① Transversality

Motivation: Regular Value Theorem

X, Y manifolds, $f: X \to Y$.

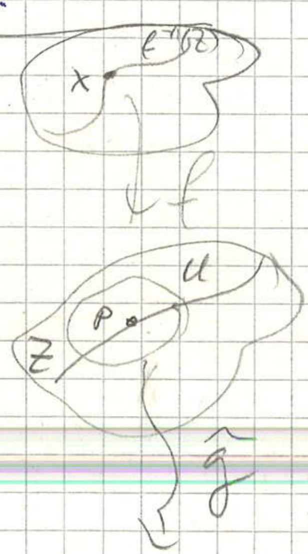
$f^{-1}(\{p\})$ is submanifold in X for $p \in Y$ if p is regular.

($\Leftrightarrow \forall x \in f^{-1}(\{p\}): df_x$ is surjective.)

Can observe: single points are 0-dim manifolds

$f^{-1}(Z)$ for $Z \subseteq Y$ submanifold

When is this submanifold in X?



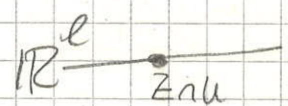
Let $x \in f^{-1}(Z)$, $p := f(x)$.

$Z \subseteq X$ submanifold $\Rightarrow$ get a chart on a neighbourhood U of p:

$$g = \overbrace{(\underbrace{g_1, \dots, g_l}_{\operatorname{codim} Z}, \underbrace{g_{l+1}, \dots, g_n}_{\dim Z})}^{\dim Y}$$

$$U \xrightarrow{\sim} \tilde{U} \subseteq \mathbb{R}^n,$$

$$g_1|_{Z \cap U}, \dots, g_l|_{Z \cap U} \equiv 0$$



Define $\tilde{g} := (g_1, \dots, g_l)$.

Then $\tilde{g}: U \to \mathbb{R}^l$,

$$\boxed{U \cap Z = \tilde{g}^{-1}(\{0_{\mathbb{R}^l}\})}$$

$x \in V := f^{-1}(U)$.

Look at $\tilde{g} \circ f|_V : V \to \mathbb{R}^l$.

Then $V \cap f^{-1}(Z) = (\tilde{g} \circ f|_V)^{-1}(\{0_{\mathbb{R}^l}\})$

$\Rightarrow$ This is submanifold if $0_{\mathbb{R}^l}$ is regular value of $(\tilde{g} \circ f)|_V$

② $\Leftrightarrow \operatorname{Im}(d\tilde{g}_{f(\tilde{x})} \circ df_{\tilde{x}}) = \mathbb{R}^l \quad \forall \tilde{x} \in V \cap f^{-1}(Z)$

Now $T_pZ = \ker(d\tilde{g}_p)$ ($\tilde{g}$ maps Z to 0).
$\tilde{g}$ is still submersion, so ~~above condition translates to~~: $\operatorname{Im}(df_{\tilde{x}})$ has to span the rest of $T_{f(\tilde{x})}Y$:

$V \cap f^{-1}(Z)$ ~~has~~ is a submanifold of X if $\forall \tilde{x} \in V \cap f^{-1}(Z)$:

$$\operatorname{Im}(df_{\tilde{x}}) + T_{f(x)}Z = T_{f(x)}Y.$$

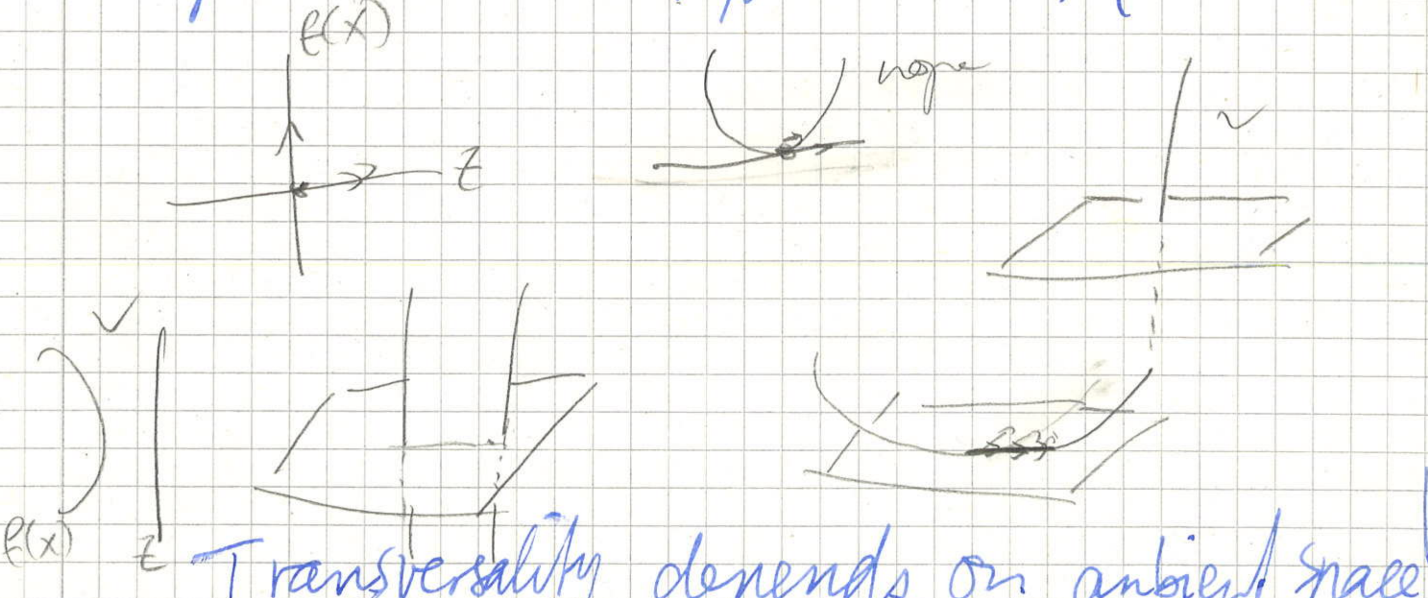
Going from local to global ⇝ summarize

Theorem / Definition:
If $\forall x \in f^{-1}(Z)$:

$$\operatorname{Im}(df_x) + T_{f(x)}Z = T_{f(x)}Y$$

then $f^{-1}(Z) \subseteq X$ is a sub-mnf of $\operatorname{codim}_X(f^{-1}(Z)) = \operatorname{codim}_Y(Z)$.
We say $f \pitchfork Z$ (f transversal to Z)
Proof: above, regular value theorem

Examples: Main Example $X \subseteq Y$, $f: X \hookrightarrow Y$



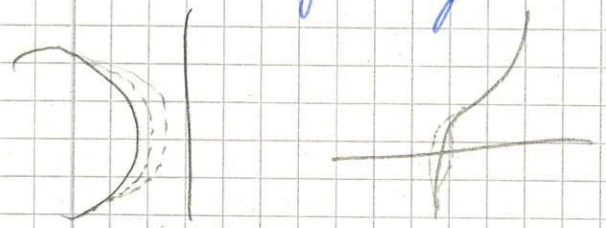
Transversality depends on ambient space!

3 Have observed already

Some properties of Transversality

stability (under certain conditions)

If f is transversal and only slightly disturbed, it remains stable.



$\rightsquigarrow$ transversality is "observable", since we always have small errors in "nature."

generic (generate)

meaning: Can deform any non-transversal f by arbitrarily small amount to make f transversal.

$\rightsquigarrow$ non-transversal maps cannot be observed.

First step

Transversality Theorem

X, S, Y manifolds, $Z \subseteq Y$ submnf.

$F: X \times S \to Y$ smooth, X possibly with boundary.

$F \pitchfork Z \Rightarrow$ almost all $s \in S$:

$f_s := F(\cdot, s) \pitchfork Z$.

Idea: S is parameter for distortions of f.

Proof: $W := F^{-1}(Z) \subseteq X \times S$, $\pi: X \times S \to S$.

$\boxed{\pi|_W: W \to S \quad \text{almost all } s \in S \text{ are regular values}}$ empty $\Leftarrow$

via Sard's Theorem. (special case regular)

- Let $s \in S$ regular for $\pi|_W \rightsquigarrow$ show $f_s \pitchfork Z$.

Take a point $f_s(x) = z \in Z$.

④ $F \pitchfork Z \Rightarrow \mathrm{Im}(dF_{(x,s)}) + T_z Z = T_z Y$

$\Leftrightarrow \forall a \in T_z Y \ \exists b \in T_{(x,s)}(X \times S)$:

or +, does not matter

$$dF_{(x,s)}(b) \cancel{+} - a \in T_z Z$$

$$T_{(x,s)}(X \times S) \cong T_x X \times T_s S$$

$$b = (v, w) \qquad dF_{(x,s)}(v)$$

If $w = 0$, then even $\overset{dF_{(x,s)}(v)}{\overset{\|}{df_s(v)}} - a \in T_z Z$.

If not: Use regularity of s for $\pi|_W$:

$$(d\pi|_W)^{-1}(w) = (\tilde{v}, w) \in \cancel{T_z Z}\ T_{(x,s)} W$$

$$dF_{(x,s)}(v - \tilde{v}, 0) + \underbrace{dF_{(x,s)}(\tilde{v}, w)}_{\in T_z Z} - a \in T_z Z$$

$$\Rightarrow dF_{(x,s)}((v - \tilde{v}, 0)) - a \in T_z Z$$

$$df_s(v - \tilde{v}) - a \in T_z Z$$

$$\Rightarrow f_s \pitchfork Z. \qquad \square$$

Next step: How to obtain such F, relation to f?

Driving example: $Y = \mathbb{R}^n$. $S = \mathbb{R}^n$ or unit ball $B \subseteq \mathbb{R}^n$

$$F(x, s) = f_s(x) = f(x) + s.$$

Clearly $F \pitchfork Z$ for any Z due to submersion.

Regular f_s is homotopic to f via

$$g(x, t) := f_{t \cdot s}(x).$$

$\Rightarrow$ can be generalized to arbitrary Y.

Transversality Homotopy Theorem:

$\forall f : X \to Y$, $Z \subseteq Y$ submanifold (X possibly boundary)

$\exists g : X \to Y$, $g \pitchfork Z$ and g homotopic to f.

Idea:

$Y \subseteq \mathbb{R}^n$. Get ε-nghb. of ~~Y~~ Y ~~...~~ in $\mathbb{R}^n$

(Y^ε) s.t. $Y^\varepsilon \twoheadrightarrow Y$ is smooth, well-defined

Then can vary f with $S = \varepsilon \cdot B$ and project down to Y.

5 ~~(generic)~~

~~Even stronger result~~

What Theorem does not say:

Can control the "amount of distortion" by the "size of s" $\in S$.

Can make theorem even stronger:

Definition:

$f: X \to Y$, $Z \subseteq Y$ submf, $C \subseteq X$ arbitrary.

$f \pitchfork Z$ on C if $\forall x \in C \cap f^{-1}(Z)$:

$\mathrm{Im}(df_x) + T_{f(x)} Z = T_{f(x)} Y$

Extension Theorem:

If $f \pitchfork Z$ on C, C closed (X possibly with boundary), then can find

$g \pitchfork Z$ homotopic to f and

$g_{|U} = f_{|U}$ for open neighbourhood U of C.

Proof: Partition of unity.

Meaning: