An Albanese construction for Campana's $\mathscr{C}$-pairs

Transcendental aspects of algebraic geometry

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Outline

- $\mathscr{C}$-Pairs and their Differentials
- Irregularities
- Morphisms of $\mathscr{C}$-Pairs
- The Albanese of a $\mathscr{C}$-Pair

$\mathscr{C}$-Pairs and their Differentials

Definition: A $\mathscr{C}$-Pair is a pair (X, D) of a normal analytic variety X and Weil $\mathbb{Q}$-divisor D with standard coefficients,

$$D = \sum_i \frac{m_i - 1}{m_i} \cdot D_i, \quad \text{all } m_i \in \mathbb{N}^+ \cup \{\infty\}.$$

Today: All $m_i \in \mathbb{N}$.

Idea: Given a compact Kähler space X, then $\mathscr{C}$-pairs $\left(X, \sum \frac{m_i-1}{m_i} \cdot D_i\right)$ interpolate between extreme cases

- All $m_i = 1$ $D = 0$, get classic geometry of the compact space X.
- All $m_i = \infty$ D is reduced, get geometry of the logarithmic pair.

$\mathscr{C}$-Pairs and their Differentials

Sheaves of Adapted Differentials

Want: Want sheaf of differential forms with logarithmic poles of order $\frac{m_i-1}{m_i}$ along the divisor D_i,

$$\Omega^1_X \subseteq \Omega^1_{(X,D)} \subseteq \Omega^1_X(\log D)$$

Have: On suitable covers of $\gamma : \widehat{X} \to X$, it is possible to define a sheaf

$$\gamma^*\Omega^1_X \subseteq \text{“adapted differentials”} \subseteq \gamma^*\Omega^1_X(\log D)$$

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More precisely: Given $\mathscr{C}$-pair (X, D) and quasi-finite morphism $\gamma : \widehat{X} \to X$ between analytic varieties of equal dimension, get reflexive sheaves

$$\gamma^{[*]}\Omega_X^{[p]} \subseteq \Omega^{[p]}_{(X,D,\gamma)} \subseteq \gamma^{[*]}\Omega_X^{[p]}(\log D),$$

uniquely characterized by suitable universal properties.

Pedro Núnez: “There exists a unique presheaf on the category of spaces over X [...] that is a sheaf with respect to the qfh Grothendieck topology.”

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Irregularity and Augmented Irregularity

Irregularity: Given $\mathscr{C}$-pair (X, D) where X is compact Kähler and a cover $\gamma : \widehat{X} \twoheadrightarrow X$, define

$$q(X, D, \gamma) := h^0\left(\widehat{X},\, \Omega^{[1]}_{(X,D,\gamma)}\right)$$

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$$q^+(X, D) := \sup_{\gamma} q(X, D, \gamma)$$

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Question:

- Possible that $q^+(X, D) \gg 0$, but $q(X, D, \gamma) = 0$ as long as …
  - … γ is solvable?
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Albanese Irregularities

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$$\begin{array}{ccccc} \widehat{X} & \longrightarrow & \mathrm{Alb}(\widehat{X}) & \xrightarrow{\text{quotient}} & B_\gamma \\ \downarrow & & & & \\ X & & & & \end{array}$$

where B_γ is the maximal Lie group quotient of $\mathrm{Alb}(\widehat{X})$ such that all differentials coming from B_γ are adapted. Then, define

$$q_{\mathrm{Alb}}(X,D,\gamma) := \dim B_\gamma \quad \text{and} \quad q^+_{\mathrm{Alb}}(X,D) := \sup_\gamma q_{\mathrm{Alb}}(X,D,\gamma)$$

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Example: Inequality in first line can be strict.

Question: Can the inequality in the second line ever be strict?

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Morphisms of $\mathscr{C}$-Pairs

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Setting: Let (X, D_X) and (Y, D_Y) be $\mathscr{C}$-pairs, and let $\varphi : X \to Y$ be any morphism.

Slogan: Call φ a morphism of $\mathscr{C}$-pairs if adapted differentials pull back to adapted differentials.

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$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\varphi}} & \widehat{Y} \\ {\scriptstyle \gamma_X}\downarrow & & \downarrow{\scriptstyle \gamma_Y} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

and every number p, the differential of $\widehat{\varphi}$ maps adapted differentials to adapted differentials,

$$\widehat{\varphi}^* \Omega^{[p]}_{(Y, D_Y, \gamma_Y)} \to \Omega^{[p]}_{(X, D_X, \gamma_X)}.$$

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Example:

- X smooth, $D \subset X$ a smooth prime divisor with multiplicity $\frac{m-1}{m}$
- $\varphi : \mathbb{C} \to X$ an entire curve.

Then, φ is a morphism between $\mathscr{C}$-pairs $(\mathbb{C}, 0)$ and (X, D) if every point of intersection between D and the curve has multiplicity $\geq m$.

Morphisms of $\mathscr{C}$-Pairs

(Non-)Examples

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Non-Example: Let $\varphi : X \to Y$ be the morphism from a Kummer K3 to the torus quotient. Then, φ is not a $\mathscr{C}$-morphism between $(X, 0)$ and $(Y, 0)$.

But: φ is a morphism between $\left(X, \frac{1}{2} \cdot E\right)$ and $(Y, 0)$.

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Main Features:

- Natural, functorial.
- Ties in with notions of the minimal model program.
- Categorical quotients exist for actions of finite groups.
- $\mathscr{C}$-Morphisms between torus quotients come from group actions.

The Albanese of a $\mathscr{C}$-Pair

Goal: Generalize results in the direction of Lang's conjecture, hyperbolicity, rational points and entire curves, where the best results are known only for subvarieties of Abelian varieties or compact tori.

The Albanese of a $\mathscr{C}$-Pair

Definition following Serre

Definition: Let (X,D) be a $\mathscr{C}$-pair where X is compact and normal, $\lfloor D \rfloor = 0$. An Albanese of (X,D) is a torus quotient pair (Q, D_Q) and a $\mathscr{C}$-morphism

$$a : (X,D) \to (Q, D_Q)$$

such that any other $\mathscr{C}$-morphism from (X,D) to a torus quotient pair factors uniquely via a.

The Albanese of a $\mathscr{C}$-Pair

Main Results

Existence Theorem: An Albanese exists if and only if $q^+_{\mathrm{Alb}}(X, D) < \infty$. □

Bloch-Ochiai Theorem: If $q_{\mathrm{Alb}}(X, D) > \dim X$, then every $\mathscr{C}$-entire curve $g : (\mathbb{C}, 0) \to (X, D)$ is algebraically degenerate. □

Conjecture: A weaker variant of the Albanese exists unconditionally.

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Existence Theorem: An Albanese exists if and only if $q^+_{\text{Alb}}(X, D) < \infty$.

Proof: Given a pair (X, D) and a Galois cover $\gamma : \widehat{X} \to X$, look at diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\beta}} & B_\gamma \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\text{quotient}} \\ X & \xrightarrow[\beta]{} & B_\gamma \big/ \operatorname{Gal}\gamma. \end{array}$$

Prove that β is a morphism of $\mathscr{C}$-pairs (X, D) and $(B_\gamma, 0) \big/ \operatorname{Gal}(\gamma)$. Then, take the limit over the category of all covers. □

The Albanese of a $\mathscr{C}$-Pair

Construction of the Albanese

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Bloch-Ochiai Theorem for $\mathscr{C}$-pairs: Let (X, D) be a $\mathscr{C}$-pair where X is a compact, normal Kähler space. If $q_{\mathrm{Alb}}(X, D) > \dim X$, then every $\mathscr{C}$-entire curve $g : (\mathbb{C}, 0) \to (X, D)$ is algebraically degenerate.

Proof: Interpret Noguchi's work in parabolic Nevanlinna theory as a Nevanlinna theory for $\mathscr{C}$-pairs. Then, follow ideas of Kawamata from his proof of the classic Bloch-Ochiai theorem. □

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The classic Albanese has more than one meaning...

Thank you for your time!