Differential Topology - Chapter 1: Smooth Manifolds

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November 13, 2018

In the following we will outline several fundamental definitions to be used over the course of this seminar while following along J.W. Milnor's *Topology from the Differentiable Viewpoint*. Furthermore we will regard $\mathbb{R}^n$ to be under the euclidean topology throughout this chapter.

Definition. Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$ be open subsets. A map $f : U \to V$ is called *smooth* if for all natural numbers n all of the partial derivatives $\frac{\partial^n f}{\partial x_{i_1} \dots \partial x_{i_n}}$ exist and are continuous.

We will now generalize this definition to apply it to functions with more varied domains.

Definition. Let $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^l$ be arbitrary subsets. A map $f : X \to Y$ is called *smooth* if for every $x \in X$ a neighbourhood $U \subseteq \mathbb{R}^k$ of x and a smooth map $g : U \to \mathbb{R}^l$ exist such that g coincides with f throughout $U \cap X$.

Remark. For two smooth functions $f : X \to Y$ and $g : Y \to Z$ their composition $g \circ f : X \to Z$ is also smooth.

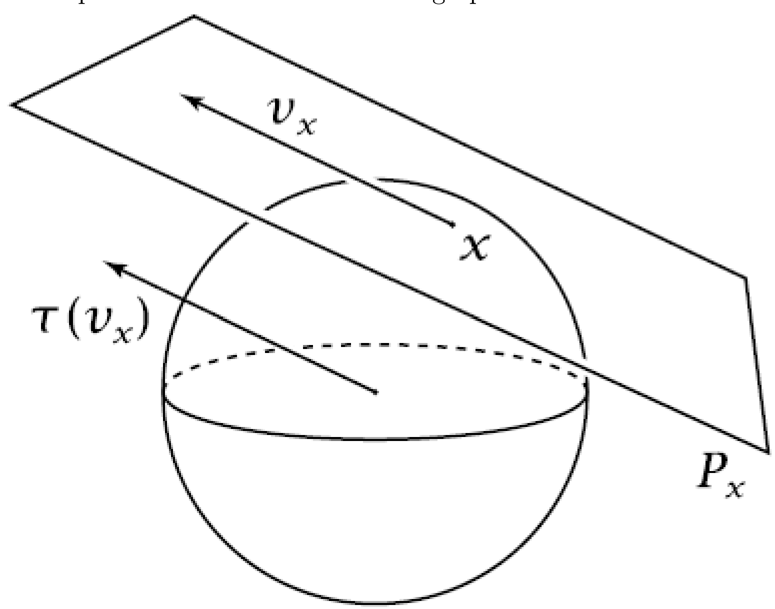
Definition. Let $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^l$ be arbitrary subsets. A map $f : X \to Y$ is called a *diffeomorphism* if it is a homeomorphism between X and Y and if both f and f^{-1} are smooth.

Definition. A set $M \subseteq \mathbb{R}^k$ is called a *smooth manifold* of dimension m if for every $x \in M$ a neighbourhood $W \cap M$ (where $W \subseteq \mathbb{R}^k$ is open) of x exists that is diffeomorphic to an open subset $U \subseteq \mathbb{R}^m$

Remark.

- A diffeomorphism $f : U \to W \cap M$ is called a *parametrization* of the region $W \cap M$
- A diffeomorphism $f^{-1} : W \cap M \to U$ is called a *system of coordinates* on $W \cap M$
- We will occasionally need to look at smooth manifolds of dimension 0. It is interesting to note that M being such a manifold is, per definition, equivalent to each $x \in M$ having a neighbourhood $W \cap M$ s.t. $W \cap M = \{x\}$

We now wish to define the notion of the derivative df_x for smooth maps $f : M \to N$ where M and N are smooth manifolds. To do this, we'll first have to find a linear subspace to associate with each $x \in M \subseteq \mathbb{R}^k$ - the tangent space, TM_x. Then we will be able to define df_x as a linear map from TM_x to $TN_{f(x)}$. Elements of the tangent space will be called *tangent vectors* to M at x.
The tangent space can be described a bit more intuitively by first considering the m-dimensional hyperplane in $\mathbb{R}^k$ which best approximates M near x and then considering TM_x to simply be a parallel hyperplane through the origin. An example for this can be seen in the graphic below.



Before we are able to define the tangent space and df_x in general, we will once more have to take a closer look at the case of maps between open subsets.

Definition. Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$ be open subsets. The tangent space for every $x \in U$ is then defined as

$$TU_x := \mathbb{R}^k.$$

Furthermore we can define the derivative for every smooth $f : U \to V$ as

$$\begin{aligned} df_x \colon \mathbb{R}^k &\to \mathbb{R}^l \\ h &\mapsto \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}. \end{aligned}$$

Remark. Using this definition df_x is a linear map. This becomes fairly obvious once one realizes that the map is equivalent to the Jacobian Matrix of f evaluated at x.

We will now note a few fairly important and fundamental properties the derivative as defined above possesses. Let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$ be open subsets and $x \in U$:

1. **(Chain Rule)** Let $W \subseteq \mathbb{R}^m$ be an open subset.
   If $f : U \to V$ and $g : V \to W$ are smooth maps, then

   $$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

2. If I is the identity map on U, then dI_x is the identity map on $\mathbb{R}^k$. More generally, if $U' \subseteq U$ is an open subset, $i : U' \to U$ its inclusion and $x' \in U'$, then $di_{x'}$ is also the identity map on $\mathbb{R}^k$.

3. If $L : \mathbb{R}^k \to \mathbb{R}^l$ is a linear map and $x \in \mathbb{R}^k$ then $dL_x = L$.

We can use these properties to easily prove a theorem:

Theorem. *If f is a diffeomorphism between open sets $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^l$, then $k = l$ and df_x is regular for all $x \in U$.*

Proof. We know $f^{-1} \circ f$ is the identity map on U. Thus (2) combined with the chain rule implies $df^{-1}_{f(x)} \circ df_x$ is the identity map on $\mathbb{R}^k$ for all $x \in U$. Analogously $df_x \circ df^{-1}_{f(x)}$ is the identity on $\mathbb{R}^l$. It follows that df_x is regular and thus a bijective linear map between $\mathbb{R}^k$ and $\mathbb{R}^l$, so $l = k$. $\square$

A partial converse to this theorem also holds and is probably already known to most.

Theorem (Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^k$ be open and $f : U \to \mathbb{R}^k$ be smooth. If df_x is regular for $x \in U$, then an open set $U' \subseteq U$ containing x exists such that $f(U')$ is open and $f : U' \to f(U')$ is a diffeomorphism.*

An elegant proof of this can be found in Michael Spivak's *Calculus on Manifolds*.
We've now reached the point at which we're able to generalize the definition of the tangent space to arbitrary smooth manifolds.

Definition. Let $M \subseteq \mathbb{R}^k$ be a smooth manifold of dimension m and $x \in M$. Now let

$$g : U \to M$$

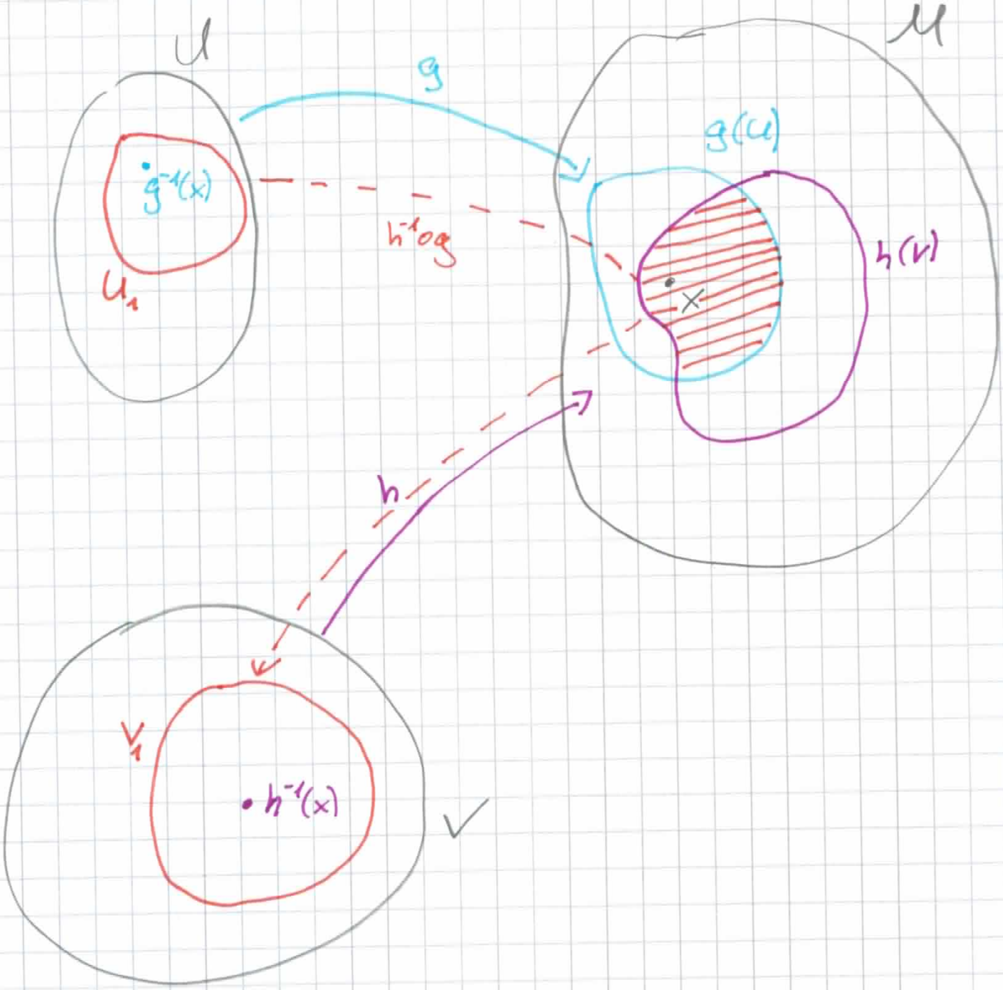
be a choice of diffeomorphism (parametrization), such that $U \subseteq \mathbb{R}^m$ is open and $g(U) \subseteq M$ is a neighbourhood of x. Now let $dg_{g^{-1}(x)}$ be the derivative as defined above. Then the tangent space TM_x is defined:

$$TM_x := dg_{g^{-1}(x)}(\mathbb{R}^m)$$

This definition does not depend on the choice of parametrization, but since this is not immediately obvious, we will have to prove it.

Proof. Let $h : V \to M$ be another choice of parametrization, s.t. $V \subseteq \mathbb{R}^m$ is open and $h(V) \subseteq M$ is a neighbourhood of x.
Then $h^{-1} \circ g : \mathbb{R}^m \to \mathbb{R}^m$ maps some neighbourhood U_1 of $g^{-}(x)$ diffeomorphically onto a neighbourhood V_1 of $h^{-1}(x)$. (See sketch on the next page)

Sketch (Proof of the fact that TM_x is well-defined)



So now the commutative diagram

$$\begin{tikzcd} & \mathbb{R}^k & \\ U_1 \arrow[ur, "g"] \arrow[rr, "h^{-1}\circ g"'] & & V_1 \arrow[ul, "h"'] \end{tikzcd}$$

gives rise to the commutative diagram

$$\begin{tikzcd} & \mathbb{R}^k & \\ \mathbb{R}^m \arrow[ur, "dg_{g^{-1}(x)}"] \arrow[rr, "d(h^{-1}\circ g)_{g^{-1}(x)}"'] & & \mathbb{R}^m \arrow[ul, "dh_{h^{-1}(x)}"'] \end{tikzcd}$$

It thus follows that $dg_{g^{-1}(x)}(\mathbb{R}^m) = dh_{h^{-1}(x)}(\mathbb{R}^m)$ and therefore TM_x is well-defined. $\square$

We now have a working definition of a tangent space, but we still have to show it is actually of the correct dimension.

Theorem. *For M and x as before, TM_x is an m-dimensional vector space.*

Proof. Let M, x, g and U be defined as above. Since $g^{-1} : g(U) \to U$ is smooth, per definition there is an open set $W \subseteq \mathbb{R}^k$ and a smooth map $F : W \to \mathbb{R}^m$ such that $x \in W$ and F coincides with g^{-1} on $W \cap g(U)$. If we now define $U_0 := g^{-1}(W \cap g(U))$ the following diagram commutes:

$$\begin{tikzcd} & W \arrow[dr, "F"] & \\ U_0 \arrow[ur, "g"] \arrow[rr, hook, "inclusion"'] & & \mathbb{R}^m \end{tikzcd}$$

It follows that

$$\begin{tikzcd} & \mathbb{R}^k \arrow[dr, "dF_x"] & \\ \mathbb{R}^m \arrow[ur, "dg_{g^{-1}(x)}"] \arrow[rr, "identity"'] & & \mathbb{R}^m \end{tikzcd}$$

also commutes. It is now obvious that $dg_{g^{-1}(x)}$ has rank m and therefore TM_x is of dimension m. $\square$

We have now reached the point at which we are able to achieve our original goal of defining df_x for functions between smooth manifolds.

Definition. Let $M \subseteq \mathbb{R}^k$ be an m-dimensional smooth manifold, $N \subseteq \mathbb{R}^l$ an n-dimensional smooth manifold, $f : M \to N$ a smooth map and $x \in M$.
Since f is smooth an open set $W \subseteq \mathbb{R}^k$ containing x and a smooth map

$$F : W \to \mathbb{R}^l$$

exist such that F coincides with f on $W \cap M$. The derivative

$$df_x : TM_x \to TN_{f(x)}$$

is then defined as

$$df_x := dF_x\big|_{TM_x}.$$

We now need to prove two things to show the validity of this definition: First, that $dF_x(TM_x) \subseteq TN_{f(x)}$ and second that it is well-defined.

Proof. Let $h : V \to N$ be a parametrization such that $V \subseteq \mathbb{R}^k$ is open and $h(V)$ is a neighbourhood of $f(x)$. Now choose a parametrization $g : U \to M$ such that $U \subseteq \mathbb{R}^m$ is open, $x \in g(U)$, $g(U) \subseteq W$ and $f(g(U)) \subseteq h(V)$.
Then $h^{-1} \circ f \circ g : U \to V$ defines a smooth map and the following diagram commutes:

$$\begin{tikzcd}
W \arrow[r, "F"] & \mathbb{R}^l \\
U \arrow[u, "g"] \arrow[r, "h^{-1}\circ f\circ g"] & V \arrow[u, "h"']
\end{tikzcd}$$

This implies that the following diagram of the derivatives commutes as well:

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^l \\
\mathbb{R}^m \arrow[u, "dg_{g^{-1}(x)}"] \arrow[r, "d(h^{-1}\circ f\circ g)_{g^{-1}(x)}"] & \mathbb{R}^n \arrow[u, "dh_{h^{-1}(f(x))}"']
\end{tikzcd}$$

It follows that

$$dF_x(TM_x) = dF_x(dg_{g^{-1}(x)}(\mathbb{R}^m)) \subseteq dh_{h^{-1}(f(x))}(\mathbb{R}^n) = TN_{f(x)}$$

Furthermore, since

$$dF_x \circ dg_{g^{-1}(x)} = dh_{h^{-1}(f(x))} \circ d(h^{-1} \circ f \circ g)_{g^{-1}(x)}$$

df_x is independent from the choice of F and thus well defined. □

Remark. Once again the derivative has two fundamental properties worth noting. Let M, N be smooth manifolds and $x \in M$.

1. **(Chain Rule)** Let P be a smooth manifold and $f : M \to N$ and $g : N \to P$ be smooth maps. Then the following holds:

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

2. If I is the identity map on M, then dI_x is the identity on TM_x.
   If $M \subseteq N$ and i is the inclusion map, then $TM_x \subseteq TN_x$ with inclusion di_x.

These properties lead to an analogous theorem to the one their previous counterparts lead to with the same proof behind it.

Theorem. *If M, N are smooth manifolds and $f : M \to N$ is a diffeomorphism, then the derivative $df_x : TM_x \to TN_{f(x)}$ is regular and $dim(M) = dim(N)$.*

Sources

Milnor, J. W. (1965) *Topology from the Differentiable Viewpoint.* USA: The University Press of Virginia.
Spivak, M. (1971), *Calculus On Manifolds: A Modern Approach To Classical Theorems Of Advanced Calculus.* USA: Westview Press.

Pictures: https://ncatlab.org/nlab/files/TangentSpaceToSphere.png