Oriented Manifolds

Definition: 1) Let V be a n-dimensional real vector space.
An orientation on V is a choice of an ordered basis modulo base change with a matrix with positive determinant.
The equivalence class of the standard basis is called standard orientation.
2) Let M be a smooth Manifold with boundary. An orientation on M is a choice of orientation on $T_x M$ $\forall x \in M$, such that for each $x \in M$ $\exists$ a parametrisation $g: U \longrightarrow M$ s.t. dg_x maps the standard basis onto the chosen parametrisation. (This is called orientation preserving).

Remark 1) each vectorspace admits exactly 2 orientations.
2) each connected orientable Manifold admits exactly 2 orientations.

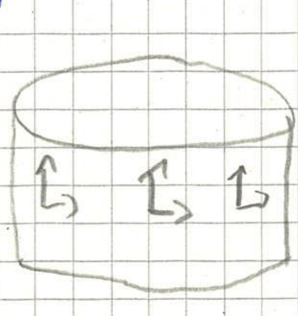
Examples
1) S^1

$$g: (-1,1) \longrightarrow S^1 \qquad t \longmapsto (t, \sqrt{1-t^2})$$

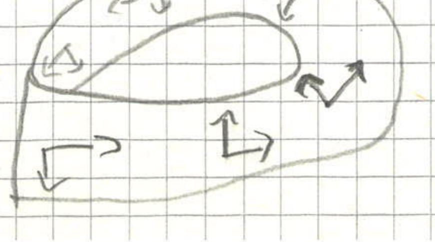
For $(x,y) \in S^1$ $\quad T_{x,y} S^1 = \langle \binom{y}{x} \rangle \leq \mathbb{R}^2$ $\quad$ choose $\binom{y}{x}$ as orientation

$dg_{x,y} = (1, \frac{x}{y})$ is orientation preserving.

2) $(0,1) \times S^1$



3) Moebius strip (non orientable)



Brouwer Degree

Definition: Let M, N be oriented n-dimensional manifolds w.o. boundary, M compact, N connected $\quad f: M \to N$

(i) for $x \in M$ regular point

$$\operatorname{sign} df_x := \begin{cases} 1 & \text{if } df_x \text{ is orientation preserving} \\ -1 & \text{if } df_x \text{ is orientation reversing} \end{cases}$$

(ii) for $y \in N$ regular value

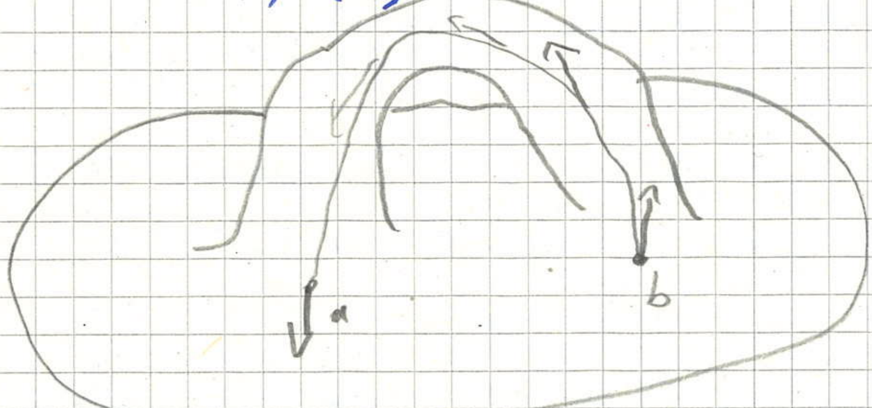
$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x$$

Theorem 1 $\deg(f, y)$ does not depend on y. (write $\deg f$)

Theorem 2. If f, g are homotopic $\deg f = \deg g$

Lemma 1: If M is the boundary of a manifold X and $f: M \to N$ extends to a smooth map $F: X \to N$, then $\deg(f, y) = 0$ for all regular values.

Proof: Assume y is regular value for F and f.
$F^{-1}(y)$ is a finite union of circles and arcs and boundary points of arcs are the only points of M. We view one such arc A with $\partial A = \{a\} \cup \{b\}$



Since X, N determine the orientation of A so if a is v.l.o.g. positive oriented b has to be negative oriented. Now sum over all arcs.

If y is not regular for F and f then use the procedure of the proof

Lemma 2: Let f, g be 2 smooth maps $M \to N$ with homotopy map $F: [0,1] \times M \to N$, $F(0) = f$, $F(1) = g$

Then $\deg(f, y) = \deg(g, y)$ $\forall y \in N$ regular

Proof $[0,1] \times M$ can be oriented as product manifold with boundary $\{0\} \times M$ ~~and~~, $\{1\} \times M$ with orientation M at 1 and $-M$ at 0.

so $\deg(F|_{\partial([0,1]\times M)}, y) = \deg(g, y) - \deg(f, y)$

And from Lemma 1 follows: $\deg(F|_{\partial([0,1]\times M)}, y) = 0$

Proof of Theorem 1

Let y, z be regular values. $\exists$ a map $h: N \to N$ smoothly homotopic to the identity with $h(y) = z$

$\Rightarrow \deg(f, y) = \deg(h \circ f, z) = \deg(f, z)$

Proof of Theorem 2 follows from Lemma 2

<u>Examples</u>

(1) S^n admits a smooth everywhere non zero vectorfield $\Leftrightarrow$ n is odd

Proof: (i) $(x_1, \dots, x_{2k}) \longmapsto (x_2, -x_1, \dots, x_{2k}, -x_{2k-1})$ is a smooth v.f.

(ii) assume n is even and $v: S^n \to S^n$ a smooth v.f. For $v(x) \in T_x S^n$ we have $v(x) \cdot x = 0$. We define $F: S^n \times [0, \pi] \to S^n$

$(x, \theta) \longmapsto x \cdot \cos\theta + v(x) \cdot \sin\theta$

$\langle F(x,\theta), F(x,\theta) \rangle = 1$

and $F(x, 0) = x$, $F(x, \pi) = -x$ $\Rightarrow$ id is homotopic to $-$id ↯

(since $\deg(\mathrm{id}) = 1$ and $\deg(-\mathrm{id}) = -1$)

(2) View $S^1 \subseteq \mathbb{C}$ $p_k: S^1 \to S^1$ $z \mapsto z^k$

$p_k^{-1}(1)$ consists of the k'th unit roots at which dp_k is orientation preserving if $k > 0$ and reversing if $k < 0$ (and not a regular value if $k = 0$) so $\deg(p_k) = k$