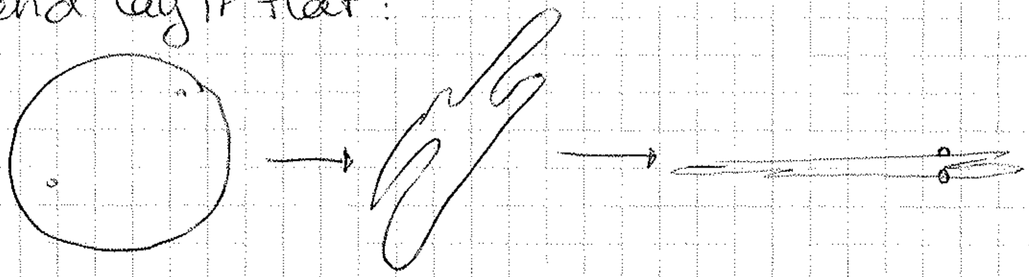
# Borsuk-Ulam Theorem

Thm (Borsuk-Ulam)
$\forall f: S^n \to \mathbb{R}^n$ continuous, $\exists x \in S^n$ s.t. $f(x) = f(-x)$.

Illustrations:

1. $n=2$: take a ball, deflate and crumple it, and lay it flat:



By the thm, there are two points lying on top of another that were diametrically opposite (antipodal)

2. At any given time, there are two antipodal places on earth s.t. the temperature and pressure are the same.

Equivalent versions:

(BU1a) $\forall f: S^n \to \mathbb{R}^n$ continuous, $\exists x \in S^n$ s.t. $f(x) = f(-x)$

(BU1b) $\forall f: S^n \to \mathbb{R}^n$ continuous s.t. $f(-x) = -f(x)$ $\forall x \in S^n$ (antipodal) $\exists x \in S^n$ s.t. $f(x) = 0$.

(BU2a) There is no antipodal mapping $f: S^n \to S^{n-1}$

(BU2b) There is no continuous mapping $f: B^n \to S^{n-1}$ that is antipodal on the boundary, i.e. $f(-x) = -f(x)$ $\forall x \in S^{n-1} = \partial B^n$

(LS-c) For any cover $F_1, \dots, F_{n+1}$ of the sphere $S^n$ by $n+1$ closed sets, $\exists F_i$ s.t. $F_i \cap (-F_i) \neq \emptyset$

(LS-o) $\forall$ cover $U_1, \dots, U_{n+1}$ of $S^n$ by $n+1$ open sets, there is at least one set containing a pair of antipodal pts

Proof:

(BU1b) $\Rightarrow$ (BU1a): $g(x) = f(x) - f(-x)$ antipodal

$\overset{(BU1b)}{\Longrightarrow} \exists x$ s.t. $g(x) = 0 \;\cancel{\Leftrightarrow}\; = f(x) - f(-x)$

(BU1b) $\Rightarrow$ (BU2a): If it exists, then it ~~is now~~ contradicts (BU1b)

(BU2a) $\Rightarrow$ (BU1b): Assume $f: S^n \to \mathbb{R}^n$ antipodal cont. no zero,

then $g(x) = \dfrac{f(x)}{\|f(x)\|}$ contradicts BU2a

(BU2b) $\Rightarrow$ (BU2a): $\pi: (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$

is a homeomorphism between the upper hemisphere $U$ of $S^n$ with $B^n$. Then, ~~an antipod~~ $f: S^n \to S^{n-1}$ antipodal would yield to $g: B^n \to S^{n-1}$ antipodal on $\partial B^n$ by $g(x) = f(\pi^{-1}(x))$

(BU2a) $\Rightarrow$ (BU2b): For $g: B^n \to S^{n-1}$ as in (BU2b) we define ~~$f(x)$~~ $f(x) = g(\pi(x))$ and $f(-x) = -g(\pi(x))$ for $x \in U$

$f: S^n \to S^{n-1}$ continuous and antipodal ↯

(BU1a) $\Rightarrow$ (LS-c) $F_1, \dots, F_{n+1}$ a closed cover

we define $f: S^n \to \mathbb{R}^n$ by $f(x) := (\mathrm{dist}(x, F_1), \dots, \mathrm{dist}(x, F_n))$

$\exists x \in S^n$ (BU1a) s.t. $f(x) = f(-x) = y$. If $y_i = 0$ then $\substack{x \\ -x} \in F_i$

Otherwise $x, -x \in F_{n+1}$.

(LS-c) $\Rightarrow$ (BU2a)

Result: $\exists$ a covering of $S^{n-1}$ by closed sets $F_1, \dots, F_{n+1}$ s.t. no $F_i$ contains antipodal points. Then if $\exists f: S^n \to S^{n-1}$ antipodal, $f^{-1}(F_1), \dots, f^{-1}(F_{n+1})$ would contradict (LS-c)

(LS-c) $\Rightarrow$ (LS-o): Use the fact $\forall$ open cover $U_1, \dots, U_{n+1}$ $\exists$ closed cover $F_1, \dots, F_{n+1}$ s.t. $F_i \subset U_i$.

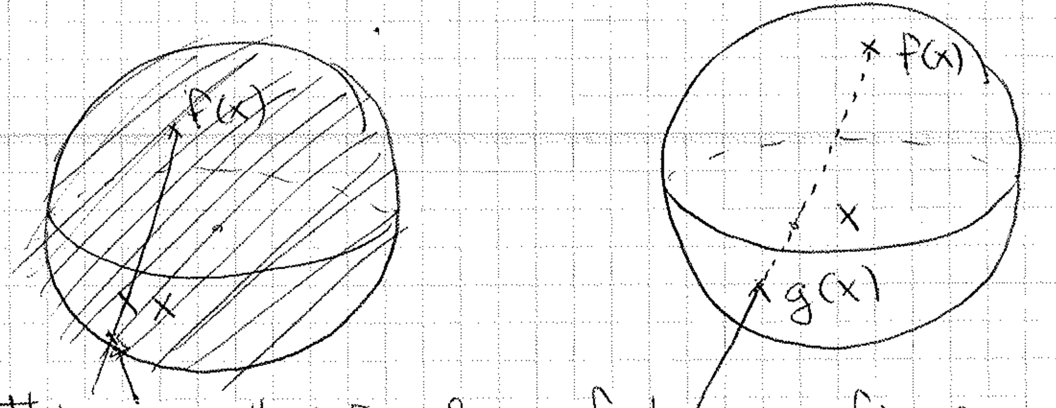
$\forall x$ choose $V_x$ nbh s.t. $\overline{V_x} \subset U_i$ for some $i$ and use compactness.

(LS-o) ⇒ (LS-c): $F_1, \dots, F_{n+1}$ a closed cover
define $U_i^{\varepsilon} := \{x \in S^n : dist(x, F_i) < \varepsilon\}$.
Let $\varepsilon \to 0$ and use compactness of the sphere.
We obtain $x_n$ in $S^n$ s.t. $\lim_{j\to\infty} dist(x_j, F_i) = \lim_{j\to\infty} dist(-x_j, F_i) = 0$
and select a convergent subsequent. ($F_i$ closed so the limit still in $F_i$)

## Borsuk-Ulam ⇒ Brouwer fixpoint theorem

Recall (Brouwer)
$\forall f: B^n \to B^n$ continuous, ~~there~~ $\exists x \in B^n$ s.t. $f(x) = x$

Proof (BU2b) ⇒ Brouwer:
For contradiction, assume $\exists f: B^n \to B^n$ s.t. $f$ has no fixed point. Define $g: B^n \to S^{n-1}$ s.t. $g(x)$ is the point ~~of~~ on $S^{n-1}$ that intersects w/ the ray from $f(x)$ to $x$.

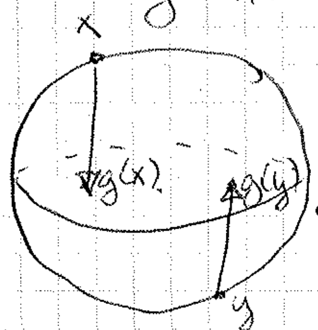



This is well defined as $f$ has no fixed point and $g|_{\partial B^n = S^{n-1}} = id_{\partial B^n}$.
Then $g$ antipodal on $S^{n-1}$ which contradicts (BU2b) □

Proof: Suppose $F_i$ contains no antipodal points $\forall i = 1, \dots, n+1$.
Then since $F_i$ closed and the distance between two points is strictly smaller than 2, $\exists \varepsilon_0 > 0$ s.t. the diameter of $F_i$ is at most $2 - \varepsilon_0$. $U_i^{\varepsilon_0/2}$ contradicts (LS-o)

$U_i^{\varepsilon_0/2} = \{x \in S^n : dist(x, F_i) < \varepsilon\}$

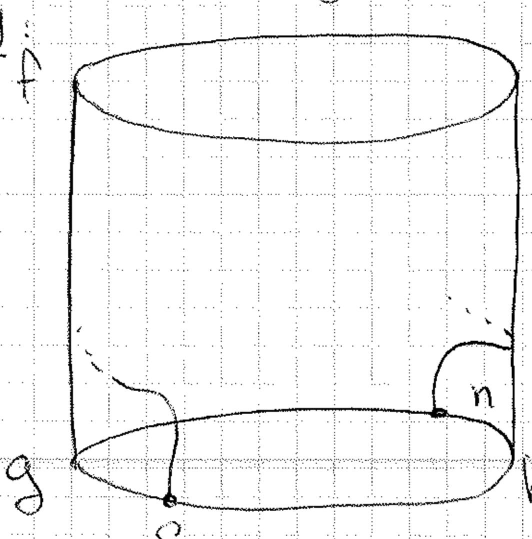
<u>Proof</u> (Borsuk-Ulam) (BU1b) Let $f: S^n \to \mathbb{R}^n$ be a continuous antipodal map.
Let $g: S^n \to \mathbb{R}^n$ be the "north-south projection" map.
i.e. $g(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.




Then $g$ has exactly two zeroes the north pole $n = (0, \dots, 0, 1)$ and the south pole $s = (0, \dots, 0, -1)$.

We consider the space $X := S^n \times [0,1]$ and the homotopy $F: X \to \mathbb{R}^n$ between $g$ and $f$ given by
$F(x,t) := (1-t)g(x) + tf(x)$.

<u>$n=1$</u>:

X


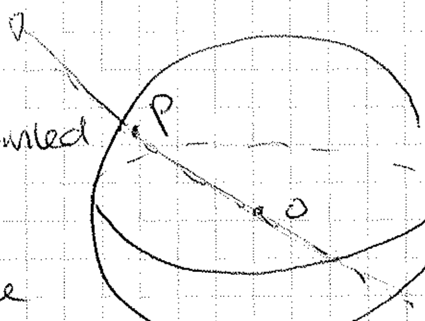

The antipodality on $S^n$ $x \mapsto -x$ is extend to the map $\nu$ on $X$ by $\nu(x,t) = (-x,t)$.
As $f, g$ antipodal, $F$ is antipodal w/ respect to $\nu$ i.e. $F(\nu(x,t)) = -F(x,t)$.
For contradiction, let's suppose that $f$ has no zeros. We consider the set $Z := F^{-1}(0)$. If $0$ is a regular value of $F$, then $Z$ is a 1-dimensional compact manifold ($n+2-(n+1)$). If not, by Sard's theorem we can assume that $0$ is actually one (by Sard we can find a regular value as close as $0$ as we want)

Then the components of $Z$ are cycles and path. Moreover, the endpoints of the path lies on the bottom or top copy of $S^n$ ($t=0$ et $t=1$) and are $0$ of $f$ ~~and~~ or $g$ and the cycles don't reach the top or the bottom.
As $f$ supposed to have no zero and $g$ have two ~~zeroes~~, there must be a path $\gamma$ connecting $n$ to $s$. But $Z$ is invariant under $\nu$ so $\gamma$ must behave symmetrically ↯
A symmetric path from $n$ to $s$ does not exist in $X$.

<u>Application of the Borsuk-Ulam theorem:</u>
<u>The Ham sandwich theorem</u>

$A_1, \dots, A_n \subset \mathbb{R}^n$ bounded measurable w/ ~~La~~ mesure $\mu$. Then $\exists$ hyperplane $H \subset \mathbb{R}^n$ s.t. $H$ divides $A_i$ into $A_i^+$ & $A_i^-$ with $\mu(A_i^+) = \mu(A_i^-)$ $\forall i = 1, \dots, n$

<u>Proof:</u>
Let's consider $S^{n-1} \subset \mathbb{R}^n$.
$\forall p \in S^{n-1}$ we consider the hyperplanes perpendicular to the vector between the origin and $p$. We define the positive side of such a hyperplane by the side pointed at by the vector $\overline{OP}$.




By the intermediate value thm, $\exists$ a hyperplane $\forall p \in S^{n-1}$ that bisects $A_n$.



(There's a hyperplane s.t. all $A_n$ is on the positive side and one so that it is on the negative $\Rightarrow$ there must be on one in between that cut $A_n$ in half).

$\Rightarrow \forall p \in S^{n-1}$ we have a hyperplane $\pi(p)$ that bisects $A_n$

(Note that the plane for $p$ and $-p$ is the same w/ opposite sides)

We define $f: S^{n-1} \to \mathbb{R}^{n-1}$ by

$f(p) = (\mu(A_1^+), \dots, \mu(A_{n-1}^+))$

where $\mu(A_i^+)$ is the volume of $A_i$ on the positive side of $\pi(p)$.

Without proof: $f$ continuous

By Borsuk-Ulam theorem, $\exists p \in S^{n-1}$ s.t. $f(p) = f(-p)$.

But $\pi(p)$ and $\pi(-p)$ are the same hyperplane w/ opposite sides $\Rightarrow$

$$f(p) = (\mu(A_1^+), \dots, \mu(A_{n-1}^+))$$
$$= (\mu(A_1^-), \dots, \mu(A_{n-1}^-)) = f(-p)$$

regarding the plane $\pi(p)$. $\square$

Borsuk-Ulam (Winding number)

$f: S^k \to \mathbb{R}^{k+1} - \{0\}$ smooth map s.t. $f(-x) = -f(x)\ \forall x \in S^k$

Then $W_2(f, 0) = 1$.

Corollary: $f$ as above, then $f$ intersects every line through zero at least once.

~~Proof~~ Borsuk-Ulam classic

Pf: Define $g(x) = (f(x), 0)$ $\quad g: S^k \to \mathbb{R}^{k+1}$ and take the $x_{k+1}$ axis for $e$.