Topology I

Problem Sheet 8

Exercise 1.

(b) Describe handle decompositions of the Klein bottle and of $\mathbb{R}P^2 \# \mathbb{R}P^2$ consisting of exactly one 0-handle, and translate these handle decomposition into words as discussed in the lecture. Use modifications of this words to show that the Klein bottle is homeomorphic to the connected sum of two copies of $\mathbb{R}P^2$.

(c) Which handle decomposition of which surface is described by the word

$$a\,b\,a\,c\,d\,e\,c\,f\,g\,h\,d\,g^{-1}f^{-1}h^{-1}b^{-1}e^{-1}?$$

Exercise 2.
From every closed, non-orientable surface, one can obtain an orientable surface with boundary by cutting along a single simple closed curve (i.e. a closed curve without self-intersections).

Exercise 3.

(a) Classify compact 1-manifolds (possibly with boundary).
Hint: Proceed here and in the next exercise similarly to the proof discussed in the lecture.

(b) Classify compact surfaces with non-empty boundary.

(c) Show that two closed surfaces are homeomorphic if and only if they are homotopy equivalent. Does this also hold for compact surfaces with boundary?

Exercise 4.

(a) Embeddings $[-1,1] \to \mathbb{R}$ are precisely the strictly monotone functions $[-1,1] \to \mathbb{R}$.

(b) Let $\varphi_1, \varphi_2 \colon [-1,1] \to \mathbb{R}$ be strictly increasing functions. Show that there exists a homeomorphism $h \colon \mathbb{R}^2_- \to \mathbb{R}^2_-$ with $h = \mathrm{Id}$ outside a compact set and $h \circ \varphi_1 = \varphi_2$, where we view $\mathbb{R}$ as the boundary of $\mathbb{R}^2_- = \{(x,y) \in \mathbb{R}^2 \,|\, y < 0\}$.
Hint: First construct such a homeomorphism $\mathbb{R} \to \mathbb{R}$.

(c) Use this to fill in the details of Lemmas 5.9 and 5.10 from the lecture, i.e. show that we can assume without loss of generality that all 1-handles are attached to the boundary of the 0-handle, and that the attachments depend only on the monotonicity behavior and images of the attaching maps.

Bonus Exercise.
Every compact 3-manifold admits a handle decomposition.
Hint: Proceed analogously to the proof given for surfaces in the lecture.
Puzzle Exercise: Every smooth, compact n-manifold admits a handle decomposition.

Bonus Exercise.
Let G be an arbitrary finitely presented group and $n \geq 5$. Construct a closed n-manifold M with $\pi_1(M) \cong G$.
Puzzle Exercise: Does this statement also hold in other dimensions?

Puzzle Exercise.
Compute the fundamental group of the 3-manifold described by the planar Heegaard diagram shown in Figure 1. Which 3-manifold is described by this Heegaard diagram?

Puzzle Exercise.
Describe a planar Heegaard diagram of the 3-torus $T^3 := S^1 \times S^1 \times S^1$.
Hint: Recall that the 3-torus can be obtained from a cube $I \times I \times I$ by identifying opposite faces. Then construct a handle decomposition of T^3 and translate it into a planar Heegaard diagram.

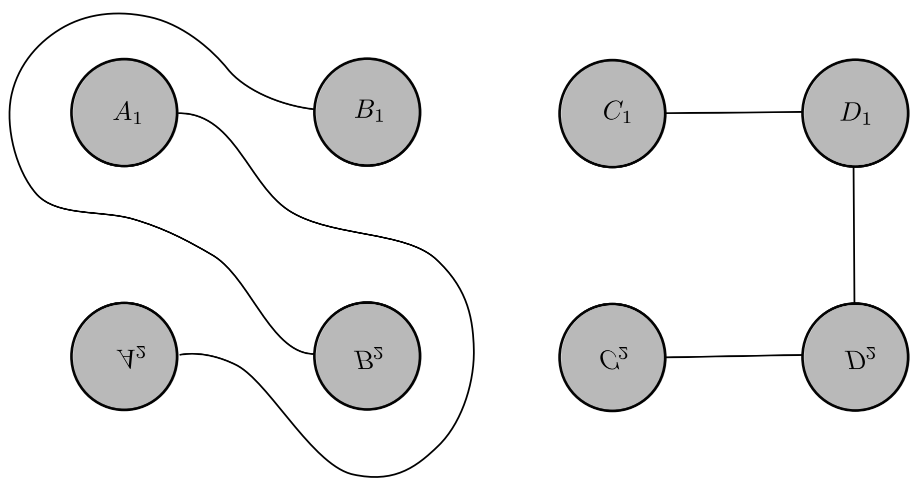

Figure 1: The attaching disks of the 1-handles are identified in pairs via reflection along the horizontal centerline of this planar Heegaard diagram.

These exercises will be discussed in the session on Thursday, June 12.