Weak Specialness and Potential Density. Peternell/Cetraro 1/7/24

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- Suitably base-changing we get $f_d : X_d \to B_d = \mathbb{P}_1$ with the orbifold base (B_d, Δ_{f_d}) of general type, X_d WS : OMC applies.

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- The corollary applies to the (WS) threefolds fibered over $\mathbb{P}_1$ deduced by base-change from the Lafon threefold.

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- Special implies WS. Reverse true for $n \leq 2$ only, by Lafon, [B-T] threefolds.

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- Lafon threefold : $\Delta_f^* \neq \Delta_f$ happens too with Enriques surfaces (which are WS). Question raised for K3's in [C2005].

The Lafon threefold.

- It is Y, defined in affine coordinates (x, y, z, u, t) by :
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- $\Delta_f^* = 0$ if the smooth fibres X_b have $|\chi(X_b| = 1$ ([ELW 2007]).

Weak specialness and fibrations.

- **Proposition :** $f : X \to B$ a fibration s.t $\Delta_f^* = 0$, $u : X' \to X$ étale. Let $f \circ u = u' \circ f'$, $f' : X' \to B'$, $u' : B' \to B$ be the Stein factorisation of $f \circ u$.
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- Let $f_d : X_d \to B_d = \mathbb{P}_1$ be deduced from the Lafon fibration $f : X \to B = \mathbb{P}_1$ by a generic base-change $g : B_d = \mathbb{P}_1 \to B$ of degree d (ramified over the smooth fibres of f only). **Then X_d is WS.**

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- $(\mathbb{P}_1, \Delta)(\mathbb{Q}) := \{x \in \mathbb{Q} | (x, t)_p \geq m_t, \forall p \ s.t : (x, t)_p > 0, t = 0, 1, \infty\}$. The **divisible version** is :
- $(\mathbb{P}_1, \Delta)^{Div}(\mathbb{Q}) := \{x \in \mathbb{Q} | (x, t)_p \equiv 0[m_t], \forall p, t = 0, 1, \infty\}$.
- $(\mathbb{P}_1, \Delta)^{Div}(k) \subset (\mathbb{P}_1, \Delta)(k)$.

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- $Card(\{0 < a^{[m]} \leq B\}) \sim C(m).B^{\frac{1}{m}}$ as $B \to +\infty$ ([E-S,1935]). $C(2) = \frac{\zeta(3/2)}{\zeta(3)}$, for example.

abc implies OM

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- *abc* claims that, for any $\varepsilon > 0, \exists C_\varepsilon > 0$ s.t : $Rad(\alpha.\beta.\gamma) \geq C_\varepsilon.M^{1-\varepsilon}, \forall(\alpha, \beta, \gamma \in \mathbb{Z})$ s.t : $\alpha + \beta = \gamma$, where $M := Max(\{|\alpha|, |\beta|, |\gamma|\})$.

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- Thus : $M^{(m_0^{-1}+m_1^{-1}+m_\infty^{-1})} = M^{1-\varepsilon'} \geq Rad(uv(u - v)) \geq C_\varepsilon.M^{1-\varepsilon}, \forall\varepsilon > 0$.
- Choose $\varepsilon < \varepsilon'$ (e.g : $\varepsilon := 43^{-1}$), divide by $M^{1-\varepsilon'}$, we get : $1 \geq C_\varepsilon.M^{\varepsilon'-\varepsilon}$ implies the claimed finiteness, since : $M \leq C_\varepsilon^{-(\varepsilon'-\varepsilon)^{-1}}$.