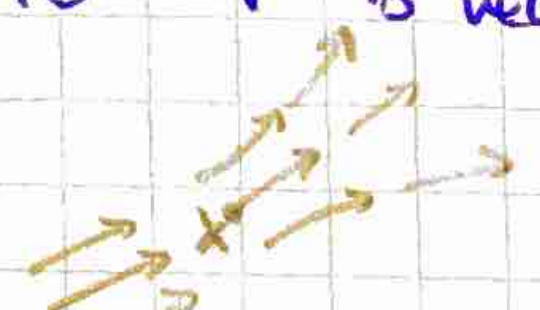
Vectorfields and Euler number

As further application of the concept of degree - which we heard last week - we will study vector fields on other manifolds.

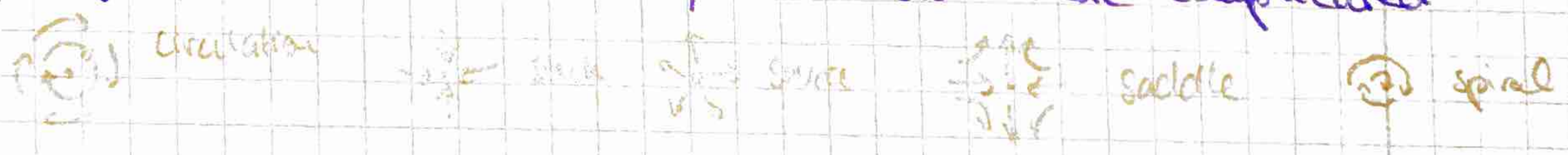
Def A vector field on a manifold X in $\mathbb{R}^N$ is a smooth assignment of a tangent vector to each point in X
which is a smooth map $\vec{v}: X \to \mathbb{R}^N$ s.t. $\vec{v}(x) \in T_x X$

all interesting behavior occurs at "zeros", the points $x \in X$ where $v(x) = 0$
Since for $v(x) \neq 0$ v is nearly constant in magnitude and direction near x. Like:



However when $v(x) = 0$ the direction of v may change radically in any small neighborhood ~~around~~ of x
The field may circulate around x | have a source | sink | saddle
may spiral in toward x or away or even more complicated

Pics



To investigate the relation between v and (topology of) X we will look at the directional change around its zeros
~~Therefore we look at the fct. $\bar{v}(x) = \frac{v(x)}{\|v(x)\|}$ where v~~

Therefore we consider first an open set $U \subset \mathbb{R}^m$ and smooth vector field $v: U \to \mathbb{R}^m$ with isolated zero at point $z \in U$
the function $\bar{v}(x) = \frac{v(x)}{\|v(x)\|}$ maps a small sphere centered at z into the unit sphere.

Def the degree of $\bar{v}(x)$ is called index i of v at zero z.

In two-dim case i ~~is~~ simply counts the ~~~~ # of times v rotates completely while we walk counterclockwise
counterclock = +1 clockwise = -1

Examples

In the plane of complex numbers the polynomial z^k defines a smooth vector field with zero of index k at the origin and $\bar{z}^k \Rightarrow i = -k$

We have to show that this concept of index is invariant under diffeomorphism of U

To explain what this means we'll look at a more general situation of a map $f: M \to N$ with with a vector field on each manifold

Definition The vector fields v (on M) and v' (on N) ~~to~~ correspond under f if ~~df~~ $d_x f$ carries $v(x)$ into $v'(f(x))$ $\forall x \in M$

If f is a diffeomorphism v' is uniquely defined by v

$$v' = df \circ v \circ f^{-1}$$

Lemma 1 Suppose the vector field v on U corresponds to v' on U' under a diffeomorphism $f: U \to U'$ $\quad (v' = df \circ v \circ f^{-1})$

THEN the index of v at an isolated zero z is equal to the index of v' at $f(z)$

following Lemma 1. the concept for index for a vector field w on arbitrary manifold M is ~~as~~

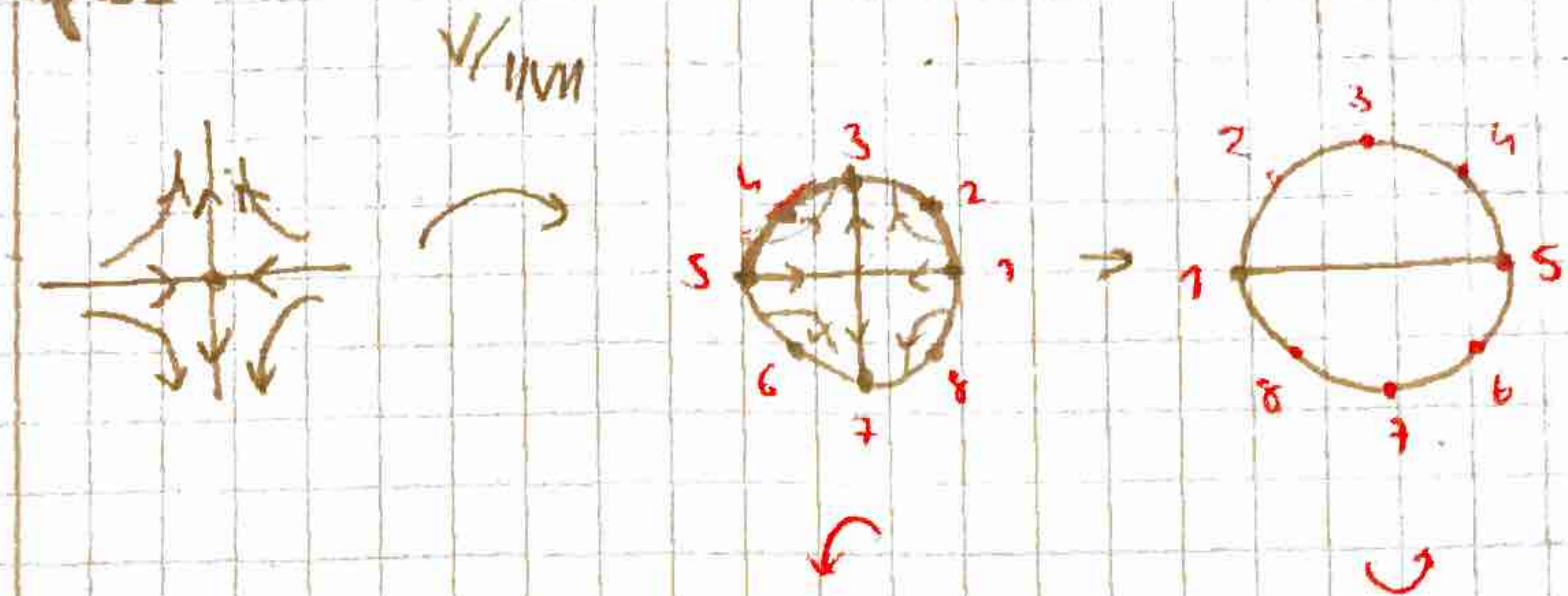
if $g: U \to M$ is parametriz. of a neighbd of z in M

then i of w at z is equal to i of $dg^{-1} \circ w \circ g$ on U at zero $g^{-1}(z)$

$\Rightarrow$ i well defined

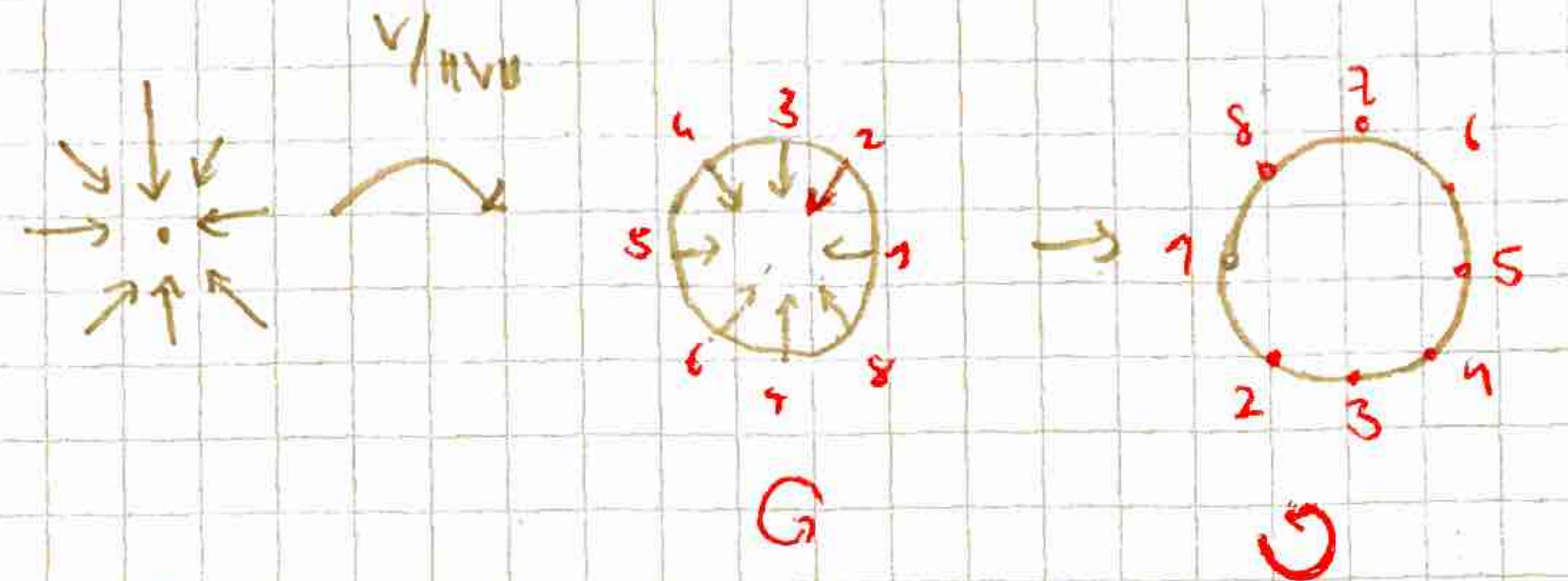
Lemma 2 Any orientation preserving diffeo f of $\mathbb{R}^m$ is smoothly isotopic to the identity

Examples



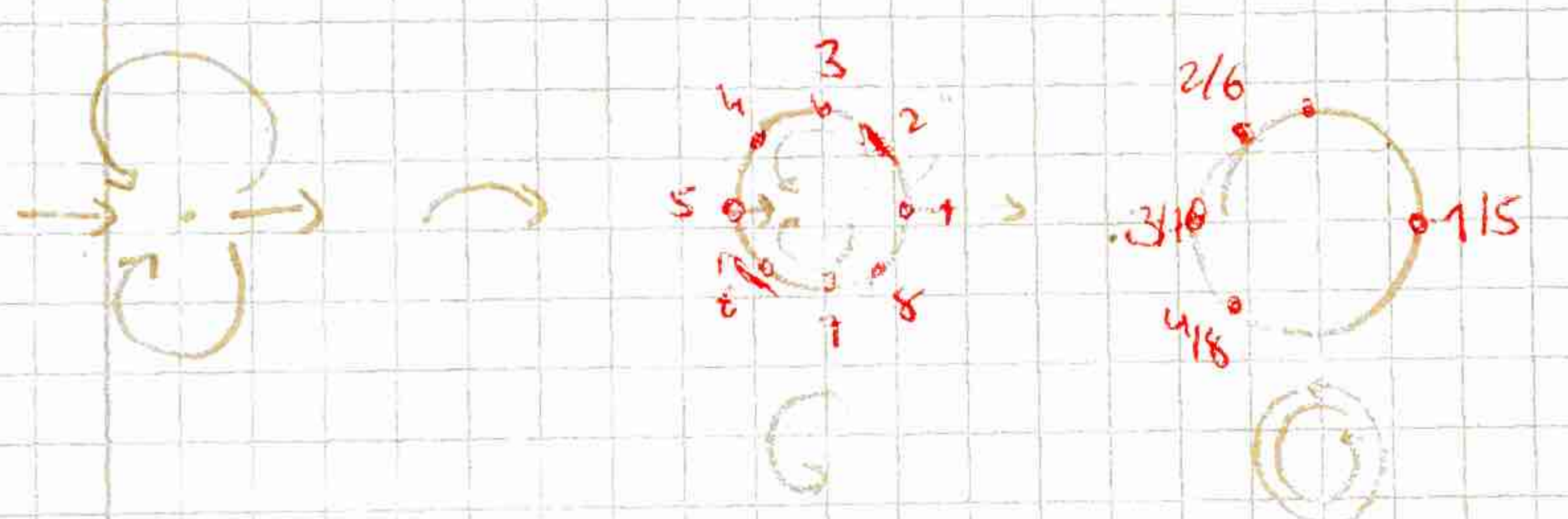
$\sim \quad z \mapsto z^{-1}$

index $= -1$



$z \mapsto z$

deg $= 1$



$= +2$ deg $= +2$

Main Theorem of the talk

Poincaré-Hopf-Theorem

M be compact manifold ~~wect~~

then: $\forall M \ \exists \ \chi(M) \in \mathbb{Z} \ \forall v$ with isolated zeros:

$$\sum i = \chi(M)$$

i = index of zero of v

$\boxed{\chi(M) = \text{euler number}}$

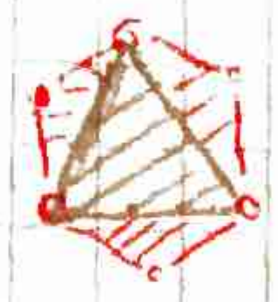
Particularly the index sum is a topological invariant of M

it does not depend on the particular choice of vector field.

For the people who have seen it before or are interested

$$\chi(M) = \sum_{k=0} (-1)^k \cdot \operatorname{rank} H_i(M) \quad (\text{ith homology group})$$

$\Rightarrow \chi(\Delta) =$ # corners/vertices − # edges + # faces

$\chi(\Delta^2) =$  $3-3+1=1$ ✓

$6-6+1=1$ ✓

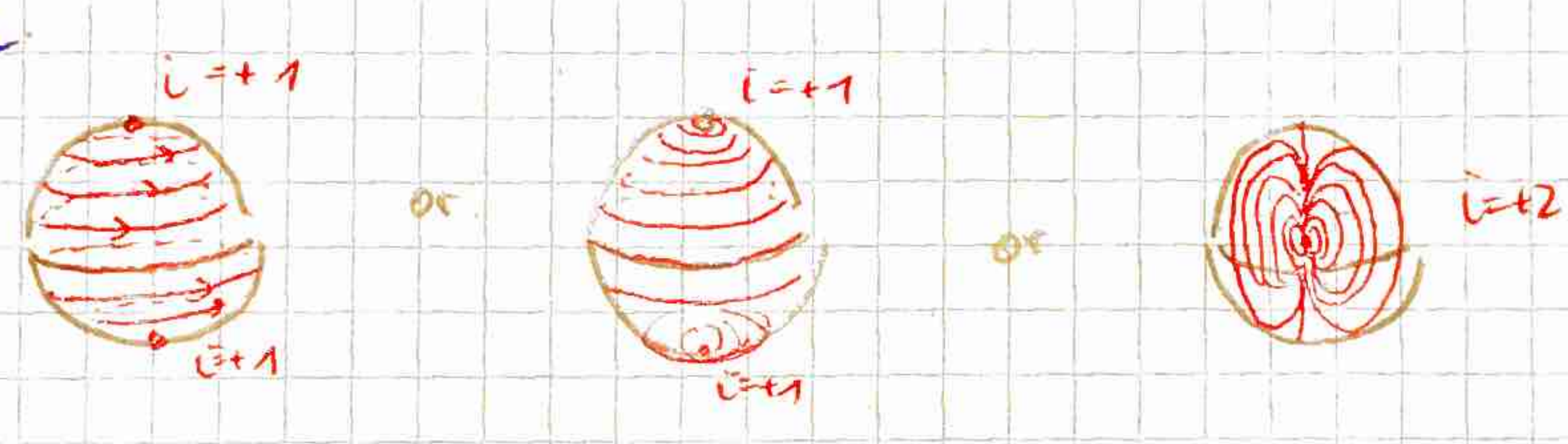
$\chi(M)$ same under subdivision

$\chi(S^1) = \triangle = 3-3+0$ $\Rightarrow \chi(S^2) =$  $4-6+4=\underline{2}$

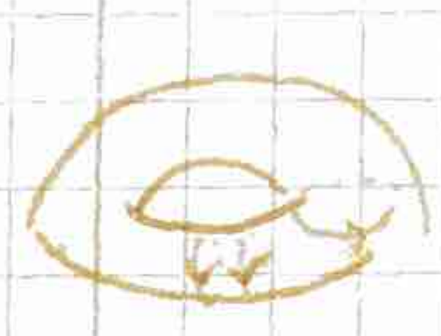
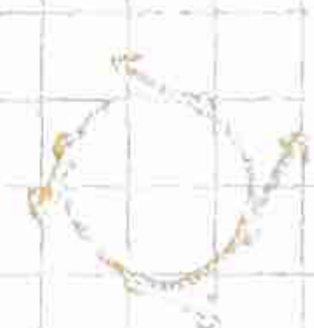
χ is the alternating sum $\sum(-1)^k \cdot h_k$ where h_k denotes the number of k-simplices in the complex.

As we've seen $\chi(S^2)=2$ so there has to be at least one zero with ~~an~~ index 2 ~~of~~ (Satz vom Igel / hairy ball theorem) (there can also be more zeros but the $\sum$ of indices has to be 2)

for example



In the case of the torus the $\chi(T^2)=0$ " it is possible to comb a hairy donut flat / the torus doesn't have to have ~~an~~ a zero

 or $\chi(S^1)=0$ 

<u>Remark:</u> Let M be a compact manifold and w a smooth vector field on M with isolated zeros.

If M has a boundary then w is required to point outward at all boundary points.

<u>Def.</u> Let $X \subset \mathbb{R}^m$ be a compact m-manifold with boundary. <u>The Gauss mapping</u> $g: \partial X \to S^{m-1}$ assigns to each $x \in \partial X$ the outward unit normal vector at x.

~~Kon~~ Lemma 3 (Hopf)

If $v: X \to \mathbb{R}^m$ is a smooth vector field with isolated zeros, v points out of X along the boundary

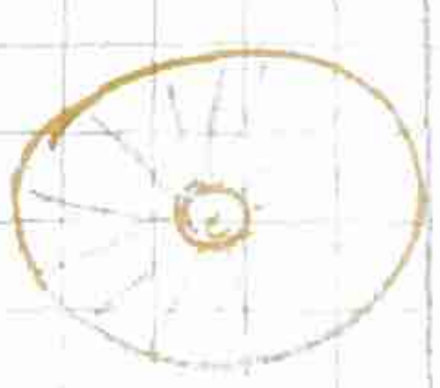
THEN: $\sum i = \deg(g)$

$\sum i$ doesn't depend on v

Example If a vector field on the disk D^m point outward along the boundary then $\sum i = +1$

Proof. removing an ε-ball around each zero $\Rightarrow$ we get new manifold with boundary

$\bar{v}(x) = \frac{v(x)}{\|v(x)\|}$ maps the manifold into S^{m-1}



$\sum$ degrees of $\bar{v}$ restricted to the various boundary components $= 0$

But $\bar{v}|\partial X$ is homotopic to g and the degrees of the other boundary comp $= -\sum i$ (each small sphere gets wrong orientation)

Therefore $\deg(g) - \sum i = 0 \Leftrightarrow \deg(g) = \sum i$ □

Remark

$\deg(g)$ known as "curvatura integra" of ∂X

$\chi(x)$"

(+)

Definition the vector field is nondegenerate at zero z if the linear transformation $d_z v$ is non singular

Lemma 4 Index of v at a nondegenerate zero is either $+1$ or -1 according to the determinante of $d_z v$ pos or neg.

~~Lemma 5 the determinate $d_z w$~~

(+) consider vector field v on open set $U \subset \mathbb{R}^m$ and think of v as mapping $U \to \mathbb{R}^m$ so that $d_z v: \mathbb{R}^m \to \mathbb{R}^m$ is defined,

⌈ more generally consider a zero z of vect. f. w on a manifold $M \subset \mathbb{R}^k$. Think of w as a map from $M \to \mathbb{R}^m$ s.o. that $d_z w : T_z M \to \mathbb{R}^k$ is defined ⌋

Lemma 5 $d_z w$ carries $T_z M$ into subspace $T_z M \subset \mathbb{R}^m$ and therefore can be considered as lin. transformation from $T_z M$ to itself.

If this lin. transf. has determinant $D \neq 0$ then z is an isolated zero of w with index $+1$ or -1 according as D is positive or negative

(~~Theorem 1~~)

⌈ Now consider a compact, boundaryless manifold $M \subset \mathbb{R}^k$

Let N_ε denote the closed ε-neigh. of M (ε suff small $\to$ N_ε smooth manifold with boundary) ⌋

Theorem 1 For any vector field v on M with only nondegen. zeros the $\Sigma\iota$ is equal to the dg. (Gauss mapping) $g : \partial N_\varepsilon \to S^{k-1}$

Proof. For $x \in N_\varepsilon$ let $r(x) \in M$ the closest point on M

Note $x - r(x)$ perpendicular to the tang. space of M at $r(x)$

(ε suff small $\to$ $r(x)$ smooth and well defined

We consider the squared dist. fct. $\varphi(x) = \|x - r(x)\|^2$

with $\operatorname{grad}\varphi = 2(x - r(x))$

then for each $x \in \partial N_\varepsilon = \varphi^{-1}(\varepsilon^2)$ the outward unit vector is given by

$$g(x) = \frac{\operatorname{grad}\varphi}{\|\operatorname{grad}\varphi\|} = \frac{2(x - r(x))}{2\varepsilon} = \frac{x - r(x)}{\varepsilon}$$

We're extending the vector field v to a vec f w by

$$w(x) = (x - r(x)) + v(r(x))$$

Then w points outward along the boundary (since $w(x) \cdot g(x) > \varepsilon$)

Computing derivative of w at a zero $z \in M$

we get: $d_z w(h) = d_z v(h)$ for $h \in T_z M$

$d_z w(h) = h$ for $h \in T_z M^\perp$

$\Rightarrow$ Det. of w = Det of v $\quad i$ of w at zero z = i of v at z

$\xrightarrow{\text{Lemma } 3}$ $\sum i = \deg(g)$ $\boxtimes$

3 steps Poincaré Hopf Sketch

Step 1 Identification of the invariant $\sum i$ with $\chi(M)$

Step 2 Proving theorem for vector field w. deg. zeros

Step 3 Manifolds with boundary.

Step 1

Sufficient to construct one example of a nondeg. vector field on M with $\sum i = \chi(M)$

pleasant way: acc. to Marston Morse: always possible to find a real-valued fct on M whose "gradient" is a nondeg. vector field

$\Rightarrow$ Morse furthermore showed that the sum of indices associated with such a gradient field $= \chi(M)$

Step 2 Consider first vec f. v on openset U with isolated zero z.

If $\lambda: U \to [0,1]$

$\lambda(N_1) = 1$ $\quad N_1$ nghbd of z

$\lambda(\text{outside a slightly larger nghb}) = 0$

If y is suffic. small regular value of v

then $v'(x) = v(x) - \lambda(x) y$ is nondegenerate within N

(if y suff small $\Rightarrow$ v' will have no zeros at all within $N - N_1$)

The sum of the indices at the zeros within N can be evaluated as the degree of the map $\bar{v}: \partial N \to S^{m-1}$
and hence does not change during this alteration.

More generally consider vect. fields on a compact manifold M.
Applying this argument locally we see that any vector field with isolated zeros can be replaced by a nondeg. vect. field without altering the integer $\sum i$

Step 3

If $M \subset \mathbb{R}^n$ has boundary then any v.f. v which points outward along ∂M can be extended over the nghb N_ε so as to point outward along ∂N_ε

Problem N_ε is not smooth manifold / only C^1 manifold
The extension w from earlier will only be contin. v.f. near ∂M

The argument can be carried out

a) by showing that our strong differentiability assump. aren't necessary

b) by other methods.

Proof Lemma 2

Assume $f(0)=0$ $\quad d_0f(x) = \lim_{t\to 0} \frac{f(tx)}{x}$

We will construct a smooth isotopy $F: \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$

$$F(x,t) = f(tx)/t$$
$$F(x,0) = d_0f(x) \qquad F(x,1) = f(x)$$

to show F smooth even for $t \to 0$
we write f as

$$f(x) = x_1 g_1(x) + \dots + x_m g_m(x)$$

where $g_1, \dots, g_m$ suitable smooth functions

$$F(x,t) = \frac{f(tx)}{t} = \frac{x_1 g_1(tx)}{t} + \dots + \frac{x_m g_m(tx)}{t}$$

Since g_k smooth $\Rightarrow \lim_{t\to 0} \frac{g_k(tx)}{t} = g_k'(x)$ smooth for $\forall k$

$\Rightarrow F(x,0) = d_0f(x) = x_1 g_1'(x) + \dots + x_m g_m'(x)$ $\quad$ since g_k smooth
$\rightarrow g_k'$ smooth
$\Rightarrow F(x,0)$ smooth

$\Rightarrow f$ isotopic to d_0f which is clearly isotopic to identity

Proof Lemma 1

We assume $z = f(z) = 0$ and U convex.

1) case: f preserves orientation $\qquad$ like lemma 2

$f_t : U \to \mathbb{R}$
f_0: identity
f_1: f $\qquad$ and $f_t(0) = 0 \ \forall t$

Let $v_t = df_t \circ v \circ f_t^{-1}$
all defin. and non zero on a suff small sphere around 0

the index $v = v_0$ at 0 must be the same to the index $v' = v_1$

$(v_1 = df_1 \circ v_0 \circ f_1^{-1} = df \circ v_0 \circ f^{-1} = v_0)$ at zero

2) f doesn't presve orientation

Suff. to show for special case of a reflection ρ

Then $v' = \rho \cdot v \cdot \rho^{-1}$