb{C}$. Describe the homomorphism

$$f_\star\colon \pi_1(S^1, 1) \to \pi_1(S^1, f(1)),$$

for the following maps $f\colon S^1 \to S^1$:

(a) $f(e^{i\theta}) = e^{i(\theta+\pi/2)}$,

(b) $f(e^{i\theta}) = e^{in\theta}$, for $n \in \mathbb{Z}$,

(c) $f(e^{i\theta}) = \begin{cases} e^{i\theta}, & \text{if } 0 \leq \theta \leq \pi, \\ e^{i(2\pi-\theta)}, & \text{if } \pi \leq \theta \leq 2\pi. \end{cases}$

Exercise 4.

(a) Describe a space that is path-connected but not locally path-connected.

(b) Describe a space that is locally path-connected but not semi-locally simply connected.

(c) Describe a space that is semi-locally simply connected but not locally simply connected.

Hint: These exercises can be solved easily using the right tools (e.g. internet search engines or a book on set-theoretic topology). However, to truly become familiar with these concepts, I recommend that you avoid using such aids while working on these exercises.

Bonus Exercise.
Prove the homotopy lifting property for paths, i.e., prove Lemma 4.12 from the lecture.
Hint: Proceed similarly to the proof of Lemma 4.11.

These exercises will be discussed in the session on Thursday, May 22.