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# Klt degenerations of projective spaces

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Very classical problem:

$\mathcal{X} \sim$ normal cplx space

$\downarrow$ flat family of cpt cplx spaces

$\Delta$

s.t. for $t \neq 0$ $X_t \simeq \mathbb{P}^n$

Q: What can you say about the central fibre $X_0$?

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1st try: Assume $X_0$ smooth.

By Ehresmann's thm $X_0 \overset{\text{diffeo}}{\simeq} \mathbb{P}^n$

Thm (Siu 1992) If $X_0$ smooth, then $X_0 \overset{\text{biholom}}{\simeq} \mathbb{P}^n$.

Setup today:

$\mathcal{X} \to \Delta$ is a projective family so $\exists A \to \mathcal{X}$ rel. ample line bundle

and $X_t \simeq \mathbb{P}^n$

$X_0$ Fano variety with klt singularities (not nec. $\mathbb{Q}$-factorial)

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Goals:

1) Classify the central fibre $X_0$, i.e. classify the singular Fanos that admit a smoothing to $\mathbb{P}^n$

2) Give sufficient conditions s.t. $X_0 \simeq \mathbb{P}^n$

Remarks / History:

- If $\dim X_0 = 2$ work towards 1) was started by Manetti (1991) and solved by Hacking-Prokhorov (2008)

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Thm: (Hacking-Prokhorov) If $X$ is a klt del Pezzo surface with a smoothing to $\mathbb{P}^2$, then it is a weighted proj space $\mathbb{P}(a^2, b^2, c^2)$ where $(a,b,c)$ is a solution of the Markov eqn $a^2+b^2+c^2 = 3abc$.

- Thm (Fujita 1973) In our setup assume that $A$ can be chosen s.t. $A|_{X_t} \simeq \mathcal{O}_{\mathbb{P}^n}(1)$. Then $X_0 \simeq \mathbb{P}^n$.

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Thm (H-Novelli 2013) In our setup assume that for every rat. curve $C \subset X_0$ we have $-K_{X_0} \cdot C \geq \dim X_0 + 1$ (*)

Then $X_0 \simeq \mathbb{P}^n$.

Recall Thm (Cho-Miyaoka-Shepherd Barron, Kebekus ~2002)

$X_0$ Fano manifold s.t. $\forall C \subset X_0$ rat. curve $-K_{X_0} \cdot C \geq \dim X + 1$

Then $X_0 \simeq \mathbb{P}^n$.

Crucial point for H-Novelli: if $\mathcal{X} \supset X_t \simeq \mathbb{P}^n \supset l_t$ line then $-K_{\mathcal{X}} \cdot l_t = n+1$

$\downarrow$

$\Delta$ then by (*) $l_t \rightsquigarrow \underset{X_0}{l_0}$ irred. rat. curve

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Ex (Pinkham's thesis 1974)

Let $\nu: \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ 2nd Veronese embedding, and $\mathcal{W} \subset \mathbb{P}^6$ the cone over $\nu(\mathbb{P}^2)$ with vertex $P$.

If $H_t \cap \mathcal{W}$ is a hyperplane section, then

$\begin{cases} H_t \cap \mathcal{W} \simeq \mathbb{P}^2 & \text{if } P \notin H_t \\ \text{singular cone } \mathcal{C} & \text{if } P \in H_t \end{cases}$

More precisely $\mathcal{C} \simeq \mathbb{P}(1,1,4) = \mathbb{F}_4$ + contract the negative section

If $l_t \subset \mathbb{P}^2$ is a line it degenerates to $l_0 = l' + l''$ lines in the ruling of the cone.

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Thm 1 (H, Peternell 2024) In our setup if $T_{X_0}$ is semistable, then $X_0 \simeq \mathbb{P}^n$.

Thm 2 (HP '24) In our setup, assume $\dim X_0 = 3$

1) If $X_0$ has terminal sings, then $X_0 \simeq \mathbb{P}^3$

2) If $X_0$ has canonical sings, then

a) $X_0 \simeq \mathbb{P}^3$

b) $X_0$ = cone obtained by contracting neg. section of $\mathbb{P}(\mathcal{O}_Q \oplus K_Q)$ $Q$ quadric surface

c) $X_0$ = special variety $\subset \mathbb{P}^{34}$

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Special variety $X_0 \subset \mathbb{P}^{34}$ is Gorenstein, not $\mathbb{Q}$-fact.

$\rho(X_0) = 1$, rk $Cl(X_0) = 2$

$-K_{X_0} = 4H$ with $H$ Weil divisor s.t. $h^0(X_0, H) = 4$

but $H = 3F + B$ $B$ = fixed part of $|H|$

$|F|$ pencil of dP surfaces.

$\exists X' \to X_0$ small modification

$\downarrow$ ~ loc. triv. fibration with fibre $F$ dP surface of degree 4 with a $D_5$-singularity

$\mathbb{P}^1$

$-K_F = 4l$

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Remark: Asher-de Vleming-Yuchen have classified Gorenstein Fano threefolds with smoothing to $\mathbb{P}^3$, and obtain same list.

Idea of proofs: combination of Fujita and H-Novelli:

if $\mathcal{X} \to \Delta$ klt degeneration of $\mathbb{P}^n$

$\exists \mathcal{H} \to \mathcal{X}$ reflexive sheaf s.t. $\mathcal{H}|_{X_t} = \mathcal{O}_{\mathbb{P}^n}(1)$

and $\mathcal{H}|_{X_0}$ induces Weil divisor $H_0 \to X_0$ $\mathbb{Q}$-Cartier s.t. $h^0(X_0, H_0) \geq n+1$

$\rightsquigarrow$ get: rational map $\varphi_0: X_0 \dashrightarrow \mathbb{P}^d$

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- If $X_0$ = Pinkham's example then $H_0 = \mathcal{O}_{\mathbb{P}(1,1,4)}(2)$

so $\mathrm{Bs}|H_0|$ = vertex of cone

and $\varphi_0$ = projection from the vertex, so $\varphi_0$ contracts lines of the ruling.

Prop (HP)

- If the general fibre of $\varphi_0$ is finite, then $X_0 \simeq \mathbb{P}^n$
- If it is not finite, then it contracts the degenerations of lines $\underset{\mathbb{P}^n}{l_t} \rightsquigarrow \underset{X_0}{l_0}$