Ball quotients and moduli spaces

Joint work with S. Casalaina-Martin, S. Grushevsky, S. Kondō, R. Laza, Y.Maeda

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0. Introduction

Many moduli problems can be viewed from two perspectives, namely

- As a GIT quotient space
- Via Hodge theory.

The interplay between these two perspectives is often very interesting and can be subtle.

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This talk is primarily concerned with moduli spaces $\mathcal{M}$ which are naturally isomorphic (via Hodge theory) to an open subset of a ball quotient

$$\mathcal{M} \hookrightarrow \mathbb{B}^n/\Gamma$$

Examples are:

- Deligne-Mostow varieties (moduli of weighted sets of points in $\mathbb{P}^1$)
- Certain moduli spaces of $K3$ surfaces (with automorphisms)
- Non-hyperelliptic curves of genus 4
- Moduli space of cubic surfaces
- Moduli space of cubic threefolds.

Many of these spaces are related to each other (e.g. as ball sub-quotients).

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From both points of view one has natural compactifications, namely

- The GIT quotient

$$\mathcal{M} \hookrightarrow \mathcal{M}^{\mathrm{GIT}}$$

- The Baily-Borel compactification

$$\mathcal{M} \subset \mathbb{B}^n/\Gamma \hookrightarrow \overline{\mathbb{B}^n/\Gamma}^{\mathrm{BB}}.$$

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- The GIT quotient
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- The Baily-Borel compactification
$$\mathcal{M} \subset \mathbb{B}^n/\Gamma \hookrightarrow \overline{\mathbb{B}^n/\Gamma}^{\mathrm{BB}}.$$

These inclusions sometimes (but not always) extend to an isomorphism
$$\phi : \mathcal{M}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^n/\Gamma}^{\mathrm{BB}}.$$

Typically, these spaces can be quite singular. For GIT quotients Kirwan has introduced the *Kirwan blow-up* $\mathcal{M}^K$ as a partial resolution and for ball quotients one can take the *toroidal compactification* $\overline{\mathbb{B}^n/\Gamma}^{\mathrm{tor}}$ (which is canonical in this case).

We thus have a diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{K}} & \overset{f}{\dashrightarrow} & \overline{\mathbb{B}^n/\Gamma}^{\mathrm{tor}} \\ \Big\downarrow p & & \Big\downarrow \pi \\ \mathcal{M}^{\mathrm{GIT}} & \underset{\phi}{\overset{\cong}{\longrightarrow}} & \overline{\mathbb{B}^n/\Gamma}^{\mathrm{BB}} \end{array}$$

where f is a birational map.

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We thus have a diagram

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where f is a birational map.

Question

How are the partial resolutions $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^n/\Gamma}^{\mathrm{tor}}$ related? In particular:

- How does the topology of $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^n/\Gamma}^{\mathrm{tor}}$ compare?
- Is f an isomorphism?
- Are the spaces $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^n/\Gamma}^{\mathrm{tor}}$ K-equivalent?
- Are they derived equivalent?

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Example 1: Cubic surfaces

Let

$$\mathcal{M}_{\rm surf} = \text{Moduli space of cubic surfaces.}$$

By a result of Allcock, Carlson and Toledo this also has a ball quotient model. Let

$$\mathcal{E} \subset \mathbb{Q}(\sqrt{-3}).$$

be the ring of *Eisenstein integers*. We equip the lattice

$$\Lambda_{\rm surf} := 4\mathcal{E} + \mathcal{E}(-1)$$

with the standard hermitian form with signature $(4,1)$.

$$\Gamma_{\rm surf} := \mathrm{U}(\Lambda_{\rm surf})$$

be the integral unitary group. This group acts on the 4-dimensional ball

$$\mathbb{B}^4 = \{z \in \mathbb{P}(\Lambda_{\rm surf} \otimes \mathbb{C}) \mid |z_0|^2 + \dots |z_3|^2 > |z_4|^2\}.$$

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The result of Allcock, Carlson and Toledo then says that there is an open embedding

$$\mathcal{M}_{\mathrm{surf}} \hookrightarrow \mathbb{B}^4/\Gamma_{\mathrm{surf}}.$$

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The result of Allcock, Carlson and Toledo then says that there is an open embedding

$$\mathcal{M}_{\mathrm{surf}} \hookrightarrow \mathbb{B}^4/\Gamma_{\mathrm{surf}}.$$

The difference

$$(\mathbb{B}^4/\Gamma_{\mathrm{surf}})\backslash\mathcal{M}_{\mathrm{surf}} = D_{A_1}$$

is the *discriminant* consisting of cubic surfaces with at most A_1-singularies. Moreover, this inclusion extends to an isomorphism

$$\mathcal{M}_{\mathrm{surf}}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^4/\Gamma_{\mathrm{surf}}}^{\mathrm{BB}}.$$

The Baily-Borel compactfication $\overline{\mathbb{B}^4/\Gamma_{\mathrm{surf}}}^{\mathrm{BB}}$ has a unique cusp and under the above isomorphism this corresponds to the unique polystable point, namely the $3A_2$-cubic:

$$S = \{x_0^3 + x_1x_2x_3 = 0\}.$$

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Example 2: Cubic threefolds

$\mathcal{M}_{\text{3fold}}$ = Moduli space of cubic threefolds.

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Example 2: Cubic threefolds

$$\mathcal{M}_{\mathrm{3fold}} = \text{Moduli space of cubic threefolds.}$$

Allcock, Carlson and Toledo constructed a ball quotient model:

$$\Lambda_{\mathrm{3fold}} := \mathcal{E}_1 + 2\mathcal{E}_4 + \mathcal{H}.$$

These are hermitian lattices with underlying $\mathbb{Z}$-lattices

$$\Lambda_{\mathrm{3fold},\mathbb{Z}} = A_2(-1) + 2E_8(-1) + 2U.$$

The group

$$\Gamma_{\mathrm{3fold}} := \mathrm{U}(\Lambda_{\mathrm{3fold}})$$

acts on the 10-dimensional ball $\mathbb{B}^{10} \subset \mathbb{P}(\Lambda_{\mathrm{3fold}} \otimes \mathbb{C})$ and

$$\mathcal{M}_{\mathrm{3fold}} \hookrightarrow \mathbb{B}^{10}/\Gamma_{\mathrm{3fold}}.$$

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$$\mathcal{M}_{\text{3fold}} \hookrightarrow \mathbb{B}^{10}/\Gamma_{\text{3fold}}.$$

The discriminant consists of cubic threefolds with singularities of type $A_i, i \leqslant 5, D_4$ and chordal cubic (i.e. the secant variety of a rational normal curve of degree 5 on $\mathbb{P}^4$).

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Here the relation between the GIT compactification and the Baily-Borel compactification is more subtle (CM-L):

$$\begin{array}{ccc} & \mathcal{M}^{\mathrm{K}}_{\mathrm{3fold}} & \\ & \downarrow & \\ & \widehat{\mathcal{M}}_{\mathrm{3fold}} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathcal{M}^{\mathrm{GIT}}_{\mathrm{3fold}} & \dashrightarrow & \overline{\mathbb{B}^{10}/\Gamma}^{\mathrm{BB}} \end{array}$$

Here we have

- $\widehat{\mathcal{M}}_{\mathrm{3fold}}$ is the partial Kirwan blow-up in the chordal cubic
- $\mathcal{M}^{\mathrm{K}}_{\mathrm{3fold}}$ is the full Kirwan blow-up
- q is a small semi-toric modification contracting a $\mathbb{P}^1$ whose image c_{2A_5} is one of the two cusps of $\overline{\mathbb{B}^{10}/\Gamma_{\mathrm{3fold}}}^{\mathrm{BB}}$
- The other cusp c_{3D_4} corresponds to the $3D_4$-cubic threefold.

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Example 3: 8 points in $\mathbb{P}^1$

Let

$$\mathcal{M}_8 = \text{Moduli space of 8 (unordered) points in } \mathbb{P}^1.$$

This is also the moduli space of hyperelliptic curves of genus 3. It is related to the *Gaussian integers*

$$\mathscr{G} = \mathbb{Z}[i] \subset \mathbb{Q}(\sqrt{-1})$$

Using this one can define a hermitian lattice

$$\Lambda_8 = \mathscr{G}_1 + 2\mathscr{G}_2$$

of rank 6 and signature $(1, 5)$ with underlying integral lattice

$$\Lambda_{8,\mathbb{Z}} = U + U(2) + 2D_4(-1).$$

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One has an open inclusion

$$\mathcal{M}_8 \hookrightarrow \mathbb{B}^5/\Gamma_8.$$

where

$$\Gamma_8 = \mathrm{U}(\Lambda_8).$$

This extends to an isomorphism

$$\mathcal{M}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^5/\Gamma_8}^{\mathrm{BB}}.$$

The discriminant $D = (\mathbb{B}^5/\Gamma_8) \backslash \mathcal{M}$ parameterizes non-reduced 8-tuples with at most 3 points coming together. The cusp corresponds to the unique properly polystable point with multiplicity $(4, 4)$.

- This is the maximal Deligne-Mostow variety based on the Gaussian integers (an *ancestral space*).

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Example 3: 12 points in $\mathbb{P}^1$

Let

$$\mathcal{M}_{12} = \text{Moduli space of 12 (unordered) points in } \mathbb{P}^1.$$

and consider the lattice

$$\Lambda_{12} = 2\mathscr{E}_4 + \mathscr{H}$$

whose underlying integral lattice is $\Lambda_{12,\mathbb{Z}} = 2E_8(-1) + 2U$. Then

$$\mathcal{M}_{12} \hookrightarrow \mathbb{B}^9/\Gamma_{12}$$

where $\mathbb{B}^9 \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ and $\Gamma_{12} = \mathrm{U}(\Lambda_{12})$. Again, this inclusion extends to an isomorphism

$$\mathcal{M}_{12}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{BB}}.$$

- This is the maximal Deligne-Mostow variety (*ancestral variety)* based on the Eisenstein integer.

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Example 4: genus 4 curves

The moduli space

$$\mathcal{M}_4^{\mathrm{nh}} = \text{Moduli space of non hyperelliptic curves of genus 4.}$$

also has a 9-dimensional ball quotient model

$$\mathcal{M}_4^{\mathrm{nh}} \hookrightarrow \mathbb{B}^9/\Gamma_4^{\mathrm{nh}}.$$

- By Kondō's work the two ball quotients $\mathbb{B}^9/\Gamma_{12}$ and $\mathbb{B}^9/\Gamma_4^{\mathrm{nh}}$ are commensurable.
- Both $\mathbb{B}^9/\Gamma_{12}$ and $\mathbb{B}^9/\Gamma_4^{\mathrm{nh}}$ appear as sub-ball quotients in the ball quotient model of the moduli space of cubic threefolds as the *hyperelliptic locus* and the *nodal divisor* respectively.
- The moduli space of elliptically fibered $K3$ surfaces with a non-symplectic automorphism of order 3 is also an open subset of $\mathbb{B}^9/\Gamma_4^{\mathrm{nh}}$.

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Level cover 1: cubic surfaces

Many of the moduli spaces discussed above have natural level covers. In the case of cubic surfaces this is

$$\mathcal{M}_{\mathrm{surf}}^{\mathrm{m}} = \text{Moduli space of marked cubic surfaces.}$$

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Level cover 1: cubic surfaces

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$$\mathcal{M}_{\mathrm{surf}}^{\mathrm{m}} = \text{Moduli space of marked cubic surfaces.}$$

This is also an open subset of a ball quotient

$$\mathcal{M}_{\mathrm{surf}}^{\mathrm{m}} \hookrightarrow \mathbb{B}^4/\Gamma_{\mathrm{surf}}^{\mathrm{m}}.$$

Here $\Gamma_{\mathrm{surf}}^{\mathrm{m}} \lhd \Gamma_{\mathrm{surf}}$ is the stable unitary group and

$$\Gamma_{\mathrm{surf}}^{\mathrm{m}}/\Gamma_{\mathrm{surf}}^{\mathrm{m}} \cong W(E_6) \times \{\pm 1\}.$$

There is a natural (smooth) compactification, namely the *Naruki compactification* $\overline{\mathcal{N}}$ and one can ask how this is related to the toroidal compactification.

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This was answered by

Theorem (Gallardo, Kerr, Schaffler (2021))

The Naruki compactification $\overline{\mathcal{N}}$ is isomorphic to the toroidal compactification of the ball quotient. More precisely, there is a $W(E_6)$-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{m}}_{\mathrm{surf}} & \hookrightarrow & \mathbb{B}^4/\Gamma^{\mathrm{m}}_{\mathrm{surf}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{N}} & \xrightarrow{\sim} & \overline{\mathbb{B}^4/\Gamma^{\mathrm{m}}_{\mathrm{surf}}}^{\mathrm{tor}} \end{array}$$

Remark

They also proved that $\overline{\mathcal{N}}$ is isomorphic to the KSBA compactification where the lines are given weight $1/9 + \varepsilon$.

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Level cover 2: 8 points in $\mathbb{P}^1$

Here we have a natural cover given by considering *ordered tuples* of points.

$$\mathcal{M}_{8,\mathrm{ord}} = \text{Moduli space of 8 ordered points in } \mathbb{P}^1.$$

Clearly we have an S_8-cover to the moduli space of unordered tuples.

$$\mathcal{M}_{8,\mathrm{ord}} \to \mathcal{M}_8 = \mathcal{M}_{8,\mathrm{ord}}/S_8.$$

The cover $\mathcal{M}_{8,\mathrm{ord}}$ is also a GIT moduli space – with $\mathrm{SL}(2,\mathbb{Z})$ acting on $(\mathbb{P}^1)^8$ – and thus we have a GIT compactification $\mathcal{M}_{8,\mathrm{ord}}^{\mathrm{GIT}}$ as well as a Kirwan blow-up $\mathcal{M}_{8,\mathrm{ord}}^{\mathrm{K}}$.

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This moduli space is also an open set of a ball quotient:

$$\mathcal{M}_{8,\mathrm{ord}} \hookrightarrow \mathbb{B}^5/\Gamma_{8,\mathrm{ord}}$$

where $\Gamma_{8,\mathrm{ord}} \lhd \Gamma_8$ is the *stable* unitary group, i.e., the group acting trivially on the discriminant. We have $\Gamma_8/\Gamma_{8,\mathrm{ord}} \cong S_8$. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{8,\mathrm{ord}}^{\mathrm{GIT}} & \xrightarrow[\phi_{\mathrm{ord}}]{\cong} & \overline{\mathbb{B}^5/\Gamma_{8,\mathrm{ord}}}^{\mathrm{BB}} \\ \downarrow & & \downarrow \\ \mathcal{M}_8^{\mathrm{GIT}} & \xrightarrow[\phi]{\cong} & \overline{\mathbb{B}^5/\Gamma_8}^{\mathrm{BB}} \end{array}.$$

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As in the unordered case one can consider the Kirwan blow-up as well as the toroidal compactification.

Theorem (Gallardo-Kerr-Schaffler (2021))

There is a natural S_8-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{8,\mathrm{ord}}^{\mathrm{K}} & \xrightarrow{\cong} & \overline{\mathbb{B}^5/\Gamma_{8,\mathrm{ord}}}^{\mathrm{tor}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{8,\mathrm{ord}}^{\mathrm{GIT}} & \xrightarrow[\phi]{\cong} & \overline{\mathbb{B}^5/\Gamma_{8,\mathrm{ord}}}^{\mathrm{BB}} \end{array}$$

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As in the unordered case one can consider the Kirwan blow-up as well as the toroidal compactification.

Theorem (Gallardo-Kerr-Schaffler (2021))

There is a natural S_8-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{8,\mathrm{ord}}^{\mathrm{K}} & \xrightarrow{\cong} & \overline{\mathbb{B}^5/\Gamma_{8,\mathrm{ord}}}^{\mathrm{tor}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{8,\mathrm{ord}}^{\mathrm{GIT}} & \xrightarrow[\phi]{\cong} & \overline{\mathbb{B}^5/\Gamma_{8,\mathrm{ord}}}^{\mathrm{BB}} \end{array}$$

However, note that

Remark

This does not imply that we have a corresponding isomorphism in the unordered case

$$f : \mathcal{M}_8^{\mathrm{K}} \dashrightarrow \overline{\mathbb{B}^5/\Gamma_8}^{\mathrm{tor}}.$$

Level cover 3: 12 points in $\mathbb{P}^1$

Here we have again a natural cover given by considering *ordered tuples* of points.

$$\mathcal{M}_{12,\mathrm{ord}} = \text{Moduli space of 12 ordered points in } \mathbb{P}^1.$$

Question

Does $\mathcal{M}_{12,\mathrm{ord}}$ have a ball quotient model?

This case does not appear in the Deligne-Mostow list, but it could be a ball quotient via a different period map.

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Moduli spaces of n-pointed curves

Classically, one considers

$$\mathcal{M}_{g,n,\mathrm{ord}} = \text{Moduli space of ordered } n\text{-pointed genus } g \text{ curves}$$

and its Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n,\mathrm{ord}}$ of stable curves.

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and its Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n,\mathrm{ord}}$ of stable curves.

There are also a series of other compactifications due to Hassett

$$\mathcal{M}_{g,\mathcal{A},\mathrm{ord}} = \text{Hassett space of ordered weighted } n\text{-pointed genus.}$$

Here $\mathcal{A}$ is an n-tuple of *weights*

$$\mathcal{A} = (a_1, \dots, a_n) \in \mathbb{Q}^n,\ 0 < a_i \leqslant 1.$$

This has a natural compactification $\overline{\mathcal{M}}_{g,\mathcal{A},\mathrm{ord}}$.

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The Hassett moduli spaces represent the functor given by families $\pi : \mathcal{C} \to S$ of stable curves with n sections

$$s_1, \dots s_n : S \to \mathcal{C}$$

with the following properties:

(1) $2g - 2 + a_1 + \dots + a_n > 0$

(2) If the sections $s_{i_1}, \dots, s_{i_r}$ are not disjoint, then $a_{i_1} + \dots + a_{i_r} \leqslant 1$

(3) $K_\pi + a_1 s_1 + \dots a_n s_n$ is π-ample.

The Hassett spaces are of particular interest in connection with the minimal model program for $\overline{\mathcal{M}}_{g,n,\mathrm{ord}}$.

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We shall now specialize to genus $g = 0$ again and, to simplify notation by dropping the genus from it:

$$\mathcal{M}_{g,n,\mathrm{ord}} \rightsquigarrow \mathcal{M}_{n,\mathrm{ord}}, \quad \mathcal{M}_{g,\mathcal{A},\mathrm{ord}} \rightsquigarrow \mathcal{M}_{\mathcal{A},\mathrm{ord}}.$$

We shall now specialize to genus $g = 0$ again and, to simplify notation by dropping the genus from it:

$$\mathcal{M}_{g,n,\mathrm{ord}} \rightsquigarrow \mathcal{M}_{n,\mathrm{ord}}, \quad \mathcal{M}_{g,\mathcal{A},\mathrm{ord}} \rightsquigarrow \mathcal{M}_{\mathcal{A},\mathrm{ord}}.$$

Kiem and Moon have studied log canonical models of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{n,\mathrm{ord}}$ and the following sequence of maps

$$\overline{\mathcal{M}}_{n,\mathrm{ord}} \to \overline{\mathcal{M}}_{\mathcal{A}^{\lfloor n/2\rfloor-3}_{n,\mathrm{ord}}} \to \ldots \to \overline{\mathcal{M}}_{\mathcal{A}^{1}_{n,\mathrm{ord}}} \to \mathcal{M}^{\mathrm{GIT}}_{n,\mathrm{ord}}.$$

where the Hassett weights are given by the n-tuple

$$\mathcal{A}^i_n = \left(\frac{1}{\lfloor n/2\rfloor + 1 - i} + \varepsilon\right)^n.$$

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Theorem (Kiem-Moon)

If n is even, then

$$\overline{\mathcal{M}}_{\mathcal{A}^1_n,\mathrm{ord}} \cong \mathcal{M}^{\mathrm{K}}_{n,\mathrm{ord}} \to \mathcal{M}^{\mathrm{GIT}}_{n,\mathrm{ord}}$$

is the Kirwan blow-up at the $\frac{1}{2}\binom{n}{n/2}$ *cusps corresponding to the polystable points.*

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Theorem (Kiem-Moon)

If n is even, then

$$\overline{\mathcal{M}}_{\mathcal{A}_n^1,\mathrm{ord}} \cong \mathcal{M}^{\mathrm{K}}_{n,\mathrm{ord}} \to \mathcal{M}^{\mathrm{GIT}}_{n,\mathrm{ord}}$$

is the Kirwan blow-up at the $\frac{1}{2}\binom{n}{n/2}$ *cusps corresponding to the polystable points.*

One can also consider the unordered case (by dividing by S_n):

$$\overline{\mathcal{M}}_n \to \overline{\mathcal{M}}_{\mathcal{A}_n^{\lfloor n/2 \rfloor -3}} \to \ldots \to \overline{\mathcal{M}}_{\mathcal{A}_n^1} \to \mathcal{M}_n^{\mathrm{GIT}}.$$

leading to the natural question whether this also holds n the unordered case:

Question (A)

Is it correct that

$$\overline{\mathcal{M}}_{\mathcal{A}_n^1} \cong \mathcal{M}_n^{\mathrm{K}} \to \mathcal{M}_n^{\mathrm{GIT}}?$$

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Deligne-Mostow varieties

Question

Which configurations of weighted points in $\mathbb{P}^1$ can be parameterized by (open subsets of) ball quotients (via the period map of hypergeometric functions)?

Here we are only interested in the cases where

- The group is arithmetic
- The moduli space is non-compact.

According to Mostow and Thurston this happens in 42 cases.

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More precisely we are looking at

$$\mathcal{O} = \text{ring of integers in } \mathbb{Q}(\sqrt{-d}), d > 0$$

$$\Lambda = \mathcal{O}^{n+1},$$

which is equipped with a hermitian form of signature $(n, 1)$ and

$$\mathbb{B}^n = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (x, \overline{x}) > 0\}$$

which is acted on by an arithmetic group

$$\Gamma \subset \mathrm{U}(\Lambda).$$

Remark

The only rings of integers which appear in the Deligne-Mostow list are $\mathcal{O} = \mathcal{G}$ (Gaussian integers, 13 cases) and $\mathcal{O} = \mathcal{E}$ (Eisenstein integers, 29 cases).

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Deligne-Mostow varieties come in two flavours, depending on certain numerical conditions on the weights $\underline{w} = (w_1, \ldots, w_n)$:

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Deligne-Mostow varieties come in two flavours, depending on certain numerical conditions on the weights $\underline{w} = (w_1, \dots, w_n)$:

- Condition INT: The moduli space of n *ordered* weighted points has a ball quotient model:

$$\mathcal{M}_{\underline{w},\mathrm{ord}} \subset \mathbb{B}^{n-3}/\Gamma_{\underline{w}}.$$

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Deligne-Mostow varieties come in two flavours, depending on certain numerical conditions on the weights $\underline{w} = (w_1, \ldots, w_n)$:

- Condition INT: The moduli space of n *ordered* weighted points has a ball quotient model:

$$\mathcal{M}_{\underline{w},\mathrm{ord}} \subset \mathbb{B}^{n-3}/\Gamma_{\underline{w}}.$$

- Condition $\Sigma\mathrm{INT}$: A non-trivial quotient of the moduli space of n weighted points has a ball quotient model

$$\mathcal{M}_{\underline{w},\Sigma} \subset \mathbb{B}^{n-3}/\Gamma^{\Sigma}_{\underline{w}}$$

  where $\Gamma^{\Sigma}_{\underline{w}} = (S[\underline{w}] \times \Gamma_{\underline{w}})$ and $S[\underline{w}] \subset S_n$ is a subgroup defined by the weights $\underline{w}$.

Condition INT implies condition $\Sigma\mathrm{INT}$.

Example

- $n = 8$, $\underline{w} = (1/4)^8$. Here INT holds:

$$\mathcal{M}_{\underline{w},\mathrm{ord}} = \mathcal{M}_{8,\mathrm{ord}} \subset \mathbb{B}^5/\Gamma_{\underline{w}} = \mathbb{B}^5/\Gamma_{8,\mathrm{ord}}.$$

This is the Gaussian ancestral space ($\mathcal{O} = \mathcal{G}$).
$[(1 - (1/4 + 1/4))^{-1} = (1/2)^{-1} = 2 \in \mathbb{Z}]$

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$$\mathcal{M}_{\underline{w},\mathrm{ord}} = \mathcal{M}_{8,\mathrm{ord}} \subset \mathbb{B}^5/\Gamma_{\underline{w}} = \mathbb{B}^5/\Gamma_{8,\mathrm{ord}}.$$

  This is the Gaussian ancestral space ($\mathcal{O} = \mathcal{G}$).
  $[(1-(1/4+1/4))^{-1} = (1/2)^{-1} = 2 \in \mathbb{Z}]$

- $n = 12$, $\underline{w} = (1/6)^{12}$. Hier $\Sigma\mathrm{INT}$ holds, $S[\underline{w}] = S_{12}$:

$$\mathcal{M}_{\underline{w},\Sigma} = \mathcal{M}_{12} \subset \mathbb{B}^9/\Gamma^{\Sigma}_{\underline{w}} = \mathbb{B}^9/\Gamma_{12}.$$

  This is the Eisenstein ancestral space ($\mathcal{O} = \mathcal{E}$).
  $[(1-(1/6+1/6))^{-1} = (2/3)^{-1} = 3/2 \in 1/2\mathbb{Z}]$

Example

- $n = 8$, $\underline{w} = (1/4)^8$. Here INT holds:

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- There are also other cases where not all weights are even one one divides by a product of symmetric groups.

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Compairing compactifications

For the Deligne-Mostow varieties one has isomorphisms

$$\mathcal{M}^{\mathrm{GIT}}_{\underline{w},\mathrm{ord}} = (\mathbb{P}^1)^n /\!/_{\underline{w}} \mathrm{SL}_2(\mathbb{C}) \cong \overline{\mathbb{B}^{n-3}/\Gamma_{\underline{w}}}^{\mathrm{BB}} \quad (\mathrm{INT})$$

and

$$\mathcal{M}^{\mathrm{GIT}}_{\underline{w},\Sigma} = (\mathbb{P}^1)^n /\!/_{\underline{w}} (S[\underline{w}] \times \mathrm{SL}_2(\mathbb{C})) \cong \overline{\mathbb{B}^{n-3}/\Gamma^{\Sigma}_{\underline{w}}}^{\mathrm{BB}} (\Sigma\mathrm{INT}).$$

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At this point the question arises how this compares to the Hassett spaces.

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Theorem (Gallardo–Kerr–Scheffler 21)

Assume that condition $\Sigma\mathrm{INT}$ *holds. Then*

$$\overline{\mathcal{M}}_{\mathcal{A}^1_{n,\mathrm{ord}}}/S[\underline{w}] \cong \overline{\mathbb{B}^{n-3}/\Gamma^{\Sigma}_{\underline{w}}}^{\mathrm{tor}}$$

where the right hand side is the unique toroidal blow-up.

Here we recall that

$$\mathcal{A}^1_n = (\frac{1}{\lfloor n/2 \rfloor} + \varepsilon)^n$$

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Here we recall that

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Now assume that INT holds and that n is even. Then comparing this with the result of Kiem and Moon we obtain:

$$\mathcal{M}^{\mathrm{K}}_{n,\mathrm{ord}} \cong \overline{\mathcal{M}}_{\mathcal{A}^1_{n,\mathrm{ord}}} \cong \overline{\mathbb{B}^{n-3}/\Gamma_{\underline{w}}}^{\mathrm{tor}}.$$

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Question

In how far does this extend to the $\Sigma\mathrm{INT}$ case?

In particular, we have:

$$\mathcal{M}^{\mathrm{K}}_{8,\mathrm{ord}} \cong \overline{\mathbb{B}^5/\Gamma_{8,\mathrm{ord}}}^{\mathrm{tor}}.$$

Question (B)

Is it true that also

$$\mathcal{M}^{\mathrm{K}}_{8} \cong \overline{\mathbb{B}^5/\Gamma_{8}}^{\mathrm{tor}}?$$

Or in the case of 12 points where $\Sigma\mathrm{INT}$ holds, but not INT:

$$\mathcal{M}^{\mathrm{K}}_{12} \cong \overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}?$$

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Results

The overall question is

Question

We know that in the level case (ordered case, marked cubic surfaces) the Kirwan compactification and the toroidal compactification agree. Does this still hold without a level structure?

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Results

The overall question is

Question

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Here we shall discuss this exemplary for the case of 12 points (joint work with Maeda, Kondō).

Theorem

The Betti numbers

$$b_i(\mathcal{M}_{12}^{\mathrm{K}}) = b_i(\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}), i \geqslant 0.$$

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We shall now investigate how different $\mathcal{M}_{12}^{\mathrm{K}}$ and $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ are as varieties.

Theorem

Neither the rational map

$$f : \mathcal{M}_{12}^{\mathrm{K}} \dashrightarrow \overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$$

nor its inverse f^{-1} extend to a morphism.

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Theorem

Neither the rational map

$$f : \mathcal{M}_{12}^{\mathrm{K}} \dashrightarrow \overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$$

nor its inverse f^{-1} extend to a morphism.

Proof. We first remark that f cannot be an isomorphism:

- The intersection of the discriminant and the Kirwan exceptional divisor is generically not transversal.
- The intersection of the discriminant and the toroidal exceptional divisor is generically transversal.

The first of these claims follows from a Luna slice computation.

If f were a morphism it must be a (small) contraction (since both spaces are normal). But this contradicts the fact that both $\mathcal{M}_{12}^{\mathrm{K}}$ and $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ are $\mathbb{Q}$-factorial.

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One can still ask the

Question

- Are the varieties $\mathcal{M}_{12}^{\mathrm{K}}$ and $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ abstractly isomorphic?
- Are the varieties $\mathcal{M}_{12}^{\mathrm{K}}$ and $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ K-equivalent?

Two projective normal $\mathbb{Q}$-Gorenstein varieties X and Y are called *K-equivalent* if there is a common resolution of singularities Z dominating X and Y birationally

$$\begin{array}{ccc} & Z & \\ {\scriptstyle f_X} \swarrow & & \searrow {\scriptstyle f_Y} \\ X & \leftarrow - - - - - - \rightarrow & Y \end{array}$$

such that $f_X^* K_X \sim_{\mathbb{Q}} f_Y^* K_Y$.

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Theorem

The varieties $\mathcal{M}_{12}^{\mathrm{K}}$ *and* $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ *are not K-equivalent and hence not isomorphic (even as abstract varieties).*

Corollary

The questions (A) and (B) have a negative answer:

$$\mathcal{M}_{12}^{\mathrm{K}} \not\cong \overline{\mathcal{M}}_{\mathcal{A}_{12}^1} \cong \overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}.$$

Remark

- Here the isomorphism comes from the result of Gallardo–Kerr–Scheffler.
- This is in contrast to the ordered case, where

$$\mathcal{M}_{12,\mathrm{ord}}^{\mathrm{K}} \cong \overline{\mathcal{M}}_{\mathcal{A}_{12}^1,\mathrm{ord}} \cong \overline{\mathbb{B}^9/\Gamma_{12,\mathrm{ord}}}^{\mathrm{tor}}.$$

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Idea of proof. It is enough to prove that

$$(K_{\mathcal{M}_{12}^{\mathrm{K}}})^9 \neq (K_{\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}})^9.$$

Using modular forms, a Luna slice computation and geometric arguments one shows that

$$K_{\mathcal{M}_{12}^{\mathrm{K}}} = -210\mathscr{L} - 9\Delta, \quad K_{\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}} = -210\mathscr{L} - 16T$$

where $\mathscr{L}$ is the Hodge line bundle, Δ is the Kirwan exceptional divisor and T is the toric boundary. It then suffices top prove that

$$(9\Delta)^9 \neq (16T)^9.$$

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Looijenga introduced the concept of *semi-toric compactifications* which generalizes toric compactifications. Recently, Alexeev and Engel characterized these as those compactifications which lie between toric compactifications and the Baily-Borel compactification, which in the case of ball quotients means

$$\overline{\mathbb{B}^n/\Gamma}^{\mathrm{tor}} \to \overline{\mathbb{B}^n/\Gamma}^{\mathrm{semitor}} \to \overline{\mathbb{B}^n/\Gamma}^{\mathrm{BB}}.$$

Oda has characterized these in terms of LMMP.

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Oda has characterized these in terms of LMMP. One can prove (where H^{K} is the closure of the discriminant locus in the Kirwan blow-up):

Theorem

The pair $(\overline{\mathcal{M}}_{12}^{\mathrm{K}}, \frac{5}{6}H^{\mathrm{K}} + \Delta)$ is not a log minimal model (of itself). Hence $\overline{\mathcal{M}}_{12}^{\mathrm{K}}$ is not a semi-toric compactification.

(The factor5/6 comes from ramification.)

For varieties with finite quotient singularities Kawamata has introduced the notion of *stacky derived equivalence* (which coincides with classical derived equivalence in the case of smooth varieties).

Theorem

The varieties $\overline{\mathcal{M}}_{12}^{\mathrm{K}}$ *and* $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ *are not stacky derived equivalent.*

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For varieties with finite quotient singularities Kawamata has introduced the notion of *stacky derived equivalence* (which coincides with classical derived equivalence in the case of smooth varieties).

Theorem

The varieties $\overline{\mathcal{M}}_{12}^{\mathrm{K}}$ *and* $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ *are not stacky derived equivalent.*

This follows from a theorem of Kawamata using that the varieties are not K-equivalent and that $-K_{\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{BB}}}$ is big.

Question

Are $\overline{\mathcal{M}}_{12}^{\mathrm{K}}$ and $\overline{\mathbb{B}^9/\Gamma_{12}}^{\mathrm{tor}}$ are derived equivalent in the classical sense?

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Outlook

The results which we have discussed here also hold in other situations

- Moduli of 8 points (Maeda, H.)
- Cubic surfaces (using the Naruki compactification) (Casalaina-Martin, Grushevsky, H., Laza + Maeda)
- Cubic threefolds (some results, work in progress with Grushevsky)
- All Deligne-Mostow varieties (work in progress with Maeda).

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Thank you for your attention

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