1 Preamble

Today our goal is to show that for any smooth map $f : S^n \longrightarrow S^m$ and its regular values V the residue class modulo 2 of $\#f^{(-1)}(y)$ does not depend on the choice of $y \in V$.

In general, this also works for $f : M \longrightarrow N$ where M is compact without boundary, N is connected and both manifolds are of the same dimension.

2 Definitions

For this topic we need two new definitions:

(*) **Definition 1.** For $X \subset R^k$ two maps $f, g : X \longrightarrow Y$ are called smoothly homotopic if a smooth map $F : X \times [0,1] \longrightarrow Y$ exists such that:
$F(x,0) = f(x)$ and $F(x,1) = g(x), \forall x \in X$.
F is called a smooth homotopy between f and g.

(**) **Definition 2.** A diffeomorphism f is smoothly isotopic to g if a smooth homotopy $F : X \times [0,1] \longrightarrow Y$ from f to g exists such that for each $t \in [0,1]$ the correspondence $x \mapsto F(x,t)$ is a diffeomorphism between X and Y.
F is called a smooth isotopy between f and g.
We call two points x, y in a manifold isotopic if a smooth isotopy carrying one onto the other exists.

Note that being smoothly homotopic is an equivalence relation.
Symmetry and reflexivity are quite easy to show.
For the transitivity we will construct a smooth bumper funcion $\phi : [0,1] \longrightarrow [0,1]$ such that:
$\phi(t) = 0$ for $0 \leq t \leq \frac{1}{3}$ and
$\phi(t) = 1$ for $\frac{2}{3} \leq t \leq 1$.
For example:
$\phi(t) = \frac{\lambda(t-\frac{1}{3})}{\lambda(t-\frac{1}{3})+\lambda(\frac{2}{3}-t)}$ where $\lambda(r) = 0$ for $r \leq 0$ and $\lambda(r) = \exp(-r^{-1})$.
Given smooth homotopies F, between f and g, and H, between g and h the formula $G(x,t) = F(x, \phi(t))$ is a smooth homotopy with:
$G(x,t) = f(x)$ for $0 \leq t \leq \frac{1}{3}$ and $G(x,t) = g(x)$ for $\frac{2}{3} \leq t \leq 1$.
Construct your $D(x,t) = H(x, \phi(t))$ similarly.
Now concatenate both to get the homotopy between f and h and due to the bumper function we arrive at a smooth homotopy.

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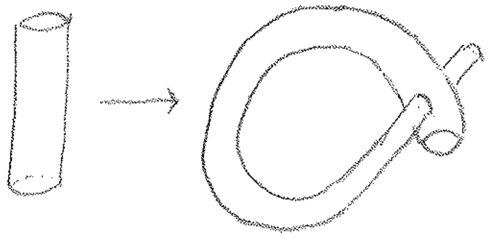

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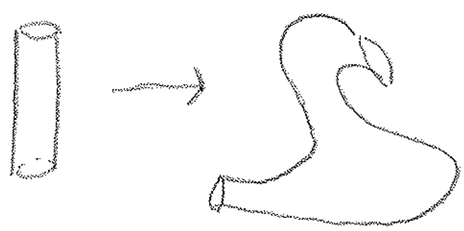
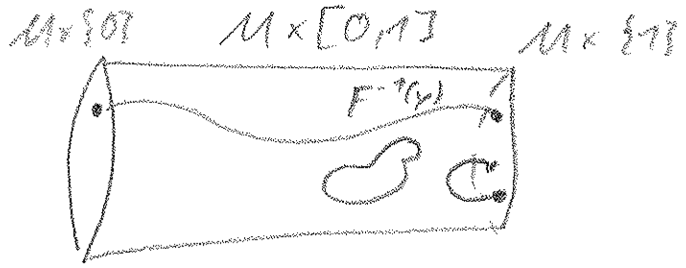


3 Homotopy Lemma

Lemma 1. *Let $f, g : M \longrightarrow N$ be smoothly homotopic maps between manifolds of the same dimension, where M is compact and without boundary. If $y \in N$ is a regular value for both f and g, then:*
$\#f^{-1}(y) \equiv \#g^{-1}(y)$ *mod(2).*

Proof. Let $F : M \times [0,1] \longrightarrow N$ be a smooth homotopy between f and g.
First case:
Suppose that y is also a regular value for F.
Now we use Lemma 4 from the last talk, which was:
If $y \in N$ is a regular value for F and the restriction $F(x)|_{boundary(M\times[0,1])}$, then $F^{-1}(y) \subset M \times [0,1]$ is a compact 1-manifold with boundary. Furthermore the
(*) $boundary(F^{-1}(y))$ is equal to:
$F^{-1}(y) \bigcap (M \times \{0\} \bigcup M \times \{1\}) = f^{-1}(y) \bigcup g^{-1}(y)$
Hence, the number of boundary points of F^{-1} is the same as $\#f^{-1}(y) + \#g^{-1}(y)$.
But in the last talk we already learned that a compact 1-Manifold always has an even number of boundary points.
Hence $\#f^{-1}(y) \equiv \#g^{-1}(y)$ mod(2).

Second case:
Now suppose that y is no regular value of F, but of f and g.
From chapter one we know that $\#f^{-1}(y)$ and $\#g^{-1}(y)$ are locally constant.
Hence, we can find neighbourhoods $V_1, V_2 \subset N$ of y, s.t for any $y^* \in V_1$ and $y^{**} \in V_2$
$\#f^{-1}(y) = \#f^{-1}(y^*)$ and $\#g^{-1}(y) = \#g^{-1}(y^{**})$ hold.
We also know that regular values are dense everywhere, so we can choose one z of F within $V_1 \bigcap V_2$.
$\#f^{-1}(y) = \#f^{-1}(z) \equiv \#g^{-1}(z) = \#g^{-1}(y)$ mod 2.
This holds because of the first case. □



4 Homogeneity Lemma

Lemma 2. *Let y and z be arbitrary interior points of the smooth, connected manifold N. Then a diffeomorphism $h : N \longrightarrow N$ exists such that h is smoothly isotopic to the identity and carries y onto z.*

Proof. First we construct a smooth isotopy from R^n to itself which:

(*) 1. leaves all points outside of the unit ball fixed, and

2. slides the origin to any desired point of the open unit ball.

For this we construct a smooth $\phi : R^n \longrightarrow R$ with:
$\phi(x) > 0$ for $||x|| < 1$ and $\phi(x) = 0$ for $||x|| \geq 1$
(For example $\phi(x) = \lambda(1 - ||x||^2$ with $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = \exp(-t^{-1})$ for $t > 0$).
Now for arbitrary $c \in R^n$ and $x^* \in R^n$ the differential equations:
$\frac{dx_i}{dt} = c_i\phi(x_1, ..., x_n), x(0) = x^*$ for $i = 1, ..., n$ have a unique solution.
It has now become apparent that c gives you the direction and ϕ the distance to shift the origin.
For example, if you want to carry the origin to p, choose $c = \frac{p}{|p|}$ and a matching t.
From now on denote $x(t) = F_t(x^*)$ and note:

1. $F_t(x^*)$ is defined and smooth for any t and x^*.

2. $F_0(x^*) = x^*$

3. $F_{s+t}(x^*) = F_s \circ F_t(x^*)$

Observe that that F_{-t} is the inverse of F_t and for $t \longrightarrow 0$ you get the identity.
It then follows that each F_t is a diffeomorphism from R^n onto itself and with a variable t smoothly isotopic to the identity.
With suitable choices of c and t, F_t will carry the origin to any point in the open ball.
Now for any connected manifold N (connected manifolds are path connected) any interior point y has a neighborhood diffeomorphic to R^n.
So our argument shows that any point in this neighborhood is isotopic to y.
In other words , each isotopy class of points in the interior of N is an open set, and the interior of N is partitioned into disjoint open isotopy classes.
But N is connected $\Leftrightarrow$ N can not be separated into open sets.
Hence it is only one set. □

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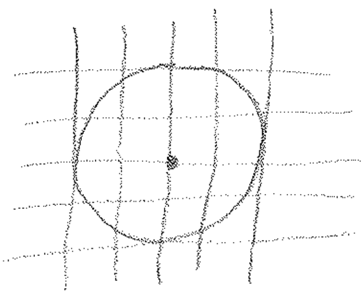

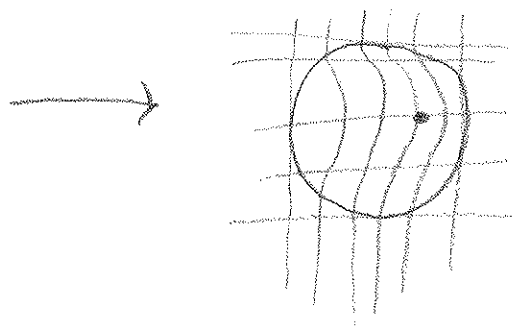

5 Main theorem

Theorem 3. *Assume that M is compact and has no boundary, N is connected and $f : M \longrightarrow N$ is smooth.*
If y and z are regular values of f then:
$\# f^{-1}(y) \equiv \# f^{-1}(z)$ *mod(2)*
This common residue class, which is called the mod 2 degree of f, depends only on the smooth homotopy class of f.

Proof. Given y, z and f as above. Let h be a diffeomorphism from N to N which is isotopic to the identity and carries y onto z.
Then z is a regular value of $h \circ f$ and $h \circ f$ is smoothly homotopic to f.
From the homotopy Lemma follows:
$\#(h \circ f)^{-1}(z) \equiv \# f^{-1}(z) \pmod 2$, but
$(h \circ f)^{-1}(z) = f^{-1}h^{-1}(z) = f^{-1}(y)$, hence
$\# f^{-1}(y) \equiv \# f^{-1}(z) \pmod 2$
Call this residue class $deg_2(f)$. Now suppose that f is smoothly homotopic to g.
Sard's theorem guarantees you a regular value y for both.
Then the homotopy Lemma gives you:
$deg_2(f) = \# f^{-1}(y) \equiv \# g^{-1}(y) = deg_2(g)$ □

6 Example

Example 4. A constant map $c : M \longrightarrow M$ has a degree that is even mod 2. The only point with a preimage is no regular value because the derivative of our map is 0.
The identity has an odd degree mod 2.
Hence there is no smooth homotopy between them for M compact and with an empty boundary.
For $M = S^n$ this implies that there is no smooth map $f : D^{n+1} \longrightarrow S^n$, which leaves the sphere pointwise fixed.
We already saw this in §2 lemma 5.
Otherwise we could use f to construct a smooth homotopy:
$F : S^n \times [0, 1] \longrightarrow S^n, F(x, t) = f(tx)$
with $F(x, 1) = f(x)$ and $F(x, 0)$ =constant, but then they would have the same degree modulo 2.

(*) **Example 5.** The projection from S^2 to D with $D = [-1, 1] \times [-1, 1] \times \{0\}$. This is a smooth map and S^2 is compact without boundary and D is connected. The main theorem gives us that the residue class is the same for all regular values. As u can see there are no points in the preimage for values outside of the disk and 2 for those in the interior. The points on the boundary are no regular values.

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