Universal covers of complex algebraic varieties

Y. Brunebarbe
with B. Bakker and J. Tsimerman

Cetraro-70th birthday of Thomas Peternell
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- In dimension 1: $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{D}$.
- In higher dimension: rather mysterious!

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Question (Shafarevich)

Let X be a smooth projective variety. Is $\tilde{X}$ holomorphically convex? i.e., does there exist a proper holomorphic map $\tilde{X} \to \tilde{Y}$ to a Stein analytic space $\tilde{Y}$?

Shafarevich conjecture

Works of Campana, Kollár, Napier, Gurjar, Shastri, Zuo, Mok, Katzarkov, Jost, Lasell, Ramachandran, etc.

Theorem (Eyssidieux-Katzarkov-Pantev-Ramachandran)

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X normal algebraic variety, such that $\pi_1(X) \subset \mathrm{GL}(n, \mathbb{C})$ for some $n \geq 1$. Then $\tilde{X}$ is dense Zariski-open in a holomorphically convex space.

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Fact: Every representation $\rho\colon \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ with torsion-free image is the restriction of a nonextendable representation.

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More generally, we give a sufficient condition on $\rho\colon \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ implying that $\tilde{X}^{\ker \rho}$ is holomorphically convex.

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Proposition(Griffiths): ρ nonextendable $\iff$ p proper $\iff$ $\tilde{p}$ proper.

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A complex space is Stein $\iff$ it admits a strictly psh exhaustion function.

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Theorem (Griffiths-Schmid)

There exists $\mathcal{D} \to \mathbb{R}_{\geq 0}$ which is strictly psh in the horizontal directions.

General strategy to prove Theorem A

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2 main steps:

1. Construct the Cartan-Remmert reduction $\tilde{X}^{\ker\rho} \to \tilde{Y}$ a priori ($\iff$ the Shafarevich morphism).
2. Construct a strictly psh exhaustion function on $\tilde{Y}$.

The Shafarevich morphism

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- When ρ is semisimple, f^{an} was constructed independently by Deng-Yamanoi and myself.
- When ρ is the monodromy of a $\mathbb{Z}$-VHS, f is the Stein factorization of the period map. It is algebraic by o-minimal GAGA (BBT) and definability of period maps (Bakker–Klingler–Tsimerman).

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$\rightsquigarrow$ Reduction to the 'maximal case'.

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$$f \circ g(Z) = \{pt\} \iff \forall \rho \in \Sigma, g^{-1}\rho\colon \pi_1(Z) \to \mathrm{GL}(n, \bar{\mathbb{Q}}_p) \text{ is bounded.}$$

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Consider the map combining period maps and non-archimedean reductions for all prime numbers (only finitely many matters):

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Proposition

Assume that Σ is nonextendable (hence Σ^{Shaf} too). Then:

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- *Every connected compact analytic subspace of $\tilde{X}^{\Sigma^{\mathrm{Shaf}}}$ is contained in a fibre of ϕ.*
- *The connected components of the fibres of ϕ are compact.*

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Therefore, ϕ has a Stein factorization

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which is the Σ-Shafarevich morphism.

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Thus, the algebraization of s is a consequence of:

Theorem (BBT)

*Let $f : X \to Z$ be a proper morphism of definable analytic **normal** spaces. Then the Stein factorization exists in the definable analytic category.*

Thanks!