Laetitia de Abreu Nunes 1

J-B separation thm

(Classical) Jordan Curve Theorem

Every simple closed curve (= Jordan curve) $=C$
in $\mathbb{R}^2$ divides the plane into two connected $=\mathbb{R}^2 \setminus C$
components, the "inside" and "outside".

↳ Not so obvious

We will prove this theorem in n-dimensions. It's the

Jordan-Brouwer Separation Theorem

First, we introduce the

Winding number

X - compact connected manifold
$f: X \to \mathbb{R}^n$ smooth map
Here, suppose $\dim(X) = n-1$
$\Rightarrow f$ might be the inclusion map of a hypersurface into $\mathbb{R}^n$

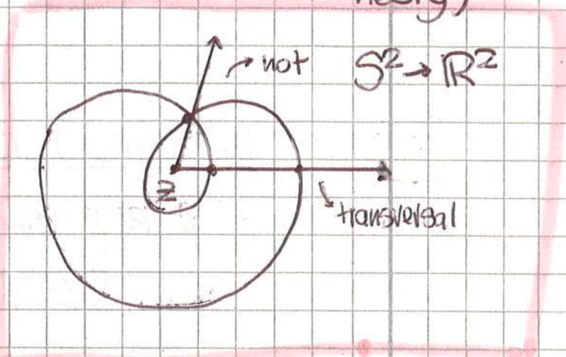
Q: How does f wrap X around in $\mathbb{R}^n$?
→ Take $z \in \mathbb{R}^n$, $z \notin f(X)$
Define $u: X \to S^{n-1}$, $u(x) = \dfrac{f(x) - z}{|f(x) - z|}$, unit vector — indicates direction from z to $f(x)$

How $f(x)$ winds around z ↔ how often $u(x)$ points in a given direction

$u: X \to S^{n-1}$ hits almost every direction vector the same number of times mod 2 (Intersection Theory)

"$\deg_2(u)$ times

! invariant !

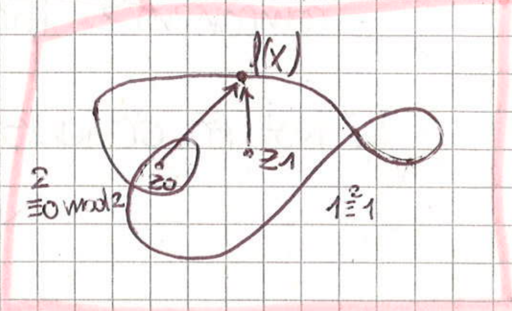


We define the mod 2 winding number of f around z to be

$$W_2(f,z) = \deg_2(u)$$

1 - inside } mod 2

0 - outside



($\deg_2(u) = I_2(u, \{y\})$, same $\forall y \in X$ (reg. value))

First, we prove

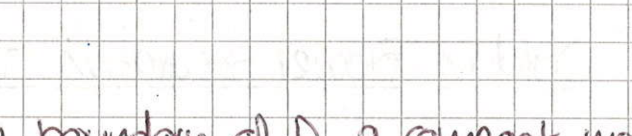
<u>Theorem</u>

Suppose X is the boundary of D, a compact manifold with boundary and let $F: D \to \mathbb{R}^n$ be a smooth map extending f ($\partial F = f$). (X compact connected manifold, f smooth map, $\dim(X) = n-1$)

Let z be a regular value of F and $z \notin f(X)$

Then ($F^{-1}(z)$ is a finite set (regularity of z))

$$W_2(f,z) = \# F^{-1}(z) \bmod 2$$



$W_2(f,z) = 0 = \# F^{-1}(z) \bmod 2$

Proof

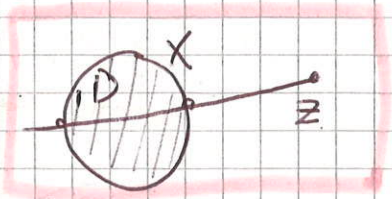
1 • $z \notin F(D) \Rightarrow W_2(f,z)=0$

⌜We can extend $u \to U: D \to \mathbb{R}^n$

$x \mapsto \dfrac{f(x)-z}{|f(x)-z|}$ (well defined)

$\partial D = X$

$\Rightarrow \deg_2(u)=0$



⌟

2 · $F^{-1}(z)=\{y_1,\dots,y_l\}$. B_i ball around each y_i

(actually B_i image of ball in $\mathbb{R}^n$...)

with B_i all disjoints from each other and $X=\partial D$

$f_i := \partial B_i \to \mathbb{R}^n$, $f_i := F|_{\partial B_i}$

$\Rightarrow W_2(f,z)=W_2(f_1,z)+\dots+W_2(f_l,z) \mod 2$

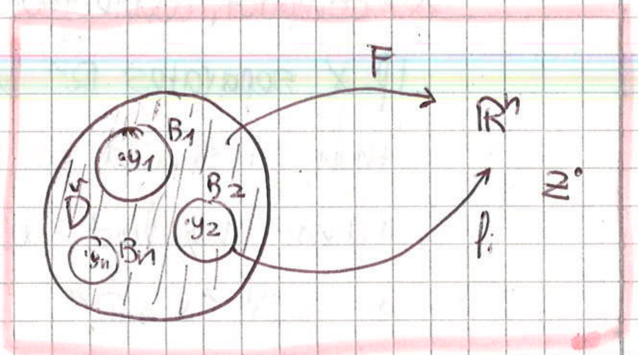
⌜Define $\check{X} = X \cup \partial B_i$

$\check{D} = D \setminus \bigcup_{i=1}^{l} Int(B_i)$

$F|_{\check{X}} : \check{X} \to \mathbb{R}^n$

extends to

$\check{D}$, $(z \notin F^{-1}(\check{D}))$



$\overset{1st}{\underset{step}{\Rightarrow}} W_2(\check{f},z)=0$, $\check{f}=\partial F|_{\check{X}}$, $\check{u} = \dots$

" $\deg_2(u|_{\check{X}}) = \underset{W_2(f,z)}{\deg_2(u)} - \underset{W_2(f_1,z)}{\deg_2(u|_{B_1})} - \dots - \underset{W_2(f_l,z)}{\deg_2(u|_{B_l})}$

$\Rightarrow W_2(f,z) = W_2(f_1,z)+\dots+W_2(f_l,z) \mod 2$ ⌟

3 · $\exists B_i$ so that $W_2(f_i, z) = 1 \ \forall i$

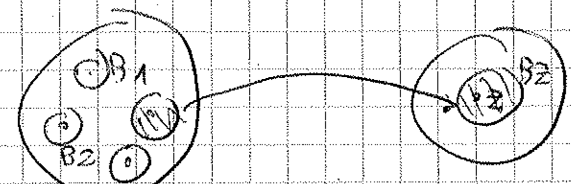
$\ulcorner$ z regular value $\Rightarrow dF_z : D \to Y$ surj. $\to$ bij (n dim)

f is loc. diffeo, ie $f_i : \partial B_i \to B_r(z)$ ↗ small sphere centered at z

$\Rightarrow u_i : \partial B_i \to S^{n-1}$ bijective

$\Rightarrow 1 = \deg_2(u_i) = W_2(f_i, z)$

(we can choose a B_i so that $\partial B_i \cong B_r(z)$ u_i bij



$\rightarrow W_2(f_1, z) \overset{2}{=} \sum_{i=1}^{l} W_2(f_i, z) \underset{\text{mod}_2}{=} \sum_{i=1}^{l} 1 \underset{\text{mod}_2}{=} l \underset{\text{mod}_2}{=} \# F^{-1}(z) \text{ mod } 2$

B_i chosen like $\uparrow 3$ $\square$

X - compact, connected hypersurface in $\mathbb{R}^n$

if X separates $\mathbb{R}^n$ into inside and outside, then it should be the boundary of a compact n-dim manifold w/ boundary (= inside)

ie $i : S^{n-1} \hookrightarrow \mathbb{R}^n$; $\omega(i, z) = 0, 1$ = inside, outside

$z \in \mathbb{R}^n, z \notin X$

(inclusion map of X around z

$\forall z \in \mathbb{R}^n \setminus X$, $u : X \to S^{n-1}$ is $u(x) = \frac{x - z}{|x - z|}$

$\rightarrow W_2(X, z)$

Jordan-Brouwer Separation Theorem

X - compact connected hypersurface in $\mathbb{R}^n$

Then, $\mathbb{R}^n \setminus X$ consists of 2 connected open sets, the "outside" D_0 and the inside "D_1"

Moreover, $\overline{D_1}$ is a compact manifold with boundary $\partial \overline{D_1} = X$

$D_0 = \{ z \mid W_2(X, z) = 0 \}$ $\quad D_1 = \{ z \mid W_2(X, z) = 1 \}$

Proof

4 • Let $z \in \mathbb{R}^n \setminus X$.

$\forall x \in X$, $\forall U$ nghbd of x in $\mathbb{R}^n$, $\exists\, p \in U$ s.t. $p \rightsquigarrow z$ by a curve not intersecting X

$V = \{ x \in X \mid \ldots \}$

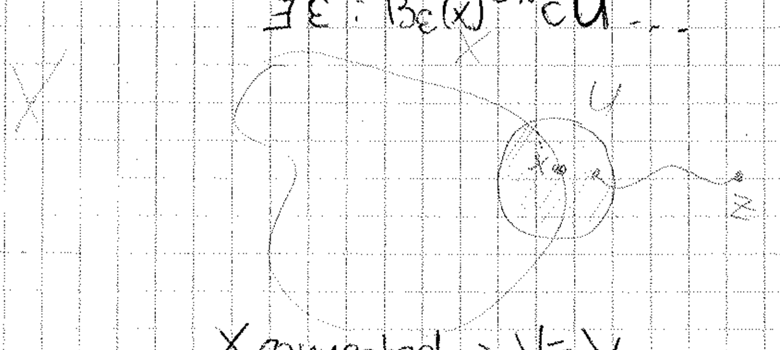
- $V \neq \emptyset$: take the straight line joining z to the closest point in X, any point arbitraly close to x lits
- V closed: Take sequence $(x_n) \in V$, $x_n \to x \in X$

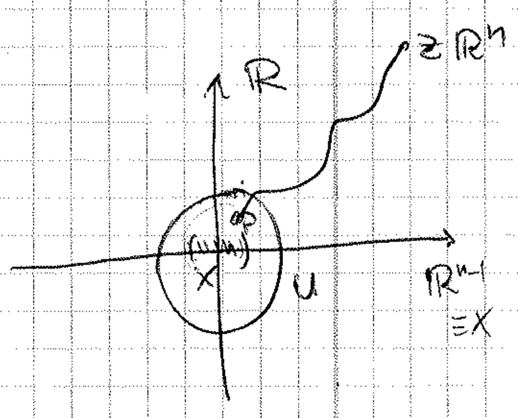
  $\forall U$ tq $x \in U$, $\exists N$ st $x_n \in U$, $\forall n \geq N$

  Take U, nbhd of x. $x_n \in U \Rightarrow \exists\, p \in U$ s.t. $p \rightsquigarrow z$ by a curve not intersecting X

- V open : Let $x \in V$. Let U nbd of x $\mathbb{R}^n$

  $\exists \varepsilon : B_\varepsilon(x)^{cX} \subset U$ ...

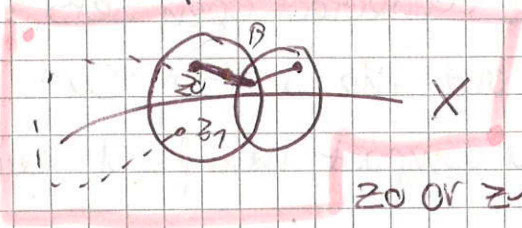




X connected $\Rightarrow V = X$ ┘

5 • $\mathbb{R}^n \setminus X$ has at most 2 connected components (≤ 2)

⌈ Let B be a small ball such that $B \setminus X$ has 2 components. Fix z_0, z_1 in opposite components



$\rightarrow$ $z_3 \in \mathbb{R}^n \setminus X$ can be joined to either z_0 or z_1 (dep side of X) by a curve not intersecting X (shown before) (4)

$\Rightarrow$ ≤ 2 components ⌋

6 • $z_0, z_1 \in$ the same connected component of $\mathbb{R}^n \setminus X$
$\Rightarrow W_2(X, z_0) = W_2(X, z_1)$

⌈ z_0 and z_1 can be joined by a curve not intersecting X: z_t ($z_0 \rightsquigarrow z_1$)

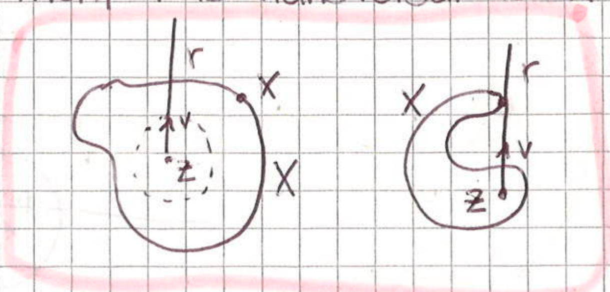
$\Rightarrow$ Homotopy $u_t(x) = \dfrac{x - z_t}{|x - z_t|}$ defined $\forall t$

$u_0(x) = \dfrac{x - z_0}{|x - z_0|}$, $u_1(x) = \dfrac{x - z_1}{|x - z_1|}$

$\Rightarrow$ Homotopic maps, same degree mod 2

$\Rightarrow W_2(X, z_0) = W_2(X, z_1)$ ⌋

7 • Let $z \in \mathbb{R}^n \setminus X$, $v \in S^{n-1}$ direction vector and $r = \{z + tv \mid t \geq 0\}$ be the ray emanating from z in direction of $\vec{v}$
Then, r is transversal to $X \iff \vec{v}$ is a regular value of $u: X \to S^{n-1}$



Proof

Ex 7 Ch 1 Sec. 5

Let $X \xrightarrow{f} Y \xrightarrow{h} Z$ sequence of smooth maps of manifolds
assume h is transversal to submanifold W of Z
$f \pitchfork h^{-1}(W) \iff h \circ f \pitchfork W$

Here define $g : \mathbb{R}^n \setminus \{z\} \to S^{n-1}$, $g(y) = \dfrac{y-z}{|y-z|}$

Then $u : X \to S^{n-1}$, $u = g \circ i$.
i inclusion map $i : X \to \mathbb{R}^n$

- $g^{-1}(\{v\}) = r$ $\quad$ ($x \in r \Rightarrow x = z + tw$, $g(x) = \dots v$, $|v| = 1$
$x \in g^{-1}(v) \Rightarrow v = \dfrac{x-z}{|x-z|}$, $t = |x-z| \dots g^{-1}(\{v\}) \subset r$)

- v reg value of g ($dg_x : T_x(\mathbb{R}^n \setminus \{z\}) \to T_v(S^{n-1})$
n dim and $n-1$ dim, if v is the direction from x to z, then v is the only direction in which g remains constant at $x \Rightarrow \ker(d_x g = \operatorname{span}(v))$
dim nullspace of $d_x g$ is 1 $\Rightarrow d_x g = \dim n-1$
$\Rightarrow g$ submersion $\forall x \in g^{-1}(v)$)

- $X \xrightarrow{i} \mathbb{R}^n \xrightarrow{g} S^{n-1}$
$i \pitchfork g^{-1}(\{v\}) \iff g \circ i \pitchfork \{v\}$
$X \pitchfork r \iff u \pitchfork \{v\}$

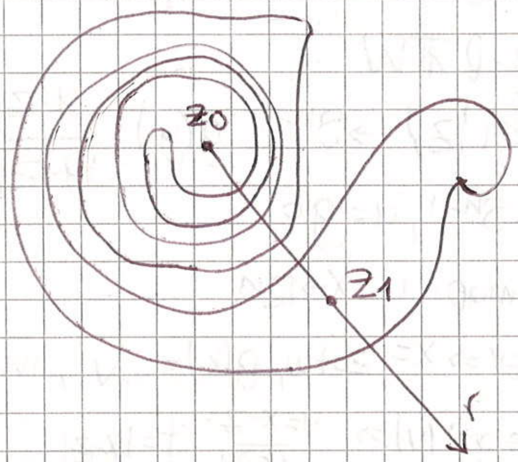
- Sard's Theorem: almost every element $\vec{v}$ of u is a reg value of u, so almost every ray from z intersects X transversally]

8. $r = \{z_0 + tv \mid t \geq 0\}$ emanating from z_0, intersects X transversally in a nonempty (finite) set

Let z_1 be another point on r (but not on X)

$z_1 \in r$ ($z_1 \notin X$), $l = \#$ r intersects X between z_0 and z_1

Then $W_2(X, z_0) = W_2(X, z_1) + l \mod 2$



$W_2(X, z_0) = W_2(X, z_1) + 5 \mod 2$

$1 = 0 + 1$

⌈ r transversal to X $\overset{7}{\Rightarrow}$ v regular value for u_0 and u_1 ($u_0, u_1 : X \to S^{n-1}$)

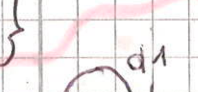
$\#u_0^{-1}(v) = \#u_1^{-1}(v) + l$

‖ # intersections of u_0 with X along v starting from z_0

$\Leftrightarrow W_2(X, z_0) = W_2(X, z_1) + l \mod 2$ ⌋

9. $\mathbb{R}^n \setminus X$ has precisely 2 components

$D_0 = \{z \mid W_2(X,z) = 0\}$ and $D_1 = \{z \mid W_2(X,z) = 1\}$



⌈ • D_0, D_1 non empty

$z_0 \in \mathbb{R}^n \setminus X$ arbitrary, ray r. Let $z_1 \in r$ such that

$l = 1 \Rightarrow W_2(X, z_0) = W_2(X, z_1) + 1 \mod 2$

$\Rightarrow W_2(X, z_0) \neq W_2(X, z_1) \overset{6}{\Rightarrow} z_0, z_1$ dont belong to the same component

• There is at least 2 connected comp.

• There is ≤ 2 (5)

} exactly 2, D_0 and D_1

10 • z very large $\Rightarrow W_2(X,z)=0$

Γ X compact $\Rightarrow$ for $|z|$ large, the image $u(X)$ on S^{n-1} lies in a small nbhd of $z/|z|$

($u(x)=\frac{x-z}{|x-z|} \simeq \frac{-z}{|z|}$)

$\Rightarrow u$ not surjective $\Rightarrow \deg_2(u)=0$

($x\in S^{n-1}$, $x\notin u(X)$ is a reg. value) zero pre-image " $W_2(X,z)$

⌋

+ Intuition D_0 = "outside"

11 $\Rightarrow$ • $\mathbb{R}^n\setminus X = D_0 \,\dot\cup\, D_1$, connected components (9)

• D_0, D_1 open:

$z\in\mathbb{R}^n\setminus X \Rightarrow \exists r>0$ s.t. $B_r(z)\cap X=\emptyset$ (X closed $\Rightarrow \mathbb{R}^n\setminus X$ open)

$y,z\in B_r(z) \overset{4}{\Rightarrow} \exists$ curve $y \rightsquigarrow z$ that doesn't intersect X

$\overset{6}{\Rightarrow} W_2(X,z)=W_2(Y,z)$

$\Rightarrow \forall z\in D_0, \exists r>0$ s.t. $B_r(z)\subset D_0$, idem for D_1

• $\overline{D_1}$ closed (by def) and D_1 bounded (10) $\Rightarrow \overline{D_1}$ bounded

↳ $\overline{D_1}$ compact

• $\partial\overline{D_1}=X$ (clear but ?)

$\mathbb{R}^n\setminus X = D_0\cup D_1 \Rightarrow D_0^c = X\cup D_1$ closed

$\Rightarrow \overline{D_1}\subset D_1\cup X$. Then $D_1\cup X\subset\overline{D_1}$ ($X\subset\overline{D_1}$)

$x\in X$, $z\in D_1$. Sequence $(z_n)\in D_1$ $z_n\to x$.

↳ U arbitrary small nbd of x and $\exists\, z_i\in U$ s.t. $z_i \rightsquigarrow z$

$z\in D_1$ and D_1 connected comp $\Rightarrow z_i\in D_1$. $\Rightarrow x\in\overline{D_1}$ (true $\forall x$)

$\Rightarrow \overline{D_1}=D_1\cup X$ and $\partial\overline{D_1}=X$

· $\bar{D}_1$ manifold with boundary

- $x \in \mathrm{int}(\bar{D}_1) \Rightarrow$ open nbhd of x is an open set in $\mathbb{R}^n$
  $\Rightarrow$ diffeom to open set in $\mathbb{R}^n$, OK
- $x \in X$
  Locally identify X with hyperplane in $\mathbb{R}^n$
  inclusion map $i: X \to \mathbb{R}^n$. $\dim(X) < n \Rightarrow i$ immersion
  $\overset{\text{local}}{\underset{\text{imm}}{\Rightarrow}} \exists$ local coord. $\{x_1, \dots, x_{n-1}\}$ around x s.t.
  $i(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, 0)$ (translation, $x = (0, \dots, 0)$)
  $\to$ In nbhd U of x, the manifold X is identified
  w the hyperplane $H = \{q_1, \dots, q_{n-1}, 0\}$
  $\to$ divides U into $H^+ = \{(q_1, \dots, q_n) \mid q_n > 0\}$,
  $H^- = \{(q_1, \dots, q_n \mid q_n < 0\}$ open

$x \in X$. $\psi: B \to U$, B open ball around 0 in $\mathbb{R}^n$, U nbd of x in $\mathbb{R}^n$ s.t. $\psi = (x_1, \dots, x_n) \Rightarrow \psi|_{U \cap X} = (x_1, x_2, \dots, x_{n-1}, 0)$
(oh $\psi(B \cap \mathbb{R}^{n-1}) = X \cap \psi(B)$)

· U separated into D_0, D_1 by X:
$z \in \mathbb{R}^n \setminus X \overset{(4)}{\Rightarrow}$ can be joined by a path to $p \in U$
$\overset{6}{\Rightarrow} W_2(X, z) = W_2(X, p) \Rightarrow z, p \in D_0$ or $z, p \in D_1$
True $\forall z \in \mathbb{R}^n \setminus X$ and $D_0 \neq \phi$, $D_1 \neq \phi$ $\Rightarrow U$ = separated into D_0 and D_1 by X

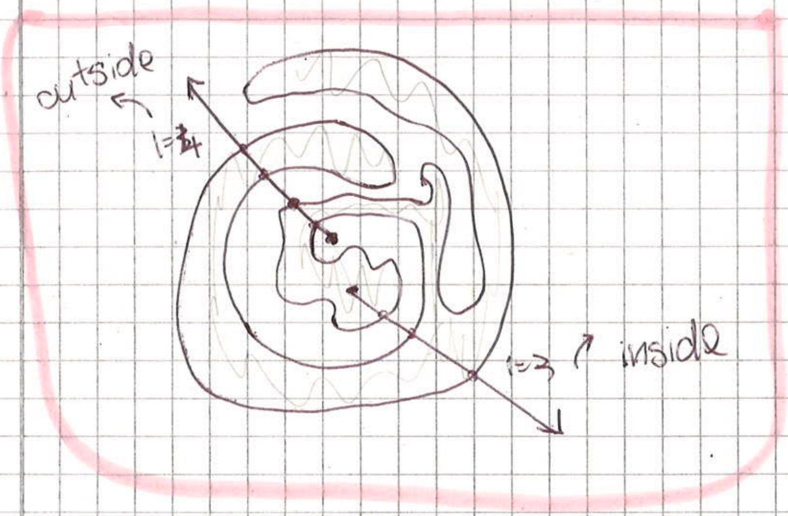
$\underbrace{\psi(B \cap (H^n \setminus \mathbb{R}^{n-1}))}$, $\psi(B \cap (-H^n \setminus \mathbb{R}^{n-1}))$ 　　H^n = upper half space in $\mathbb{R}^n$
connected (inside connected + ψ continuous)
maps entirely D_0 and D_1. Idem pour autre
ψ diffeom $\Rightarrow \psi$ surj $\Rightarrow \underbrace{\psi(B \cap (H^n - \mathbb{R}^{n-1}))}_{\subset D_1} \cap \underbrace{\psi(B \cap (-H^n \setminus \mathbb{R}^{n-1}))}_{D_0}$
$= \phi$. SPDG,
$\Rightarrow \psi|_{B \cap H^n}$ = param of $\bar{D}_1 \Rightarrow \bar{D}_1$ man. w boundary. $\square$

→ Derived process to determine wether z given lies inside or outside of X

<u>Lemma</u>

$z \in \mathbb{R}^n \setminus X$, r ray emaning from z transversal to X.

Then, z is inside of $X \Leftrightarrow r$ intersects X in an odd number of points



<u>Proof</u>

(finite)

$\boxed{\Rightarrow}$ z inside of $X \Rightarrow W_2(X, z) = 1$

$m = \# r \cap X$ finite

$\exists M_0$ s.t for $y \in r$, $|y| > M_0 \Rightarrow W_2(X, y) = 0$

M_1 = distance between z and last $r \cap X$

Choose $y \in r$ st $|y| > M_0$ and $d(y, z) > M_1$

$\Rightarrow m = l$

$\Rightarrow W_2(X, z) = W_2(X, y) + l \bmod 2$

$1 = 0 + l \bmod 2$

$\Rightarrow l = 1 \bmod 2 \Rightarrow l = \text{odd}$

$\boxed{\Leftarrow}$ Supp. $m \equiv 1 \bmod 2 \Rightarrow W_2(X, z) = 1 \ldots$

$\square$

(+ extra)

Schönflies Problem

Jordan Theorem + the inside and outside are homeomorphic to inside and outside of a standard circle