The cycle space of the K3 period domain and moduli theory for families over $\mathbb{P}^1$

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marked K3 surfaces

Recall: A compact (Kähler) surface X is called K3 surface if

- $\Omega^2_X \cong \mathscr{O}_X$, and
- $\pi_1(X) = \{e\}$.

Examples:

$X_4 \subset \mathbb{P}^3$, Kummer surfaces $\widetilde{T^2/\langle \pm 1 \rangle}$, deformations

Topology:

All K3 surfaces are deformation equivalent, hence diffeomorphic.
$\Rightarrow$ The group $H^2(X, \mathbb{Z})$ is free abelian, and together with the intersection form $\langle \cdot, \cdot \rangle_X$ isomorphic to $\Lambda := U^{\oplus 3} \oplus E_8^{\oplus 2}$. $\rightsquigarrow$ **(3,19)**

Definition

A marked K3 surface (X, ϕ) consists of a K3 surface X together with a marking $\phi\colon H^2(X, \mathbb{Z}) \xrightarrow{\cong} \Lambda$.

The period map

Crucial tool: period map

$$\mathrm{p}\colon (X,\phi) \mapsto \phi_{\mathbb{C}}\big(H^{2,0}(X)\big) \in \mathbb{P}(\Lambda_{\mathbb{C}})$$

Image of p lies in the period domain

$$\Omega := \{[\lambda] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid \langle \lambda, \lambda \rangle_{\mathbb{C}} = 0, \langle \lambda, \bar{\lambda} \rangle_{\mathbb{C}} > 0\} \subset Q.$$

For marked families $\mathscr{X} \to B$, the period map $\mathrm{p}\colon B \to \Omega$ is holomorphic.

Local Torelli Theorem (Andreotti/Weil)

Let $\mathscr{X} \to (B,0) := \mathrm{Def}(X)$ be the universal deformation of the K3 surface $X = \mathscr{X}_0$. Then, the period map $B \to \Omega$ associated with any marking ϕ of X is a (local) biholomorphism at 0.
In particular, deformations of X are unobstructed.

The moduli space of marked K3 surfaces

Together with the following

Faithfulness Theorem

For every K3 surface X, the natural action on cohomology induces an injective map

$$\mathrm{Aut}_{\mathscr{O}}(X) \hookrightarrow O\big(H^2(X, \mathbb{Z})\big).$$

the Local Torelli Theorem allows us to glue together the local deformation spaces to a fine moduli space of marked K3 surfaces

$$\widetilde{\Omega} \xrightarrow{\mathrm{p}} \Omega.$$

Because of its description in [Astérisque 126 (1985)], we will refer to this as Beauville's construction of $\widetilde{\Omega}$.

Atiyah's example $\Rightarrow$ $\widetilde{\Omega}$ is a smooth non-Hausdorff complex space.

The Global Torelli Theorem

Using twistor cycles, which we will discuss later, Todorov showed the surjectivity of the period map $\mathrm{p}\colon \widetilde{\Omega} \twoheadrightarrow \Omega$ on each connected component of $\widetilde{\Omega}$ (there are 2 of them).

Finally, Burns-Rapoport ($\oplus$ Looijenga-Peters) established the

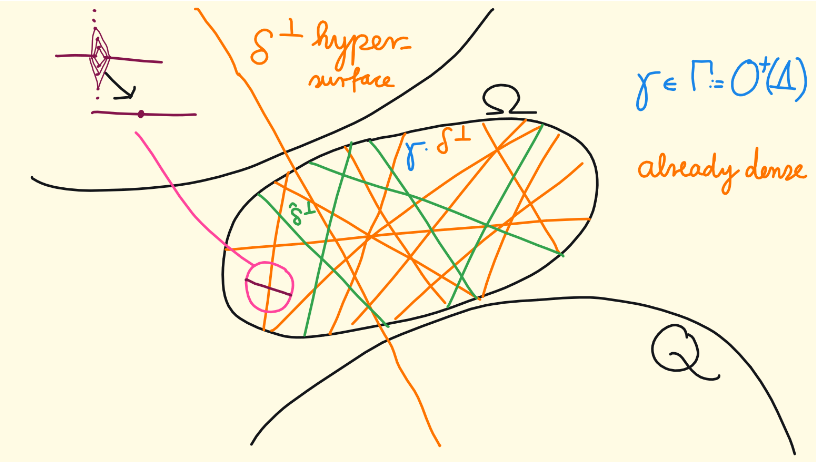
Global Torelli Theorem

For two marked K3 surfaces (X, ϕ) and (X', ϕ') there exists a unique biholomorphic map $f\colon X \to X'$ satisfying $\phi' = \phi \circ f^*$ if and only if

- $\mathrm{p}(X, \phi) = \mathrm{p}(X', \phi') \in \Omega$ (same period point), and
- $\phi_{\mathbb{C}}(\mathscr{K}_X) = \phi'_{\mathbb{C}}(\mathscr{K}_{X'}) \subset \Lambda_{\mathbb{C}}$ (same image of Kähler cone).

Together with numerical characterisation of Kähler cones, this gives the following picture.

The Burns-Rapoport space



$\delta \in \Lambda$ with $\langle \delta, \delta \rangle = -2$; i.e., a δ is a (-2)-class

The action of $\Gamma = O^+(\Lambda)$ & the base cycle

Eventually, one is interested in the moduli space of (biholomorphism classes of) K3 surfaces. Changing the marking is realised by the natural action of

$$\Gamma := O^+(\Lambda)$$

on $\widetilde{\Omega}$. Topologically, the moduli space is hence $\widetilde{\Omega}/\Gamma \simeq \Omega/\Gamma$. As

$$\Omega = SO^\circ(3,19)/SO(2) \times SO^\circ(1,19) = SO^\circ(3,19) \bullet z_0,$$

the topological space Ω/Γ is not even locally Hausdorff. ↯↯↯

The base cycle C_0

The orbit of the maximal compact subgroup $SO(3) \times SO(19)$ through z_0 is a compact complex submanifold $C_0 \subset \Omega$, yielding an embedding $\mathbb{P}^1 \hookrightarrow \Omega$. This is actually the period map for a marked **twistor family** $\mathscr{X} \to \mathbb{P}^1$ of K3 surfaces, built from a **KE-metric**.

Deforming the base cycle and its twistor space

Recall that we have

$$\mathscr{X} \twoheadrightarrow C_0 \hookrightarrow \Omega.$$

1. Compute

$$\mathscr{N}_{C_0/\Omega} \cong \mathscr{O}_{C_0}(2)^{\oplus 19}$$

$$\Rightarrow \qquad H^1\big(C_0, \mathscr{N}_{C_0/\Omega}\big) = \{0\} \quad \text{and} \quad h^0\big(C_0, \mathscr{N}_{C_0/\Omega}\big) = 57.$$

   So, $\mathrm{Dou}(\Omega)$ is smooth and of dimension 57 at the point $[C_0]$.

2. On the other hand, Brecan, Kirschner, and Schwald show: $H^2\big(\mathscr{X}, \mathscr{T}_{\mathscr{X}}\big) \neq \{0\}$, but we have $H^2\big(\mathscr{X}, \mathscr{T}_{\mathscr{X}/\mathbb{P}^1}\big) = \{0\}$, and $\mathrm{Def}(\mathscr{X})$ is smooth and of dimension 57 at the base point.

This is no coincidence and can be explained via the notions of second level families and second level period maps.

Second level families & second level period maps

Definition

A deformation of a family $f_0\colon X_0 \to C_0$ of K3 surfaces over B is a deformation $(X \xrightarrow{f} C \xrightarrow{h} B)$ of f_0 such that $X \xrightarrow{f} C$ is a family of K3 surfaces. We call $(X \xrightarrow{f} C \xrightarrow{h} B)$ a second level family over B if $C \xrightarrow{h} B$ is a smooth family of rational curves. A marking of such a family is a marking of $f\colon X \to C$.

A second level family is called embedded, if $\exists$ marking s.th. the period map $\mathrm{p}\colon C \to \Omega$ embeds each fibre C_b.

Construction

For a marked embedded second level family, the holomorphic map

$$\mathrm{P}\colon B \to \mathrm{S}(\Omega) \subset \mathrm{Dou}(\Omega), b \mapsto [C_b]$$

is called the second level period map of the family. Here, $\mathrm{S}(\Omega)$ is the open subscheme of smooth rational curves in Ω.

Local Torelli Theorem & Faithfulness

Local Torelli Theorem for total spaces

The universal family of the total space X_0 of an embedded family of K3 surfaces can be made into an embedded second level family in a natural way. For any marking of this second level family, the second level period map is a local biholomorphism at the base pt.

Faithfulness for K3s $\Rightarrow$ Faithfulness for marked families.

Faithfulness for total spaces

Let $f \colon X \to C$ be an embedded family of K3 surfaces. Then id_X is the only automorphism of the total space X that induces the identity on $H^2(X, \mathbb{Z})$.

Lemma:

Let $f \colon X \to C$ be a marked family of K3 surfaces with injective period map $C \hookrightarrow \Omega$. Then, every automorphism of X preserves f.

Two constructions of the moduli space

Now, glue deformations using the period maps and faithfulness.

Theorem

There exists a smooth complex space $\mathscr{M}$ that is a fine moduli space for marked total spaces of embedded families of K3 surfaces. Its second level period map $\mathrm{P}\colon \mathscr{M} \to \mathrm{S}(\Omega)$ is étale.

Alternative construction:

Consider the Burns-Rapoport space $\widetilde{\Omega} \to \Omega$ as a sheaf (of sets) and let

$$\Omega \xleftarrow{\mu} \mathcal{C} \xrightarrow{\nu} \mathrm{S}(\Omega)$$

be the universal family over $\mathrm{S}(\Omega)$.

Explicit realisation of $\mathscr{M}$

The moduli space $\mathscr{M}$ is isomorphic over $\mathrm{S}(\Omega)$ to the éspace étalé of the transfered sheaf $\nu_*(\mu^{-1}(\widetilde{\Omega}))$.

$\mathscr{M}$ is Hausdorff

Explicit realisation $\oplus$ the Global Torelli Theorem for K3s $\Rightarrow$

Theorem

The moduli space $\mathscr{M}$ of marked total spaces of embedded families of K3 surfaces is Hausdorff; i.e., $\mathscr{M}$ is a complex manifold.

This should be compared with $\widetilde{\Omega}$.

Complex geometry of the cycle space:

Let $\mathrm{C}_1(\Omega) \subset \mathrm{S}(\Omega)$ be the connected component containing the base cycle $C_0 \subset \Omega$ / all twistor cycles. Then, it is known that

Facts (see [Fels-Huckleberry-Wolf])

$\mathrm{C}_1(\Omega)$ is a **Kobayashi-hyperbolic Stein manifold**. In particular, the natural action of Γ on $\mathrm{C}_1(\Omega)$ is properly discontinuous.

Geometry of $\mathscr{M}$

The Stein property and Brody-hyperbolicity of $\mathrm{C}_1(\Omega)$ imply:

Proposition

Let $X \xrightarrow{f} C \xrightarrow{h} B$ be a marked second level family of $K3$ surfaces over a connected base B. Assume either that B is compact or that $B = \mathbb{C}$. If there exists a point $0 \in B$ such that the fiber $X_0 \xrightarrow{f_0} C_0$ is a twistor family, then $X \xrightarrow{h \circ f} B$ is isomorphic to the trivial family over B with fiber X_0.

Discontinuity of the Γ-action implies:

Theorem

Let $\mathscr{M}^\circ$ be the part of $\mathscr{M}$ lying over $\mathrm{C}_1(\Omega)$. Then, Γ acts properly discontinuously on $\mathscr{M}^\circ$, and the Hausdorff complex space $\mathscr{M}^\circ/\Gamma$ is a coarse moduli space for total spaces of embedded families of K3 surfaces over cycles in $C_1(\Omega)$.

Some further directions

1. Geometric meaning of cycles in $C_1(\Omega)$; cf. twistor cycles $\leftrightarrow$ KE-metrics ? Some partial answers exist.

2. Is there some notion of Mumford-Tate group / CM-points in $C_1(\Omega)$ that has meaning in terms of the families lying over the corresponding cycles ? This is related to work of Huybrechts on "brilliant families" of K3 surfaces.

3. Is the quotient $C_1(\Omega)/\Gamma$ Stein ? There is some evidence in this direction. This should have some consequences for Γ-invariant cohomology classes on Ω. Indeed, there is a "double fibration transform" induced by $\Omega \xleftarrow{\mu} \mathcal{C} \xrightarrow{\nu} C_1(\Omega)$:

$$\mathcal{F}\colon H^1(\Omega, \mathscr{O}_\Omega(-1))^\Gamma \longrightarrow H^0(C_1(\Omega), R^1\nu_*(\mu^*\mathscr{O}_\Omega(-1)))^\Gamma;$$

the $R^1\nu_*$-sheaf is isomo. to the trivial line bundle on $C_1(\Omega)$.