Minimal metrics and the Abundance conjecture

Abundance conjecture ($\Delta = \mathbb{Q}$-divisor)

(X, Δ) klt projective pair, assume $K_X + \Delta$ is nef

$\Downarrow$

$K_X + \Delta$ is semiample

(i.e. $m(K_X+\Delta)$ is basept free)

known:

- 3-folds: Miyaoka, Kawamata, Keel-Matsuki-McKernan (1980s-1990s)
- $K_X + \Delta \equiv 0$: Nakayama (2004)
- If (X, Δ) satisfies Miyaoka's inequality: Iwai, Matsumura, Müller (2024)

Higher dimensions (≥ 4)

[L-Peternell, Gongyo-Matsumura]

(X, Δ) minimal klt pair

Assume:

- $\chi(X, \mathcal{O}_X) \neq 0$
- $\exists \pi: Y \to X$ resolution: $\pi^* \mathcal{O}_X(K_X + \Delta)$ is hermitian semipositive

($\exists$ smooth metric h on $\pi^* \mathcal{O}_X(K_X+\Delta)$ s.t. $\Theta_h(K_X+\Delta) \geq 0$)

$\Rightarrow K_X+\Delta$ is semiample.

Slogan: semipositive $\Rightarrow$ semiample if $\chi(X, \mathcal{O}_X) \neq 0$

Q: where does the condition $\chi(X, \mathcal{O}_X) \neq 0$ come from?

A: maybe at the end of the talk.

More generally: we don't need to assume semipositivity; we need only "generalized algebraic singularities"

we need to work with singular metrics

Setup: X = compact complex manifold

quasi-psh function: these are locally sums of smooth and psh functions

Comparison of singularities

$\varphi_1 \precsim \varphi_2 \Leftrightarrow \exists C$ constant: $\varphi_2 \leq \varphi_1 + C$

"less singular"

$\rightsquigarrow$ equivalence relation on quasi-psh functions:

$$\varphi_1 \simeq \varphi_2 \iff \varphi_1 \preccurlyeq \varphi_2 \preccurlyeq \varphi_1$$

let α = smooth real $(1,1)$-form on X

$$PSH(X,\alpha) = \{ \varphi \text{ quasi-psh} \mid \alpha + dd^c\varphi \geq 0 \}$$

$$\left[dd^c := \frac{i}{2\pi}\partial\bar\partial \right]$$

$PSH(X,\alpha) \neq \emptyset \iff \{\alpha\} \in H^{1,1}_{BC}(X,\mathbb{R})$ is pseudoeffective

THE POINT: $\exists\, \varphi_{min} \in PSH(X,\alpha)$ which is minimal wrt singularities

proof:

$$V_\alpha := \{ \varphi \in PSH(X,\alpha) \mid \varphi \leq 0 \}$$

$$\varphi_{min} := \sup_{\varphi \in V_\alpha} \varphi$$

one can show: $\varphi_{min} \in V_\alpha$

clearly: φ_{min} is having minimal singularities. $\square$

this is an L^∞-condition

Remark: If $\varphi_1 \simeq \varphi_2 \implies$

$$\implies \mathcal{J}(\varphi_1) = \mathcal{J}(\varphi_2)$$

($\mathcal{J}(\varphi)$ = sheaf of germs of locally L^2-integrable functions on X wrt φ:

$\forall x\ \exists U \ni x : \int_U |f|^2 e^{-2\varphi} dV_\omega < \infty$

in other words:

$|\cdot|_h := |\cdot| e^{-2\varphi}$)

Notation: X compact complex manifold

T = closed positive $(1,1)$-current on X

$\Rightarrow \mathcal{J}(T)_{min}$ = multiplier ideal of any current with minimal

$\alpha + dd^c\varphi$ by defn: $\mathcal{J}(\varphi)$ sings in $\{T\}$

Theorem A: Assume MMP in dim $\leq n-1$.

Let (X,Δ) be a minimal klt projective pair of dimension n.

Let $\pi: Y \to X$ be a log resolution of (X,Δ). Write:

$$\underbrace{K_Y + \Delta_Y}_{\text{klt}} \sim_{\mathbb{Q}} \pi^*(K_X+\Delta) + E$$

(Δ_Y, E have no common components)

Let A ample on Y. Assume that $\exists D \geq 0$ on Y and a sequence of positive integers $\{m_\ell\}_{\ell\in\mathbb{N}}$ s.t. divisors

$m_\ell \xrightarrow[\ell\to\infty]{} \infty$ and

♥
$$\left\{\begin{array}{c} \mathcal{J}\left(\ell\left(K_Y+\Delta_Y+\frac{1}{m_\ell}A\right)\right)_{min} \\ \cap| \\ \mathcal{J}\left(\ell\left(K_Y+\Delta_Y\right)\right)_{min} \otimes \mathcal{O}_X(D). \end{array}\right.$$

Then: if $\chi(X, \mathcal{O}_X) \neq 0$ or if $\kappa(X, K_X+\Delta) \geq 0$, then $K_X+\Delta$ is semiample

Remark 1: we always have

$$\mathcal{J}(\ell(K_Y+\Delta_Y))_{min} \subseteq \mathcal{J}\left(\ell\left(K_Y+\Delta_Y+\frac{1}{m_\ell}A\right)\right)_{min}$$

so the assumption in the theorem mean that these multiplier ideals are asymp. the same.

Remark 2: ♥ follows from the Abundance conj.

The point: the Abundance conjecture for $\chi(X, \mathcal{O}_X) \neq 0$ is equivalent to ♥

Remark 3: ♥ is an example of asymptotic equisingular approximations.

So: to prove Abundance when $\chi(X, \mathcal{O}_X) \neq 0$, will need to show ☹

Supercanonical currents

These are minimal currents, but they are not defined by an L^∞-condition, but by an "exponential L^2-condition"

(Narasimhan–Simha '68, Tsuji '07, Berman–Demailly 2012: version of supercanonical currents)

Definition/Lemma:

X compact complex manifold

α smooth $(1,1)$-form s.t. $\{\alpha\} \in H^{1,1}_{BC}(X, \mathbb{R})$ is pseudoeffective.

Let $\mathcal{S}_\alpha := \{ \varphi \in PSH(X, \alpha) \mid \int_X e^{2\varphi} dV_\omega \leq 1 \}$

Set $\varphi_{\alpha, can} := \sup_{\varphi \in \mathcal{S}_\alpha} \varphi$.

(this is well-defined because all $\varphi \in \mathcal{S}_\alpha$ are uniformly bounded (from above)

... we can show that

$\varphi_{\alpha,\mathrm{can}}$ is really in

$PSH(X,\alpha)$

and it is minimal

<u>Theorem B</u>: Let (X,Δ) be a projective klt pair, $K_X+\Delta$ is pseudo-effective. Let $\pi: Y\to X$ be a log resolution, write

$K_Y+\Delta_Y \sim_{\mathbb{Q}} \pi^*(K_X+\Delta)+E$

Let A ample divisor on Y.

Let $\alpha\in\{K_Y+\Delta_Y\}$, $\omega\in\{A\}$ smooth $(1,1)$ forms.

Then:

(a) $\varphi_{\alpha,\mathrm{can}} = \lim_{\varepsilon\to 0} \varphi_{\alpha+\varepsilon\omega,\mathrm{can}}$

(b) each $\varphi_{\alpha+\varepsilon\omega,\mathrm{can}}$ $(\varepsilon>0)$ <u>depends only</u> on the global sections of multiples of $K_Y+\Delta_Y+\varepsilon A$.

+ some other properties which depend on the MMP,

the most important statement

all $\varphi_{\alpha+\varepsilon\omega,\mathrm{can}}$ $(0<\varepsilon\ll 1)$ are continuous away from the non-nef locus of $K_Y+\Delta_Y$!

Quick answer to:

Why $\chi(X, \mathcal{O}_X) \neq 0$?

$\chi(X, \mathcal{O}_X) \neq 0 \Rightarrow \kappa(X, K_X+\Delta) \geq 0$

$\Updownarrow$

$\kappa(X, K_X+\Delta) = -\infty \Rightarrow \chi(X, \mathcal{O}_X) = 0$

MMP

$\exists\, \pi : Y \to X$

$H^0(Y, \Omega_Y^p \otimes \pi^* \mathcal{O}_X(m(K_X+\Delta)))$

$\| $

0

$\forall p,\ \forall m$ suff. divisible

Answer: $K_X+\Delta$ is semipositiv…

$h \to \mathcal{J}(h) = \mathcal{O}_X$

$0 = H^0(Y, \Omega_Y^p \otimes \pi^* :$

$\Big\downarrow$ DPS

$H^{n-p}(Y, K_Y + \pi^*(K_X+\Delta))$

$\|$

0

$\Downarrow$

$\chi(X, \mathcal{O}_X) = 0$