1 Embeddings of manifolds

Last time we've seen [1] that the every compact manifold M^n can be embedded in $\mathbb{R}^N$ for N sufficiently large. From now on all manifolds assumed to be topological. We are going to prove that N must be bigger than $2n+1$ for compact manifolds and then extend this result to non-compact case. One common example is the Klein bottle which one becomes by attaching ends of cylinder, reversing its orientations. Another example of surface that one becomes in this way, but keeping the orientations is a torus. Such a surface can be easily constructed in $\mathbb{R}^4$ but cannot be embedded in $\mathbb{R}^3$, actually it can only be immersed in $\mathbb{R}^3$ with self-intersections. This can be proved for example using cohomology theory, the detailed proof can be found in [1] corollary 3.46.

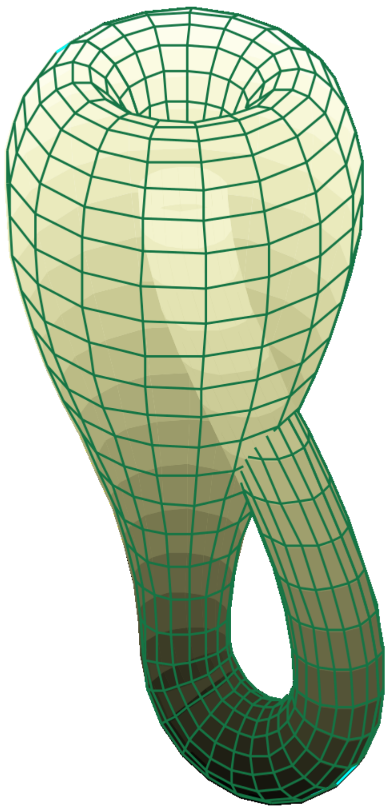

Figure 1: Klein bottle immersed in $\mathbb{R}^3$.[2]

2 Tangent bundle

To be able to prove the theorem for the compact case we need to introduce the notion of the tangent bundle of a manifold. For manifold M^n we define a tangent bundle as the set $TM^n = \bigcup_{x \in U} T_x M^n$. On this set one can define a topological space structure as follows. Let (U, φ) be the local coordinate system on M^n. Associate to tangent vector in a point $x \in M^n$ the tuple $(\varphi(x), v$, where $v = (v_1, \ldots, v_n)$ - coordinates of this vector. So we have a proper map: $TM^n = \bigcup_{x \in U} T_x M^n \to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. Sets TU cover TM^n. Declaring all maps $T\varphi$ to be homomorphisms, we endow TM^n with the structure of topological space. The charts $(TU, T\varphi)$ determine the structure of a manifold on that space.

[1] See the talk of Elif Selcen Ündar, `https://www.mathematik.hu-berlin.de/~kegemarc/WS1819SeminarDiffTopo.html`

[2] Published under GNU-Licence, `https://upload.wikimedia.org/wikipedia/commons/thumb/5/5c/Klein_bottle.svg/533px-Klein_bottle.svg.png`

a) b)

Figure 2: Fundamental polygon of a) - torus and b) - Klein bottle

3 Compact case

Theorem: Every k-dim. compact manifold M^k can be embedded in $\mathbb{R}^{2k+1}$.
Proof: Suppose we have an embedding $f : M^k \to \mathbb{R}^M$ for $M \leq 2k+1$. We will show that we can produce an embedding in $\mathbb{R}^{M-1}$. Let us introduce space $H_a := \{b \in \mathbb{R}^{\mathbb{M}} : b \perp a\}$, consisting of vectors orthogonal to a, where a is a unit vector in $\mathbb{R}^n$. Actually H_a is $T_a S^{M-1}$. Let π be the the projection of $\mathbb{R}^M$ to H. Define two maps:

$$h : M \times M \times \mathbb{R} \to \mathbb{R}^M \text{ by } h(x, y, t) \to t[f(x) - f(y)]$$

$$g : T_x M \to \mathbb{R}^M \text{ by } g(x, v) \to df_x(v)$$

First note, that if f is injective immersion, Sard's theorem implies that we can choose a point, say a in $\mathbb{R}^M$ s.t. it's belong to neither image of g and h.
Claim: $\pi \circ f$ is injective for our choice of a .
*Proof:*Choose two another distinct points x and y as some scalar t, for which $f(x) - f(y) = ta$. Since f is injective, t must be not equal to zero. But then $h(x, y, \frac{1}{t}) = a$ which is a contradiction with our choice of a.

Claim: $\pi \circ f$ is an immersion for our choice of a .
*Proof:*Let v be a nonzero vector in $T_x M$ s.t. $d(\pi \circ f)_x(v) = 0$. Because π is linear, apllying the chain rule we become $d(\pi \circ f)_x = \pi \circ df_x$. Thus $\pi \circ df_x = 0$, so $df_x(v) = ta$. We already know that f is an immersion, hence $t \neq 0$, thus $g(x, \frac{1}{t}) = a$, again a contradiction with choice of a, so the composition as an embedding. For compact manifolds injective immersions are embeddings, so the theorem is proved.

□

4 Partitions of unity

Before moving to non-compact manifolds we have to prove a corollary with aim of this theorem: **Theorem:** For any subset X of $\mathbb{R}^n$ and open covering of X $\{U_\alpha\}$ there exist a sequence of smooth functions $\{\phi_i\}$ on X, called a *a partition of unity* with the following properties:

(i) $0 \leq \phi_i(x) \leq 1$ for all $x \in X$ and all i.

(ii) Each $x \in X$ has a neighborhood on which all but finitely many functions ϕ_i are identical to zero.

(iii) Each function ϕ_i is identically zero except for some closed set contained in one of the U_α.

(iv) For each $x \in X$ $\Sigma_i \phi_i(x) = 1$.

Proof: Each set U_α in covering can be written as $M \cap W_\alpha$ for some open set W_α in $\mathbb{R}^n$. Then W is $\cup_\alpha W_\alpha$, and let $\{K_j\}$ be any nested sequence of compact sets s.t. the union of all K is W and $K_j \subset \mathrm{Int}(K_{j+1})$. For instance let $K_j = \{z \in W : |z| < j\}$. The collection of open balls in $\mathbb{R}^n$ whose closures belongs to at least one W_α forms an open cover of W. For each such ball V we select a smooth function η on $\mathbb{R}^n$, that takes one on that ball and zero outside a cIosed set contained in one of the W_α. We build a sequence of that functions as follows. For each $j \geq 3$ the compact set $K_j - Int(K_{j-1})$ is contained in the open set $W - (K_{j-2})$. The collection of all open balls that their closures belongs to $W - (K_{j-2})$ and some W_α forms an open cover of $K_j - Int(K_{j-1})$. Choose a finite subcover and then add for our sequence $\lceil \eta_i \}$ one function for each ball. Since we taked finitely subcover, for each j only finitely many functions η_i fail to vanish on K_j. The sum $\sum_{j=1}^{\inf} \eta_j$ is finite in a neighborhood of every point of W and at least one term is nonzero at any point W. Therefore $\frac{\eta_i}{\sum_{j=1}^{\inf} \eta_j}$ is a well-defined smooth function. Thus for every X we can define ϕ_i as a restriction of such function to X.

$\square$

Corrolary: On any manifold M exist a proper map $\rho : M \to \mathbb{R}$.
Proof: Let $\{U_\alpha\}$ be the collection of open subsets of M with compact closure, and let $\{\phi_i\}$ be a subordinate partition of unity. Then $\rho = \sum_{i=1}^{\infty} i\phi_i$ is well-defined smooth function. Now observe if $\rho(x) \leq j$, then one of first j functions must be nonzero at x. Then:

$$\rho^{-1}([-j, j]) \subset \bigcup_{i=1}^{j} \{x : \phi_i(x) \neq 0\}$$

$\square$

5 Non-compact case

Theorem: Every smooth k-dimensional manifold M^k can be embedded in $\mathbb{R}^{2k+1}$. **Proof:** Define a function $f : M \to \mathbb{R}^{2k+1}$ by $x \to \frac{x}{|x|^2+1}$, note that $|f(x)| < 1$ for all x. Let $\rho : M \to \mathbb{R}^{2k+1}$ be a proper function. Define $F : M \to \mathbb{R}^{2k+1}$ by $x \to (f(x), \rho(x))$. This is an injective immersion because f is. Now define a space H as in previous proof: $H := \{b \in \mathbb{R}^{\mathbb{M}} : b \perp a\}$ and project $\mathbb{R}^{2k+2}$ to H by $\pi : \mathbb{R}^{2k+2} \to H$.
Claim: The composition $\pi \circ F$ is proper.
The composition is an injective immersion for a.e. $a \in S^{2k+1}$, so we may choose an a that is neither of the sphere's two points. Given number c we assume that there exist a number d s.t. the set of points $x \in M$ where $|\pi \circ f(x)| \leq c$ is contained in the set $|\rho(x)| \leq d$. Since ρ is proper the last set is compact. If our assumption is true, then preimage of every closed ball in H under $\pi \circ F$ is a compact subset of M, that is implies that $\pi \circ F$ is proper. If the assumption is false then there exist a sequence $\{x_i\}$ in M for which $|\pi \circ F(x_i)| < c$ but $\rho(x_i) \to \infty$. Consider now the vector:

$$w_i = \frac{1}{\rho(x_i)}[F(x_i) - \pi \circ F(x_i)]$$

This is a vector that takes off the projection of $F(x_i)$ onto H, so its projective component on H is zero, and so it is a multiple of a. As $i \to \inf$:

$$\frac{F(x_i)}{\rho(x_i} \to (0, \ldots, 0, 1)$$

because $|f(x_i) < 1|$ for all i. The norm of quotient

$$\left\| \frac{\pi \circ F((x_i)}{\rho(x_i)} \right\| \leq \frac{c}{\rho(x_i)}$$

so it's converges to zero. Thus $w_i \to (0, \ldots, 0, 1)$. But each w_i is a multiple of a, therefore so is the limit, so a is on either of the poles, a contradiction.

$\square$

References

[1] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.