Bogomolov-Gieseker inequality for log terminal Kähler threefolds

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Transcendental methods in algebraic geometry

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Bogomolov-Gieseker inequality

(X, ω) compact Kähler manifold of dimension n, E vector bundle of rank r .

Bogomolov-Gieseker discriminant

$\Delta(E) := c_2(\mathrm{End}(E)) = 2rc_2(E) - (r-1)c_1^2(E) \in H^4(X, \mathbb{C})$.

Theorem (Bogomolov '78, Gieseker '79, Miyaoka '87, Uhlenbeck-Yau '86-'89)

If E is $[\omega]$-stable, then

$$\Delta(E) \cdot [\omega]^{n-2} \geqslant 0,$$

equality if and only if E is projectively flat.

Reduction to the surface case when $[\omega]$ is rational

Assume X projective, $[\omega] = c_1(H)$, H ample Cartier divisor

- Choose $H_i \in |mH|$, $m \gg 1$, $S := H_1 \cap \ldots \cap H_{n-2}$
- $\Delta(E) \cdot H^{n-2} = \frac{1}{m^{n-2}} \Delta(E|_S)$
- H_i very general $\Rightarrow E|_S$ stable (Mehta-Ramanathan)

↳ Reduce to 2-dimensional case of $(S, E|_S)$

Short intermission : Hermite-Einstein metrics

ω Kähler form, h hermitian metric on E
$\Theta_h(E) := iD_h^2 \in C^\infty(X, \Omega_X^{1,1} \otimes \mathrm{End}(E))$ Chern curvature form

Hermite-Einstein metric

h is Hermite-Einstein wrt $\omega \underset{\text{def}}{\iff} \mathrm{tr}_\omega(\Theta_h(E)) = \lambda \mathrm{Id}_E$, $\lambda \in \mathbb{R}$.

If h is HE, then

$$\begin{aligned} \Delta(E,h) \wedge \omega^{n-2} &:= (2rc_2(E,h) - (r-1)c_1^2(E,h)) \wedge \omega^{n-2} \\ &= a_n \|\mathring{\Theta}_h(E)\|^2 \omega^n \end{aligned}$$

↳ (E,h) HE $\Rightarrow \Delta(E) \cdot [\omega]^{n-2} \geqslant 0$

Proof of Bogomolov-Gieseker inequality

Donaldson-Uhlenbeck-Yau Theorem

If E is $[\omega]$-stable, it admits a HE metric h

↳ Bogomolov-Gieseker inequality

Remarks

- $\Delta(E) \cdot [\omega]^{n-2} = 0 \Leftrightarrow E$ is projectively flat.
- Generalizes to orbifolds

Log terminal varieties I

X normal analytic space

Log terminal singularities

(X, x) is log terminal if

- $\exists m \geqslant 1, \exists U$ nbd of x, $\exists \sigma \in H^0(U_{\mathrm{reg}}, mK_{U_{\mathrm{reg}}})$ non-vanishing
- $\int_{U_{\mathrm{reg}}} (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} < +\infty$

Examples

- Quotient singularities (i.e. locally $\mathbb{C}^n/G$, G finite)
- Log terminal surface singularities are quotient singularities
- $(z_0^2 + \cdots + z_n^2 = 0) \subset \mathbb{C}^{n+1}$ is log terminal but not quotient for $n \geqslant 3$.

Log terminal varieties II

X variety with log terminal singularities

Theorem (Greb-Kebekus-Kovács-Peternell '11)

X has quotient singularities in codimension two, i.e. $\exists Z \subsetneq X$ analytic subset with $\operatorname{codim} Z \geqslant 3$ such that $X \backslash Z$ is an orbifold.

Threefolds

A log terminal threefold X is an orbifold away from finitely many points.

Metrics on singular spaces

X normal analytic space

Kähler metric ω_X

It is a Kähler metric on X_{reg} which extends to a smooth Kähler metric under local embeddings $X \underset{\mathrm{loc}}{\hookrightarrow} \mathbb{C}^N$.

Orbifold metric ω_{orb}

If X is an orbifold, an orbifold metric is a Kähler metric on X_{reg} such that $p^*(\omega_{\mathrm{orb}}|_{U_{\mathrm{reg}}})$ extends to a smooth Kähler metric on V for any local uniformizing chart $p : V \to U$.

▲ At a singular point, an orbifold metric is *never* Kähler ▲

$\mathbb{Q}$-sheaves

X log terminal variety, E coherent reflexive sheaf on X.
▲ E need not be locally free in codimension two ▲

$\mathbb{Q}$-sheaf

E is a $\mathbb{Q}$-sheaf if it is an orbifold bundle in codimension two, i.e. $\exists Z \subsetneq X$ analytic subset with $\mathrm{codim} Z \geqslant 3$ such that

- $X \backslash Z$ is an orbifold, covered by local uniformizing charts $p : V \to U$.
- For any such chart, $p^*(E|_{U_{\mathrm{reg}}})$ is the restriction of a locally free sheaf on V.

Example

The tangent sheaf T_X is a $\mathbb{Q}$-sheaf.

Chern numbers

(X, ω_X) log terminal compact Kähler threefold, E $\mathbb{Q}$-sheaf on X.
$X_0 = X \backslash \{p_1, \ldots, p_r\}$ orbifold locus

h_0 orbifold metric on $E|_{X_0}$
$\omega_X = \omega_0 + i\partial\bar{\partial}\phi$, with ω_0 compactly supported on X_0.

Orbifold BG discriminant

$\Delta(E) \cdot [\omega_X] := \int_{X_0} \Delta(E, h_0) \wedge \omega_0$

It is well-defined, i.e. independent of the choice of h_0, ω_0 and ϕ.

Statement of the main theorem

Theorem (G-Păun '24)

(X, ω_X) log terminal compact Kähler threefold, E $\mathbb{Q}$-sheaf on X. If E is $[\omega_X]$-stable, then

$$\Delta(E) \cdot [\omega_X] \geqslant 0,$$

equality $\Rightarrow$ $E|_{X_{\mathrm{reg}}}$ is locally free and projectively flat.

Remarks

- Application to the abundance conjecture, cf Campana-Höring-Peternell '16
- If $[\omega_X] = c_1(H) \rightsquigarrow$ usual BG inequality for orbifold surfaces
- Alternative proof by W. Ou '24.

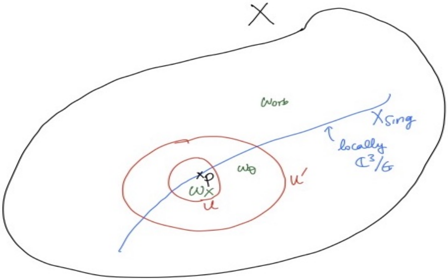
Main steps of the proof

1. Construct a suitable metric ω_θ on X, which is orbifold away from $\{p_i\}$ and Kähler at those points.
2. Appeal to [CGNPPW] to get a metric h on $E|_{X_{\mathrm{reg}}}$ which is Hermite-Einstein wrt ω_θ.
3. Compute $\Delta(E) \cdot [\omega_X]$ using h and ω_θ.

Construction of a interpolating metric

One constructs a "metric" $\omega_\theta = \omega_X + dd^c\varphi$ on X such that

$$\omega_\theta = \begin{cases} \omega_X & \text{near } p_1, \dots, p_r \\ \omega_{\mathrm{orb}} & \text{away from } p_1, \dots, p_r \end{cases}$$



The associated Hermite-Einstein metric

Thanks to [CGNPPW], there exists a HE metric h on $E|_{X_{\mathrm{reg}}}$ such that

- $\mathrm{tr}_{\omega_\theta}(\Theta_h(E)) = \lambda \mathrm{Id}_E$ on X_{reg}, for some $\lambda \in \mathbb{R}$.
- h extends to a smooth orbifold metric on $X \setminus U'$.
- $\int_{X_{\mathrm{reg}}} |\Theta_h(E)|^2_{h,\omega_\theta} \omega_\theta^3 < +\infty$.

The remaining assignment

Write $\omega_\theta = \omega_0 + i\partial\bar{\partial}\phi$ with ω_0 supported on $X \backslash U'$. Then

$$\begin{aligned}\Delta(E) \cdot [\omega_X] &= \int_{X_{\mathrm{reg}}} \Delta(E,h) \wedge \omega_0 \\ &= \underbrace{\int_{X_{\mathrm{reg}}} \Delta(E,h) \wedge \omega_\theta}_{\geqslant 0} - \int_{X_{\mathrm{reg}}} \Delta(E,h) \wedge i\partial\bar{\partial}\phi\end{aligned}$$

If $\chi_\varepsilon \to 1_{X \backslash \{p_i\}}$ is a cut-off, it is enough to show

$$\int_X \Delta(E,h) \wedge \bar{\partial}\chi_\varepsilon \wedge \partial\phi \to 0, \quad \int_X \Delta(E,h) \wedge \bar{\partial}(\chi_\varepsilon \wedge \partial\phi) = 0.$$

The blue integral

Near $p_i \in X \underset{\text{loc}}{\subset} \mathbb{C}^N$, $\phi \approx |z|^2$ hence

$$\bar{\partial}\chi_\varepsilon \wedge \partial\phi = O(1)$$

and

$$\int_X \Delta(E,h) \wedge \bar{\partial}\chi_\varepsilon \wedge \partial\phi \longrightarrow 0$$

since $\Delta(E,h) \in L^1$ and $\mathrm{Supp}(\bar{\partial}\chi_\varepsilon) \to \{p_i\}$.

The red integral

Key proposition : integration by parts

If α is a $(0,1)$-form with compact support on $X \backslash \{p_i\}$ such that $\alpha, \bar{\partial}\alpha$ are bounded wrt to ω_θ, then

$$\int_X \Delta(E,h) \wedge \bar{\partial}\alpha = 0.$$

Apply this to $\alpha = \chi_\varepsilon \wedge \partial\phi$ to get

$$\int_X \Delta(E,h) \wedge \bar{\partial}(\chi_\varepsilon \wedge \partial\phi) = 0.$$

Some elements for the key proposition

Difficulty concentrated on the annulus ∂U.

Key ingredients :

- $(X_{\mathrm{reg}}, \omega_\theta)$ admits a Green's function with $W^{1,1+\delta}$ bounds $\rightsquigarrow$ [Guo-Phong-Song-Sturm '22] + some work
- A Harnack inequality : $\exists p \gg 1$ s.t.

$$f \geqslant 0 \quad \text{and} \quad f + (\Delta_{\omega_\theta} f)_- \in L^p(X_{\mathrm{reg}}) \Longrightarrow f \in L^\infty(X_{\mathrm{reg}}).$$

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- Fine integrability properties wrt ω_θ for high order derivatives of cut-off functions for the orbifold singularities