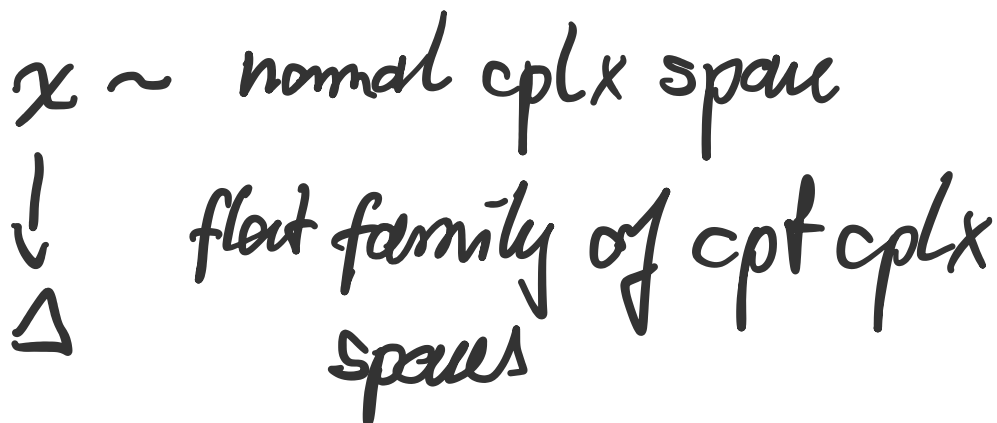


Klt degenerations of projective spaces

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Very classical problem:



s.t. for $t \neq 0$ $X_t \cong \mathbb{P}^n$

Q: What can you say about the central fibre X_0 ?

1st try: Assume X_0 smooth.

By Ehresmann's thm $X_0 \cong_{\text{diff}} \mathbb{P}^n$

Thm (Siu 1972) If X_0 smooth, then $X_0 \cong_{\text{bihol}} \mathbb{P}^n$.

Setup today:

X is a projective family so



and $X_t \cong \mathbb{P}^n$

X_0 Fano variety with klt singularities (not nec. \mathbb{Q} -factorial)

Goals:

1) Classify the central fibre X_0 , i.e. classify the singular Fano that admit a smoothing to \mathbb{P}^n

2) Give sufficient conditions s.t. $X_0 \cong \mathbb{P}^n$

Remarks / History:

• If $\dim X_0 = 2$ work towards 1) was started by Manetti (1991) and solved by Hacking-Rohrborn (2008)

Thm: If X is a klt del Pezzo surface with a smoothing to \mathbb{P}^2 , then it is a weighted proj space $\mathbb{P}(a^2, b^2, c^2)$ where (a, b, c) is a solution of the Markov eqn $a^2 + b^2 + c^2 = 3abc$.

• Thm (Fujita 1973)

In our setup assume that A can be chosen

s.t. $A|_{X_t} \cong \mathcal{O}_{\mathbb{P}^n}(1)$

Then $X_0 \cong \mathbb{P}^n$.

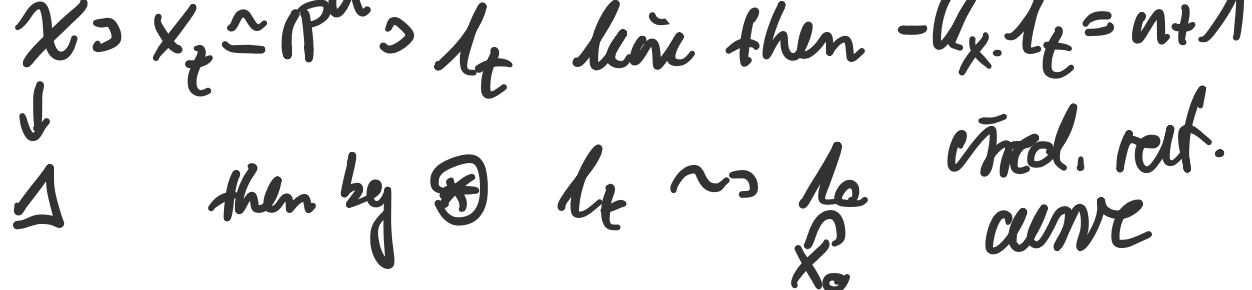
Thm (H. Nozetti 2013)

In our setup assume that for every reat. curve $C \subset X_0$ we have $-K_{X_0} \cdot C \geq \dim X_0 + 1$. Then $X_0 \cong \mathbb{P}^n$.

Recall Thm (Chen-Miyazaki-Shepherd-Barron Kebekus ~ 2002)

X_0 Fano manifold s.t. $\forall C \subset X_0$ reat. curve $-K_{X_0} \cdot C \geq \dim X_0 + 1$. Then $X_0 \cong \mathbb{P}^n$.

Crucial point for H. Nozetti: if



Ex (Pinkham's thesis 1974)

Let $\nu: \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ 2nd Veronese embedding, and $W \subset \mathbb{P}^6$ the cone over $\nu(\mathbb{P}^2)$ with vertex P .

If $H_t \cap W$ is a hyperplane section, then

$\begin{cases} H_t \cap W \cong \mathbb{P}^2 \text{ if } P \notin H_t \\ \text{singular cone } \mathcal{C} \text{ if } P \in H_t \end{cases}$

More precisely $\mathcal{C} \cong \mathbb{P}(1,1,4) = \mathbb{P}^2 + \text{contract the negative section}$

If $L_t \subset \mathbb{P}^2$ is a line it degenerates to

$L_0 = L' + L''$
 lines in the ruling of the cone.

Thm 1 (H. Peternell 2024)

In our setup if T_{X_0} is semistable, then $X_0 \cong \mathbb{P}^n$.

Thm 2 (HP 24) In our setup, assume $\dim X_0 \leq 3$

- 1) if X_0 has terminal sing, then $X_0 \cong \mathbb{P}^3$
- 2) if X_0 has canonical sing, then
 - a) $X_0 \cong \mathbb{P}^3$
 - b) $X_0 = \text{cone}$ obtained by contracting neg. section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus K_{\mathbb{P}^2})$ \mathbb{Q} quadric surface
 - c) $X_0 = \text{special variety} \subset \mathbb{P}^{34}$

Special variety $X_0 \subset \mathbb{P}^{34}$ is Gorenstein, not \mathbb{Q} -fact.

$\rho(X_0) = 1, \text{rk } \text{cl}(X_0) = 2$

$-K_{X_0} = 4H$ with H Weil divisor s.t. $h^0(X_0, H) = 4$

but $H = 3F + B$ $B = \text{fixed part of } |H|$ (F) pencil of dP surfaces.

$\exists X' \rightsquigarrow X_0$ small modification
 $\downarrow \sim$ loc. triv. fibration with fibre \mathbb{P}^1 surface of degree 4 with a D_5 -singularity
 $-K_{X'} = 4L$

Remark: Asher-de Vleming-Eckhorn have classified Gorenstein Fano threefolds with smoothing to \mathbb{P}^3 , and obtain same list.

Idea of proofs: combination of Fujita and H. Nozetti:

if $X \rightarrow \Delta$ klt degeneration of \mathbb{P}^n

$\exists H \rightarrow X$ reflexive sheaf s.t. $H|_{X_t} = \mathcal{O}_{\mathbb{P}^n}(1)$

and $H|_{X_0}$ induces Weil divisor $H_0 \rightarrow X_0$ \mathbb{Q} -Cartier

s.t. $h^0(X_0, H_0) \geq n+1$

\rightsquigarrow get: rational map $X_0 \dashrightarrow \mathbb{P}^d$

• If $X_0 = \text{Pinkham's example}$ then

$H_0 = \mathcal{O}_{\mathbb{P}(1,1,4)}(2)$

so $Bs|H_0| = \text{vertex of cone}$ and $\rho_0 = \text{projection from the vertex, so } \rho_0 \text{ contracts lines of the ruling.}$

Rep CHP)

• If the general fibre F_0 is finite, then $X_0 \cong \mathbb{P}^n$

• If it is not finite, then it contracts

