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Differentialtopology

Sard's Theorem

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The whole essay is based on Milnors "*Topology from the Differentiable Viewpoint*" and Guillemins and Pollacks "*Differential Topology*".

1 Sard's Theorem

1.1 Sard's Theorem

Let M,N be smooth manifolds and $f : U \to N$ be a smooth map, with $U \subseteq M$ open and let $C \subseteq M$ be the set of critical points.

Then $f(C) \subset N$ has measure zero (the set of critical values).

(proof will follow at the end)

While critical points and values are probably clear ($x \in U$ is critical, if and only if df_x isn't surjective), the measure of a set will be explained in the next subsection.
To take it easy, we will just look at sets with a measure zero. That are sets which can be covered with measurable sets (for example rectangles), whose total measure can get infinitely small.
Therefore we will use the following definition:

1.2 Definitions

Let $I := [a_1, b_1) \times ... \times [a_n, b_n) \subset \mathbb{R}^n$ be a rectangle.

Then we define $vol(I) := (b_1 - a_1) \cdot ... \cdot (b_n - a_n) = \prod_{i=1}^{n}(b_i - a_i)$ as the *volume of I*

And then for a subset $A \subset \mathbb{R}^n$ we say:

A has *measure zero*, if and only if, for all $\epsilon > 0$ there exists countable rectangles I_k, so that $\bigcup_{k\in\mathbb{N}} I_k \supset A$ and $[\sum_{k=1}^{\infty} vol(I_k)] < \epsilon$

There are several equivalent definitions of a set with measure zero, but we will just need this one.

(We don't need to look at the measurability, because with our definition, the outer measure is zero and therefore the set is measurable with the measure zero.)

1.3 Examples

i). All subsets of a set with measure zero has measure zero (obviously).

ii). All countable unions of sets with measure zero has measure zero, because we can w.l.o.g. look at sets A_k and the countable rectangles $I_{(k,l)}$, so that $\bigcup_{l\in\mathbb{N}} I_{(k,l)} \supset A_k$ and $[\sum_{l=1}^{\infty} vol(I_{(k,l)}] < \epsilon_k$ with $k \in \mathbb{N}$.

Now choose $\epsilon_k := \frac{\epsilon}{2^k}$ and $[\sum_{l,k=1}^{\infty} vol(I_{(k,l)})] < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$.

Now we have to generalize our definition for manifolds:

1.4 Definition

Let M be a m-dimensional manifold and A be a subset of M

> Then we say A has *measure zero in M*, if and only if for all parametrizations (U_k, φ_k) the sets $\varphi_k(U_k \cap A)^{-1} \subseteq \mathbb{R}^m$ has measure zero.

This is well-defined, because we have at most countable parametrizations and a countable union of sets with measure zero has measure zero too [like we proofed in 1.3ii)].

1.5 Remark

With Sard's Theorem we get the often used lemma:

> The set of regular values of a smooth map $f : M \to N$ $(N \backslash f(C))$ is dense in N,

which we used to proof the Brouwer fixed point Theorem and which is fundamental for the definition of the degree of a map (we needed for the definitioin at least one regular value in every open subset).

2 Examples of Sard's Theorem

2.1

Let $f : M \to N$, $f(x) = c$.

f is obviously smooth and has only one critical value $c \in N$, while every point $x \in M$ is a critical point.
The set of critical values $\{f(x) \mid x \in M\} = \{c\}$ has obviously measure zero.

2.2

Now we look at the function: $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x \cdot sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The function is smooth for all $x \in \mathbb{R}\backslash\{0\}$ and the critical values are the extrema of the function. Near $x = 0$ are infinite maxima and minima, so there are infinite critical values.

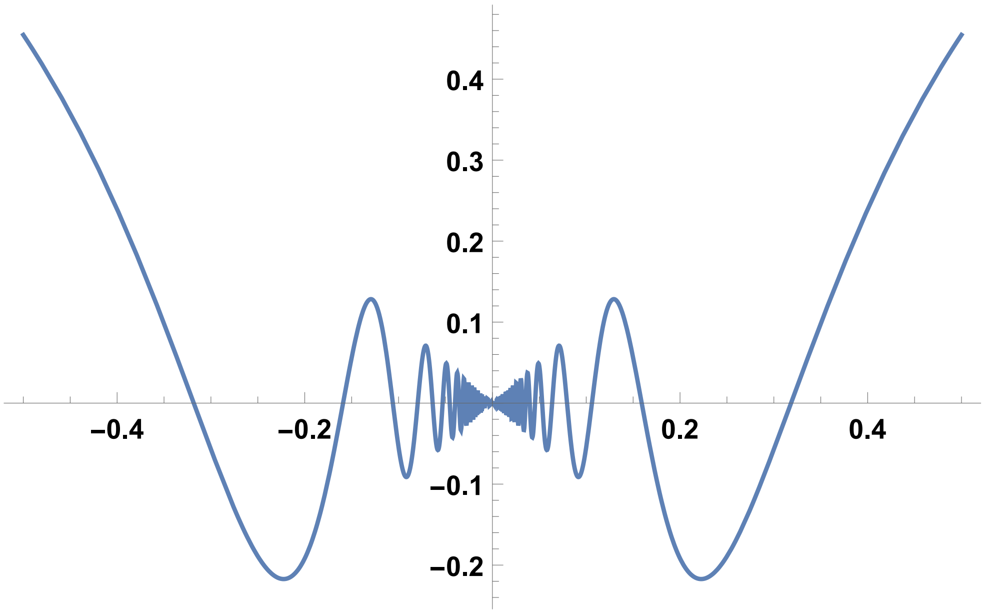

Figure 1: Function f

2.3

The Theorem of Sard says, that the set of regular values of a function $f : M \to N$ is dense in N. But we showed with the last example, that there can be infinite critical values too. Do there exists a function $f : \mathbb{R} \to \mathbb{R}$, so that the critical values are dense too? The next example is such a function:

Let $i \in \mathbb{Z}$, $\frac{1}{2} > \varepsilon > 0$ and $f_i : [i - \frac{1}{2} + \varepsilon, i + \frac{1}{2} - \varepsilon] \to \mathbb{R}$ be a map with:

$$f_i(x) = \begin{cases} q_i \cdot c_i \cdot exp(\frac{1}{(x-i)^2 - (\frac{1}{2}-\varepsilon)^2}) & x \in (i - \frac{1}{2} + \varepsilon, i + \frac{1}{2} - \varepsilon) \\ 0 & x = i \pm (\frac{1}{2} - \varepsilon) \end{cases}$$

and $c_i := exp(\frac{1}{(\frac{1}{2}-\varepsilon)^2})$.

Then f_i is smooth for all $x \in (i - \frac{1}{2} + \varepsilon, i + \frac{1}{2} - \varepsilon)$ (sometimes f_i is named $mollifier$).

$f_i' : (i - \frac{1}{2} + \varepsilon, i + \frac{1}{2} - \varepsilon) \to \mathbb{R}$, $f_i'(x) = \frac{-2(x-i)}{((x-i)^2 - (\frac{1}{2}-\varepsilon)^2)^2} \cdot f_i$ is the derivative of f_i, which is zero if and only if $x = 0$
So, $x = 0$ is a critical point and $f_i(x) = f_i(0) = q_i$ is a critical value.

To create our function f we use a bijection $\pi : \mathbb{Z} \to \mathbb{Q}$ to create maps f_i $(i \in \mathbb{Z})$ so that for every rational number q there exists a map f_i with q as a critical value. Then we define f:

$$f(x) = \begin{cases} f_i(x) & x \in (i - \frac{1}{2} + \varepsilon, i + \frac{1}{2} - \varepsilon) \\ 0 & otherwise \end{cases}$$

According to this definition, the critical values of f are the rational numbers, which are dense in $\mathbb{R}$.

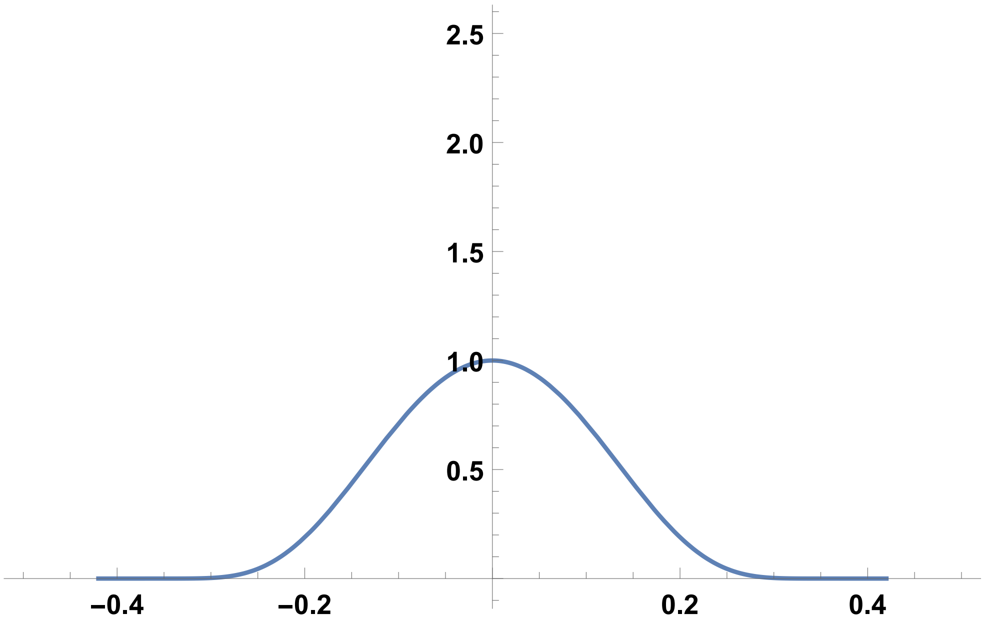

Figure 2: Function f_0 with $q_0 = 1$ and $\varepsilon = 0,08$

3 Proof of Sard's Theorem

We will use the last section to proof Sard's Theorem. At first we need two important theorems:

3.1 Fubini Theorem (for measure zero)

Let $k < n$, $c \in \mathbb{R}^k$, $V_c := \mathbb{R}^{n-k} \times \{c\}$ and A be a closed subset of $\mathbb{R}^n$ such that $A \cap V_c$ has measure zero in V_c for all $c \in \mathbb{R}^k$. Then A has measure zero in $\mathbb{R}^n$.

You can proof this theorem very easy with the real Fubini Theorem.

3.2 Second Axiom of Countability

Let τ be the set of all open sets of a manifold M. For all $U \in \tau$, there exists countabile sets $U_k \in \tau$, $k \in \mathbb{N}$, so that $U = \bigcup_{k\in\mathbb{N}} U_k$

This is an axiom for abstract manifolds. In $\mathbb{R}^n$ you can proof it easily.

3.3 Proof of Sard's Theorem

While looking at a map $f : M \to N$ (with $dim(M) = m > dim(N) = n$) and the critical points C of f, we can restict f on sets U_i so that U_i and $f(U_i)$ is diffeomorphic to sets $U_i' \subset \mathbb{R}^m$, $V_i' \subset \mathbb{R}^n$ and $\bigcup_{i\in\mathbb{N}} V_i$ is covering $f(C)$. The sets V_i are countable so:
$f(C)$ has measure zero if and only if for all $i \in \mathbb{N}$ $f(C) \cap V_i$ has measure zero.
Now, to proof the Theorem of Sard we can look at maps g_i:

$$g_i : \varphi_i^{-1}(U_i) \to \mathbb{R}^n,\ g_i(x) := \varphi_i^{-1} \circ f \circ \psi_i$$

With the parametrizations (U_i, φ_i) of M and (V_i, ψ_i) of N.

And the set of crical values of f has measure zero if and only if the set of critical values of g_i has measure zero for all $i \in \mathbb{N}$.

Now we proof, that the critical values of a map $g : U \to \mathbb{R}^n$ (with $U \subset \mathbb{R}^m$) has measure zero. This is equivalent to the Theorem of Sard (for manifolds). The proof will be by induction on m The theorem is true for $m = 0$, so we will assume that it is ture for $m-1$ and proof it for m.

Let C be the set of critical points. We define C_k as the set of all $x \in U$ such that all derivatives of f of order $\leqslant k$ vanishes at x
First we will proof:

$f(C \backslash C_1)$ **has measure zero.**

Around each $x \in (C\backslash C_1)$ we will find an open set V such that $V \cap (C\backslash C_1)$ has measure zero. Since $C\backslash C_1$ is covered by countable ones, this will proof, that $f(C\backslash C_1)$ has measure zero. Since $x \notin C_1$, we say w.l.o.g. $\partial_1 f_1(x) \neq 0$.

Now we can define $h : U \to \mathbb{R}^m$ by, $h(x) := (f_1(x), x_2, ..., x_m)$

dh_x is nonsingular, so there exists a neighborhood V of x which is diffeomorphic onto an open set V'. The composition $g := f \circ h^{-1}$ has the same critical values as f with the restriction on V.
Now we have constructed g so that $g(t, x_2, ..., x_m) = (t, y_2, ..., y_n)$. Therefore for each $t \in \mathbb{R}$ we can define the function $g_t : (\{t\} \times \mathbb{R}^{m-1}) \cap V' \to \{t\} \times \mathbb{R}^{n-1}$, with $g_t(x) = g(t, x)$. We can describe the derivative of g as:

$$dg_{(t,x)} = (\tfrac{\partial g_i}{\partial x_j}) = \begin{pmatrix} 1 & 0 \\ * & (dg_t)_x \end{pmatrix}$$

We see: $det((dg)_{(t,x)}) = det((dg_t)_x)$, so (t, x) is a critical point of g (and of f), if and only if, x is a critical point of g_t. By induction, Sard's Theorem is true for g_t and the set of critical values has measure zero. Consequently, by Fubini's theorem ($f(C\backslash C_1)$ is closed) it has measure zero for g too.

$f(C_k \backslash C_{k+1})$ **has measure zero.**

This part is similar, but easier as the last one. For each $x \in C_k\backslash C_{k+1}$ there is some $k + 1$ derivative of f which is not zero. So we find a kth derivative of f (which we will name ρ), that vanishes on C_k, but w.l.o.g. $\partial_1 \rho_1$ doesn't vanish. We define $h : V \to V'$, $h(x) = (\rho(x), x_2, ..., x_m) = (0, x_2, ..., x_m)$ (for $x \in C_k\backslash C_{k+1}$) which is a diffeomorphism. Let $g := f \circ h^{-1}$. Then g has the same critical points as f (in the neighborhood), which are in the hyperplane $\{0\} \times \mathbb{R}^{m-1}$(because of the definition of C_k and h). Let $g' : \{0\} \times \mathbb{R}^{m-1}$ be the restriction of g With the same argument we know by induction, the set of critical values of g' has measure zero and therefore g and f. Because there are countable open sets, which are covering $C_k\backslash C_{k+1}$, we are done.

And now the last part:

For $k > \frac{m}{n} - 1$, $f(C_k)$ has measure zero

Combined with the two other parts, this will proof the Theorem of Sard.

Let S be a cube with edge s. Since C_k can be covered by countably many such cubes, we have to show, that $f(C_k \cap S)$ has measure zero. From Taylor's theorem, the compactness of S and the definition of C_k we know that:

$$f(x+h) = f(x) + R(x,h), \text{ where } |R(x,h)| < a|h|^{k+1}, \text{ for } x \in C_k \cap S \text{ and } x+h \in S$$

Now we can split S in in r^m cubes with edges s/r. Let S_1 be the new cube that contains a point $x \in C_k$. Then any point in S_1 can be written as $x+h$ with $h < \sqrt{n} \cdot s/r$

Let $b := 2a(\sqrt{n} \cdot s)^{k+1}$ (constant). We see with the Taylor Theorem, that $f(S_1)$ is in the cube with the edge b/r^{k+1} centered about $f(x)$ We can do that for every $x \in C_k$).

$f(C_k \cap S)$ is contained in the union of at most r^m cubes, so the total volume of the cubes is:

$$vol \leq r^m \cdot (\frac{b}{r^{k+1}})^n = b^n \cdot r^{m-(k+1)n}$$

With r big enough the volume tends to 0, so $f(C_k)$ hast measure zero. That completes the proof.

4 References

[1] John W. Milnor: *Topology from the Differentiable Viewpoint*, 1963

[2] Victor Guillemin, Alan Pollack: *Differential Topology*, 1974