The rescaling method on subvarieties of bounded symmetric domains arising from their quotients with respect to cocompact lattices

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Kähler manifolds and isometries between them

Isometries between Riemannian manifolds

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$\sqrt{-1}\partial\overline{\partial}(\psi \circ f) = \sqrt{-1}\partial\overline{\partial}\varphi$, i.e., $\varphi(z) = \psi(f(z)) + u$, $u = 2\mathrm{Re}(h)$, $\exists$ h hol.

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If $\sqrt{-1}\partial\overline{\partial}\varphi' = \omega_g$ on U, then $\varphi' = \varphi + h + \overline{h}$ for some h holomorphic on U. Calabi [Ca53] defined locally the **diastasis** $\delta_M(x, y)$ on (M, g) by $\delta_M(x, y) := \Phi(x, x) - \Phi(x, y) - \Phi(y, x) + \Phi(y, y)$.

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Let (M,g) be complex manifold with a real-analytic Kähler metric g; $x_o \in M$, $(\mathbb{P}^N, ds^2_{FS})$, $1 \leq N \leq \infty$, be the Fubini-Study space, $o \in \mathbb{P}^N$, and $f:(M,g;x_o) \to (\mathbb{P}^N, ds^2_{FS}; o)$ be a germ of holomorphic isometry.

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Let (M, g) be a complex manifold equipped with a real-analytic Kähler metric. Let $\lambda > 0$, $1 \le N \le \infty$, and $\varphi : (M, g; x_0) \to \left(\mathbb{P}^N, \frac{1}{\lambda} ds^2_{FS}; y_0\right)$ be a germ of holomorphic isometry. Suppose for each $x \in M$, the maximal analytic extension of the diastasis $\psi_x(y) := \delta_M(x, y)$ is single-valued.

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$U \Subset \mathbb{C}^n$ bounded domain; $H^2(U) := \left\{ f \in \mathcal{O}(U) : \int_U |f|^2 dV < \infty \right\}$;
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The Bergman metric g and its associated Kähler form ω_g are given by

$$g = 2\mathrm{Re} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \otimes d\overline{z^j} \ ; \quad \omega_g = \sqrt{-1}\partial\overline{\partial} \log K(z,z) \ .$$

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On a bounded domain we have $\omega > 0$.

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Classical cases

$$D^I(p,q) = \{Z \in M(p,q,\mathbb{C}) : I - \overline{Z}^t Z > 0\} \; , \quad p,q \geq 1\,;$$

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Exceptional domains

D^V, dim 16, type E_6 ; D^{VI}, dim 27, type E_7 .

Bergman kernels for classical domains

$$K_{\mathbb{B}^n}(z,w) = \frac{c_n}{(1- < z, w >)^{n+1}} \, ;$$

$$K_{D^I(p,q)}(Z,W) = \frac{c_{p,q}}{\det\left(I_p - Z\overline{W}^t\right)^{p+q}} \, ;$$

$$K_{D^{II}(n,n)}(Z,W) = \frac{a_n}{\det\left(I_n + Z\overline{W}\right)^{n-1}} \, ;$$

$$K_{D^{III}(n,n)}(Z,W) = \frac{b_n}{\det\left(I_n - Z\overline{W}\right)^{n+1}} \, ;$$

$$K_{D^{IV}_n}(z,w) = \frac{d_n}{\left(1 - z\cdot\overline{w} + \frac{1}{4}\sum_{1\le i,j\le n} z_i^2\overline{w_j^2}\right)^n} \, .$$

Analytic continuation of holomorphic isometries up to normalizing constants with respect to the Bergman metric

Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains, and $\lambda > 0$ be a real constant. We are interested to prove extension theorems for holomorphic isometries up to normalizing constants $f : (D, \lambda\, ds^2_D; x_0) \to (\Omega, ds^2_\Omega; y_0)$.

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Interior extension

On a bounded domain U the potential function $\varphi(z) = \log K_U(z,z)$ is globally defined, hence Calabi [Ca53] applies to give interior extension results, as follows. We have a canonical holomorphic embedding $\Phi_\Omega : \Omega \to \mathbb{P}(H^2(\Omega)^\star)$. Choosing any orthonormal basis (h_i) of $H^2(\Omega)$, $\Phi_\Omega : \Omega \to \mathbb{P}^\infty \cong \mathbb{P}(H^2(\Omega)^\star)$ is given by $\Phi_\Omega(\zeta) = [h_0(\zeta), \cdots, h_i(\zeta), \cdots]$. The mapping $\Phi_\Omega \circ f : (D, ds_D^2; x_0) \to \left(\mathbb{P}(H^2(\Omega)^\star), \frac{1}{\lambda} ds_{FS}^2; \Phi_\Omega(y_0)\right)$ is a holomorphic isometry into a projective space of countably infinite dimension equipped with the Fubini-Study metric. Let $\mathbb{P}(\Lambda) \subset \mathbb{P}(H^2(\Omega)^\star)$ be the topological projective-linear span of the image of $\Phi_\Omega \circ f$, $\Lambda \subset H^2(\Omega)^\star$ being a Hilbert subspace.

Univalence of $\psi_x(y) := \delta_D(x, y)$ follows readily from the Cauchy-Schwarz inequality $|K_D(x, y)|^2 \leq K_D(x, x)K_D(y, y)$, with equality if and only if $\big(s_0(x), \cdots, s_i(x), \cdots\big)$ and $\big(s_0(y), \cdots, s_i(y), \cdots\big)$ are proportional to each other, hence $x = y$. By Calabi [Ca53], $\Phi_\Omega \circ f$ extends to a holomorphic isometry $\Psi : D \to \mathbb{P}(\Lambda)$, implying analytic continuation of Graph(f) to a complex-analytic subvariety of $D \times \Omega$.

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Let $U \subset \mathbb{C}^n$ be a bounded complete circular domain. Because of the invariance of the Bergman kernel K_U under the circle group action, i.e., $K_U(e^{i\theta}z, e^{i\theta}w)) = K_U(z, w)$ for $\theta \in \mathbb{R}$, it follows that $K_U(z, 0)$ is a constant. Denoting by $\delta_U(x, y)$ the diastasis on (U, ds^2_U) and by $\Phi(z, w)$ the polarization of the real-analytic function $\varphi(z) := \delta_U(0, z)$. We have

$$\delta_U(0, z) = \log K_U(0, 0) - \log K_U(0, z) - \log K_U(z, 0)$$
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From functional identities we will derive extension results.

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$$K_\Omega(f(z), f(z)) = A \cdot K_D(z,z)^\lambda; \quad \text{and hence}$$

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Let $\epsilon > 0$ be such that f is defined on $D_\epsilon := B(0;\epsilon) \Subset D$.

Proposition (extension of graphs of germs of holomorphic isometries)

For each $w \in D_\epsilon$, let $V_w \subset D \times \mathbb{C}^N$ be the set of all $(z, \zeta) \in D \times \Omega$ s.t.

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Idea of Proof (infinitesimal deformations of solutions to (I_w))

$$(\sharp) \qquad K_\Omega\big(f_t(z), \overline{f(w)}\big) = K_D(z,w)^\lambda \ ; \quad f_0(z) \equiv f(z) \ .$$

Assume $\frac{\partial^k}{\partial t^k} f_t(z)\big|_{t=0} \equiv 0$ for $k < \ell$ and $\eta(z) := \frac{\partial^\ell}{\partial t^\ell} f_t(z)\big|_{t=0} \not\equiv 0$. Then, $h_\alpha(f(w)) = 0; \alpha \in \mathbf{A}$, follow from expressing $\eta(z)$ in canonical coordinates of Ω of Bergman adapted to different base points along $f(D_\epsilon) \subset \Omega$.

Algebraic extension of holomorphic isometries between bounded domains with rational Bergman kernels

Theorem (Mok, *JEMS* (2012))

Let $D \Subset \mathbb{C}^n, \text{ resp. } \Omega \Subset \mathbb{C}^N$, *be bounded domains. Let* $x_0 \in D$, $\lambda \in \mathbb{R}, \lambda > 0$, *and* $f : (D, \lambda ds_D^2; x_0) \to (\Omega, ds_\Omega^2; f(x_0))$ *be a germ of holomorphic isometry.*

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The unit disk Δ is conformally equivalent to the upper half-plane $\mathcal{H}$, the unbounded realization of the unit disk by means of the inverse Cayley transform. For $\tau \in \mathcal{H}$, $\tau = re^{i\theta}$, where $r > 0$, $0 < \theta < \pi$, and for an integer $p \geq 2$, we write $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$.

Non-standard holomorphic isometries of Δ into Δ^p

Proposition (Mok [Mo12])

Let $p \geq 2$ be an integer. Equip the upper half-plane $\mathcal{H}$ with the Poincaré metric $ds^2_{\mathcal{H}} = \mathrm{Re}\frac{d\tau \otimes d\overline{\tau}}{2(Im\tau)^2}$ of constant Gaussian curvature -2 and $\mathcal{H}^p$ with the product metric.

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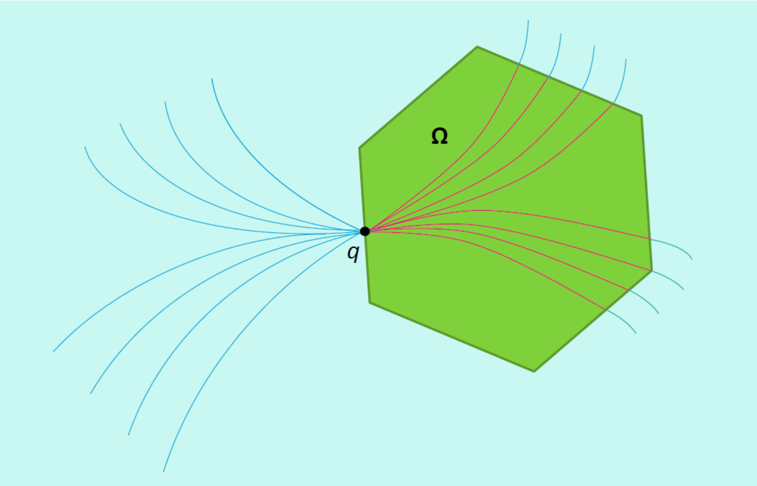
$$\sin\theta\sin\left(\frac{\pi}{p}+\theta\right)\cdots\sin\left(\frac{(p-1)\pi}{p}+\theta\right) = c_p\sin(p\theta).$$

Image of holomorphic isometry of $f : B^n \hookrightarrow \Omega$.

$$\mathcal{V}_q = \bigcup \{\text{lines } \ell \text{ on } S = G^{\mathbb{C}}/P,\ q \in \ell\} \ ;$$

$$V_q = \mathcal{V}_q \cap \Omega = f(B^n).$$



Asymptotic total geodesy of embedded Poincaré disks

Theorem (Chan-Mok, *J. Diff. Geom.* (2022))

Let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding, where* λ *is a positive real constant and* $\Omega \Subset \mathbb{C}^N$ *is a bounded symmetric domain in its Harish-Chandra realization.*

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Proposition *Let* $f_0 : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding.*

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Let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding, where λ is a positive real constant and $\Omega \Subset \mathbb{C}^N$ is a bounded symmetric domain in its Harish-Chandra realization. **Then, f is asymptotically totally geodesic at a general point $b \in \partial\Delta$.**

Proposition *Let $f_0 : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding. Suppose $Z_0 := f_0(\Delta) \subset \Omega$ is not asymptotically totally geodesic at a general point $b \in \partial Z_0$.*

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Proposition *Let* $f_0 : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding. Suppose* $Z_0 := f_0(\Delta) \subset \Omega$ *is not asymptotically totally geodesic at a general point* $b \in \partial Z_0$. **Then, there exists by rescaling a holomorphic isometric embedding** $f : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$**,** $f(\Delta) =: Z$ **with the following property.**

(†) **All tangent lines** $T_x(Z)$**,** $x \in Z$**, are equivalent under** $\mathrm{Aut}(\Omega)$**.**

Total geodesy of certain curves on tube domains

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Let Ω be an irreducible bounded symmetric domain of tube type and of rank r; $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent planes spanned by vectors of rank r.

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$\pi : \mathbb{P}T_\Omega \to \Omega$, $[\mathscr{S}] \cong L^{-r} \otimes \pi^* E^2$, where $L \to \mathbb{P}T_\Omega$ is the tautological line bundle, and E is dual to $\mathcal{O}(1)$ on M, $\Omega \Subset M$ being the Borel embedding, and

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$$(2\pi)^{-1}\sqrt{-1}\partial\overline{\partial}\log\|s\|^2 = rc_1(L, \hat{g}_0) - 2c_1(\pi^*E, \pi^*h_0),$$

where $\hat{g}_0$ and h_0 are canonical metrics. The norm $\|s(x)\|$ only depends on the isomorphism type of $T_x(Z)$. Thus, $\|s\| =$ constant o Z.

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where $\hat{g}_0$ and h_0 are canonical metrics. The norm $\|s(x)\|$ only depends on the isomorphism type of $T_x(Z)$. Thus, $\|s\| =$ constant o Z. Hence,

$$0 = rc_1(L,\hat{g}_0) - 2c_1(\pi^*E,\pi^*h_0)\,.$$

$\Leftrightarrow$ Gauss curvature $K(x) = -2/r$, and $\sigma \equiv 0$. $\square$

Application to equivariant holomorphic isometries

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Algebraic subsets of a bounded symmetric domain invariant under a discrete cocompact group action

Theorem (Chan-Mok [CM22])

Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, and $Z \subset \Omega$ be an algebraic subset. Suppose there exists a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(\Omega)$ such that $\check{\Gamma}$ stabilizes Z and $Y := Z/\check{\Gamma}$ is compact. **Then, $Z \subset \Omega$ is totally geodesic.**

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Corollary

Let $\Omega \Subset \mathbb{C}^N$ be as in Theorem, $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice. Write $X_\Gamma := \Omega/\Gamma$, and $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Y \subset X_\Gamma$ be an irreducible subvariety, and $Z \subset \Omega$ be an irreducible component of $\pi^{-1}(Y)$. Suppose $Z \subset \Omega$ is an algebraic subset. **Then, $Z \subset \Omega$ is totally geodesic.**

Pseudo-homogeneity of Z under a complex Lie group

For the Borel embedding of a bounded symmetric domain $\Omega \Subset \mathbb{C}^N \subset M$ into its compact dual manifold, we write $M = G/P$, where G is the identity component of $\mathrm{Aut}(M)$, and G_0 for the identity component of $\mathrm{Aut}(\Omega)$. $G_0 \subset G$ is a noncompact real form. We have

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There exists $\widehat{Z} \subset M$ projective such that Z is an irreducible component of $\widehat{Z} \cap \Omega$. We proceed to prove that $\forall x \in Z, \overline{Hx} = \widehat{Z}$, which implies Proposition.

Proof by the Maximum Principle

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Nadel's Semisimplicity Theorem

Theorem (Nadel, *Ann. Math.* (1990))

Let X be a compact Kähler manifold with ample canonical line bundle, and denote by $\pi : \widetilde{X} \to X$ the uniformization map. **Then, $\mathbf{Aut}_0(\widetilde{X})$ is a semisimple Lie group without compact factors.**

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Suppose there exists a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(\Omega)$ such that $\check{\Gamma}$ stabilizes Z and $Y := Z/\check{\Gamma}$ is compact. Let $H_0 \subset \mathrm{Aut}(\Omega)$ be the identity component of the subgroup of $\mathrm{Aut}(\Omega)$ which stabilizes Z. **Then, $H_0 \subset \mathbf{Aut}(\Omega)$ is a semisimple Lie group without compact factors.**

Bi-algebraicity by means of Nadel's Theorem

Maps inducing the representation $\theta : \check{\Gamma} \hookrightarrow H_0 \subset G_0 = \mathrm{Aut}_0(\Omega)$

Write $X_{\check{\Gamma}} := \check{\Gamma}\backslash\Omega = \check{\Gamma}\backslash G/K$. Without loss of generality we assume that $\imath_*\pi_1(Y) = \check{\Gamma} \subset H_0$, $\imath : Y \hookrightarrow X_{\check{\Gamma}}$, where $\imath := \imath_Y$. **By Nadel's Theorem, H_0 is a semisimple Lie group without compact factors which acts on Ω.**

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Total geodesy of $Z \subset \Omega$

Since $X_{\check{\Gamma}}$ is a $K(\pi, 1)$, the two smooth maps $f, \imath : Y \to X_{\check{\Gamma}}$ inducing the representation θ are homotopic. Recall that $L \subset H_0$ is a maximal compact subgroup, hence $\dim_{\mathbb{R}}(S_{\check{\Gamma}})$ is **minimal** among H_0-orbits on Ω.

Denote by ω the Kähler form of the canonical Kähler-Einstein metric on $X_{\check{\Gamma}}$. H_0 acts on Ω and preserves Z.

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Since $X_{\check{\Gamma}}$ is a $K(\pi, 1)$, the two smooth maps $f, \imath : Y \to X_{\check{\Gamma}}$ inducing the representation θ are homotopic. Recall that $L \subset H_0$ is a maximal compact subgroup, hence $\dim_{\mathbb{R}}(S_{\check{\Gamma}})$ is **minimal** among H_0-orbits on Ω.

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Moduli space of elliptic curves

An elliptic curve is complex-analytically a compact Riemann surface S of genus 1. In other words, $S := \mathbb{C}/L$ **for some lattice** $L \subset \mathbb{C}$. Replacing L by λL for some $\lambda \in \mathbb{C} - \{0\}$, without loss of generality we may assume $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$, i.e., $\tau \in \mathcal{H}$, where $\mathcal{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, the upper half plane. Write $S_\tau := \mathbb{C}/L_\tau$.

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For $\tau, \tau' \in \mathcal{H}$, we have $S_\tau \cong S_{\tau'}$ **if and only** if there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $L_{\tau'} = \lambda L_\tau$, **i.e.**, if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ **where** $ad - bc \neq 0$. **Thus, the set of equivalence classes of $\mathbb{C}/L$ is in one-to-one correspondence with $X = X(1) := \mathcal{H}/\mathbb{P}SL(2,\mathbb{Z})$.** $\mathbb{P}SL(2,\mathbb{Z})$ acts discretely on $\mathcal{H}$ with fixed points. **We have the j-function $j : X(1) \xrightarrow{\cong} \mathbb{C}$, and $\overline{X(1)} = \mathbb{P}^1$.**

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A suitable finite-index subgroup $\Gamma \subset \mathbb{P}SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ without fixed points and $X_\Gamma := \mathcal{H}/\Gamma$ can be compactified to a compact Riemann surface.

The j-function

On the upper half-plane $\mathcal{H} = \{\tau : \mathrm{Im}(\tau) > 0\}$ **define**

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)}$$

where $g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-4}$; $g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-6}$.

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The j-function establishes a biholomorphism $j : \mathcal{H}/\mathrm{SL}(2,\mathbb{Z}) \xrightarrow{\cong} \mathbb{C}$.

The André-Oort Conjecture

A point $\tau \in \mathcal{H}$ such that $\tau, j(\tau) \in \overline{\mathbb{Q}}$ is called a **special point** (in which case $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$ by Schneider). The notion of special points is defined for any Shimura variety $X_\Gamma = \Omega/\Gamma$, and the **André-Oort Conjecture** ascertains that the **Zariski closure of any set of special points on X_Γ is a finite union of Shimura subvarieties $X'_{\Gamma'} \hookrightarrow X_\Gamma$.**

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Lindemann-Weierstrass Theorem and Schanuel Conjecture

Lindemann-Weierstrass Theorem (1882)

Suppose $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{Q}}$ *are* $\mathbb{Q}$*-linearly independent.* **Then,** $e^{\alpha_1}, \cdots, e^{\alpha_n}$ **are algebraically independent.**

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Schanuel Conjecture (1960s)

Suppose $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ *are* $\mathbb{Q}$*-linearly independent.* **Then,**

$$\textbf{trans.deg.}_{\mathbb{Q}}\mathbb{Q}\left(\alpha_1, \cdots, \alpha_n; e^{\alpha_1}, \cdots, e^{\alpha_n}\right) \geq n.$$

The Lindemann-Weierstrass Theorem answers in the affirmative the special case of the Schanuel Conjecture where $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{Q}}$.

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Baker's Theorem (1975)

Suppose $x_1, \cdots, x_n \in \overline{\mathbb{Q}}$, *and* $\log(x_1), \cdots \log(x_n)$ *are linearly independent over* $\mathbb{Q}$. **Then** $1, \log(x_1), \cdots, \log(x_n)$ **are linearly independent over** $\overline{\mathbb{Q}}$.

The Ax-Lindemann-Weierstrass Theorem on $X_\Gamma = \Omega/\Gamma$

After Ullmo-Yafaev [UY14] in the case of cocompact lattices, and Pila-Tsimerman [PT14] in the case of Siegel modular varieties, we have

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Using the above, Tsimerman (2018) proved the André-Oort Conjecture for Siegel modular varieties $\mathcal{A}_g = \mathcal{H}_g/\mathrm{Sp}(g;\mathbb{Z})$. Recently, Pila, Shankar and Tsimerman have made available a preprint in the arXiv resolving the full André-Oort Conjecture in the affirmative.

Ax-Lindemann-Weierstrass Theorem for rank-1 lattices

Theorem (Mok, *Compositio Math.* (2019))

Let $n \geq 2$ *and* $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ *be a* **not necessarily arithmetic** *torsion-free lattice. Write* $X_\Gamma := \mathbb{B}^n/\Gamma$, $\pi : \Omega \to X_\Gamma$ *for the uniformization map. Let* $Z \subset \Omega$ *be an irreducible algebraic subset and* $\mathscr{Z} := \overline{\pi(Z)}^{\mathrm{Zar}} \subset X_\Gamma$ *be the Zariski closure of* $\pi(Z)$. **Then,** $\mathscr{Z} \subset X_\Gamma$ **is a totally geodesic subset.**

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(b) Let $\widetilde{\mathscr{Z}}$ be an irreducible component of $\pi_\Gamma^{-1}(\mathscr{Z})$. Then, at a good point $b \in \partial\widetilde{\mathscr{Z}}$, $\widetilde{\mathscr{Z}}$ **extends across** b **as the union of an analytic family of algebraic subvarieties of** $\mathbb{P}^n$.

Ax-Lindemann-Weierstrass Theorem for rank-1 lattices

Theorem (Mok, *Compositio Math.* (2019))

Let $n \geq 2$ and $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a **not necessarily arithmetic** *torsion-free lattice. Write $X_\Gamma := \mathbb{B}^n/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Z \subset \Omega$ be an irreducible algebraic subset and $\mathscr{Z} := \overline{\pi(Z)}^{\mathrm{Zar}} \subset X_\Gamma$ be the Zariski closure of $\pi(Z)$.* **Then, $\mathscr{Z} \subset X_\Gamma$ is a totally geodesic subset.**

Sketch of proof

(a) We have $\mathbb{B}^n \subset \mathbb{P}^n$, $Z^{\mathrm{open}} \subset \widehat{Z} \subset \mathbb{P}^n$. Consider $[\widehat{Z}]$ as a member of an irreducible component $\mathcal{K}$ of Chow($\mathbb{P}^n$), with universal family $\mu : \mathcal{U} \to \mathbb{P}^n$. Restrict $\mathcal{U}$ to $\mathbb{B}^n$ and take quotients wrt Γ to get $\mu_\Gamma : \mathcal{U}_\Gamma \to X_\Gamma$, tautologically foliated by $\mathscr{F}$. Proved algebraicity of $\mathcal{U}_\Gamma$ and $\mathscr{F}$ by means of L^2-estimates of $\overline{\partial}$ (Mok-Zhong, *Ann. Math.* (1989)) and Kähler geometry.

(b) Let $\widetilde{\mathscr{Z}}$ be an irreducible component of $\pi_\Gamma^{-1}(\mathscr{Z})$. Then, at a good point $b \in \partial\widetilde{\mathscr{Z}}$, $\widetilde{\mathscr{Z}}$ **extends across b as the union of an analytic family of algebraic subvarieties of $\mathbb{P}^n$**. Prove by a rescaling argument and Kähler geometry that Z is a holomorphically isometric copy of some $\mathbb{B}^m$.

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Ax-Schanuel Theorem on Shimura varieties

Theorem (Mok-Pila-Tsimerman, *Ann. Math.* (2019))

Let $\Omega \Subset \mathbb{C}^N$ *be a bounded symmetric domain,* $\Gamma \subset \mathrm{Aut}(\Omega)$ *be an arithmetic lattice, and write* $X_\Gamma := \Omega/\Gamma$*, as a quasi-projective variety. Let* $W \subset \Omega \times X_\Gamma$ *be an algebraic subvariety. Let* $D \subset \Omega \times X_\Gamma$ *be the graph of the uniformization map* $\pi_\Gamma : \Omega \to X_\Gamma$*, and* U **be an irreducible component of** $W \cap D$ **whose dimension is larger than expected**,

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$$\mathrm{codim}\, U < \mathrm{codim}(W) + \mathrm{codim}(D),$$

the codimensions being in $\Omega \times X_\Gamma$*, or, equivalently,*

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Then, the projection of U to X_Γ is contained in a totally geodesic subvariety $Y \subsetneq X_\Gamma$.

Ax-Schanuel of MPT in terms of functional transcendence

Fix a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\Omega), \pi : \Omega \to X_\Gamma$. In what follows modular functions are Γ-invariant meromorphic functions on Ω descending to rational functions on X_Γ.

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Let $V \subset \Omega$ be an irreducible complex analytic subvariety, **not contained in any weakly special subvariety $E \subsetneq \Omega$.** *Let $(z_i)_{1 \le i \le n}$ be algebraic coordinates on Ω, $\{\varphi_1, \ldots, \varphi_N\}$ be a basis of modular functions.* **Then,**

$$\textbf{trans.deg}_{\mathbb{C}}\mathbb{C}\big(\{z_i\}, \{\varphi_j\}\big) \ge n + \dim V,$$

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1. We may take the algebraic coordinates $(z_1, \cdots, z_n)$ to be the Harish-Chandra coordinates on $\Omega \Subset \mathbb{C}^n \subset \widehat{\Omega}$.
2. Here a weakly special subvariety $E \subset \Omega$ is a totally geodesic submanifold $E \subset \Omega$ such that $\pi(E) \subset X_\Gamma$ is quasi-projective.

The Abel-Jacobi map $\alpha : C \to Jac(C)$

C of genus g

$\omega_1, \dots, \omega_g$ basis of holomorphic 1-forms

$\gamma_1, \dots, \gamma_{2g}$ basis of $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

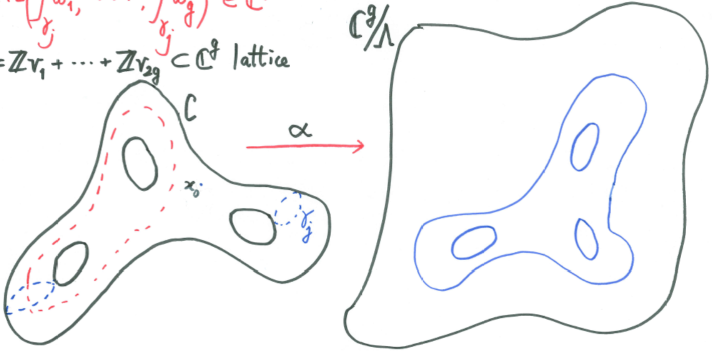
$v_j := \left(\int_{\gamma_j} \omega_1, \dots, \int_{\gamma_j} \omega_g\right) \in \mathbb{C}^g$

$\Lambda = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_{2g} \subset \mathbb{C}^g$ lattice

$\alpha(x)$

$= \left(\int_{x_0}^{x} \omega_1, \dots, \int_{x_0}^{x} \omega_g\right) \bmod \Lambda$

$\in \mathbb{C}^g/\Lambda =: Jac(C)$



Applications of Ax-Schanuel on Shimura varieties

Mok-Pila-Tsimerman has been generalized by Bakker-Tsimerman (2019) to period domains, and by Gao (2020) to mixed Shimura varieties. There have been many applications, to the **Zilber-Pink Conjecture** (beyond André-Oort), to the **Betti map for abelian schemes**, etc.

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The Uniform Mordell-Lang Conjecture

Faltings (1983) proved the Mordell Conjecture, i.e., for a smooth projective algebraic curve C of genus $g \geq 2$ defined over a number field K, there are at most **a finite number of K-rational points.**

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Let C be embedded in its Jacobian $J(C)$. Dimitrov-Gao-Habegger (2021), together with a later contribution by Kühne concerning K-rational points of small height, established the **Uniform Mordell-Lang Conjecture for curves**, proving that **the set $C(K)$ of K-rational points on C is of size uniformly bounded in terms of g, $d = [K : \mathbb{Q}]$ and the Mordell-Weil rank ρ of $J(C)$**.

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The proof has many ingredients, but it uses in an essential way Gao's work on the degeneracy of the Betti map, which in turn relies on **Ax-Schanuel on mixed Shimura varieties.**

The inverse Cayley transform on $\mathbb{B}^n$

Let $\ell \subset \mathbb{B}^n$ be the geodesic on $\left(\mathbb{B}^n, ds^2_{\mathbb{B}^n}\right)$ joining the point $(-i, 0, \cdots, 0)$ to $(i; 0, \cdots, 0)$. Let $0 \le t < 1$ and $\varphi_t \in \mathrm{Aut}(\mathbb{B}^n)$ be the transvection along ℓ mapping 0 to $(0, \cdots, 0, -it)$.

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$$\varphi_t(z_1, \cdots, z_{n-1}; z_n) = \left(\frac{\sqrt{1-t^2}z_1}{1+itz_n}, \cdots, \frac{\sqrt{1-t^2}z_{n-1}}{1+itz_n}; \frac{z_n - it}{1+itz_n}\right)$$

$$d\varphi_t(0, \cdots, 0; z_n) = \mathrm{diag}\left(\frac{\sqrt{1-t^2}}{1+itz_n}, \cdots, \frac{\sqrt{1-t^2}}{1+itz_n}; \frac{1-t^2}{(1+itz_n)^2}\right);$$

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$$\alpha_t(w) = \left(\frac{2w_1}{\sqrt{1-t^2}}, \cdots \frac{2w_{n-1}}{\sqrt{1-t^2}}, \frac{2(w_n + it)}{1-t^2} \right) + (0, \cdots, 0, it).$$

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$$\frac{2}{1-t^2}\Big(\frac{z_n - it}{1+itz_n} + it\Big) + it\Big)$$
$$= \Big(\frac{2z_1}{1+itz_n}, \cdots \frac{2z_{n-1}}{1+itz_n}; \frac{2z_n}{1+itz_n} + it\Big);$$
$$\Phi(z) = \lim_{t\to 1}\Phi_t(z) = \lim_{t\to 1}\Big(\frac{2z_1}{1+iz_n}, \cdots \frac{2z_{n-1}}{1+iz_n}; \frac{2z_n}{1+itz_n} + it\Big)$$
$$= \Big(\frac{2z_1}{1+iz_n}, \cdots, \frac{2z_{n-1}}{1+iz_n}; \frac{z_n + i}{1+iz_n}\Big) =: \mathfrak{c}(z).$$

Siegel domain representation of the complex unit ball

Write $\mathfrak{c}(z) =: \tau = (\tau_1, \cdots, \tau_n)$. As in the case of $n = 1$ we have $z_n = \frac{\tau_n - i}{1 - i\tau_n}$. For $1 \leq k \leq n-1$ we have $\tau_k = \frac{2z_k}{1+iz_n}$,

$$z_k = \frac{\tau_k}{2}(1 + iz_n) = \frac{\tau_k}{2}\left(1 + i\left(\frac{\tau_n - i}{1 - i\tau_n}\right)\right) = \frac{\tau_k}{1 - i\tau_n}.$$

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Under the inverse Cayley transform $\mathfrak{c}$ we have

$$\mathfrak{c}(\mathbb{B}^n) = \Big\{\tau = (\tau_1, \cdots, \tau_n) \in \mathbb{C}^n :$$

$$\left(\left|\frac{\tau_1}{1 - i\tau_n}\right|^2 + \cdots + \left|\frac{\tau_{n-1}}{1 - i\tau_n}\right|^2\right) + \left|\frac{\tau_n - i}{1 - i\tau_n}\right|^2 < 1\Big\}$$

$$= \Big\{\tau : (|\tau_1|^2 + \cdots |\tau_{n-1}|^2) + |\tau_n - i|^2 < |\tau_n + i|^2\Big\}$$

$$= \Big\{\tau : \mathrm{Im}(\tau_n) > \frac{1}{4}\left(|\tau_1|^2 + \cdots + |\tau_{n-1}|^2\right)\Big\} =: \mathscr{D}_n$$

The partial inverse Cayley transform on Ω

Write $\alpha = \frac{\partial}{\partial z_1}$ **and let** $\mathbb{C}^N = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$**,** $\mathcal{H}_\alpha \cong \mathbb{C}^p$**,** $\mathcal{N}_\alpha \cong \mathbb{C}^q$**, for the eigenspace decomposition corresponding to the eigenvalues** -2**,** -1 **resp.** 0 **of** $H_\alpha(\xi, \eta) := R_{\alpha\overline{\alpha}\xi\overline{\eta}}(0)$**.** We have $c = (i; 0, \cdots, 0; b)$.

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Lemma

Let $x \in \partial\Delta \times \{0\} \subset \mathrm{Reg}(\partial\Omega)$. Let Λ be a minimal rational curve passing through x. Then, $\Lambda \cap \Omega = \emptyset$ if and only if, writing $T_x(\Lambda) = \mathbb{C}\eta$, we have $\eta \in \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$.

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Proposition (Cayley projection)

Let $\Theta \subset \mathrm{Reg}(\partial\Omega)$ be a maximal boundary component. Then, for each point $z \in \Omega$ there exists a unique point $x \in \Theta$ such that $z \in \mathcal{V}_x$. Furthermore, writing $\varphi : \Omega \to \Theta$ for the continuous map defined by setting $\varphi(z) = x$ if and only if $z \in \Omega, x \in \Theta$ and $z \in \mathcal{V}_x$. Then, $\varphi : \Omega \to \Theta$ is a holomorphic submersion.

The partial inverse Cayley transform on Ω

Proposition (Cayley projection in Siegel coordinates)

Write $\mathscr{D} := \mathfrak{c}(\Omega) \subset \mathbb{C}^N$, *and let* $\varpi : \mathscr{D} \to \Omega'$ *be the Cayley projection map in Siegel coordinates,* $\varpi(z_1; z_2, \cdots, z_{p+1}; z_{p+2}, \cdots, z_N) := (z_{p+2}, \cdots, z_N)$. *Then,* $\mathcal{F}_b = \{(w_1, \cdots, w_{p+1}; b) : \mathrm{Im}(w_1) > \lambda(w_2, \cdots, w_{p+1}; b)\}$, *where* $\lambda : \mathbb{C}^p \times \Omega' \to \mathbb{R}$ *is a nonnegative real analytic function. Moreover, for each minimal rational curve* ℓ *on* $\widehat{\Omega}$ *such that* $\ell \cap \Theta$ *consists of a single point* $(i; 0; b) \in \Theta$, $\mathfrak{c}(\ell \cap \Omega)$ *is an upper half-plane given by*

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$$\mathfrak{c}(\ell \cap \Omega) = \big\{(w_1; a; b) \in \mathbb{C}\alpha \times \mathcal{H}_\alpha \times \Omega' :, \mathrm{Im}(w_1) > \lambda(a; b) \geq 0\big\}$$

for some $a = (a_2, \cdots, a_{p+1}) \in \mathbb{C}^p \cong \mathcal{H}_\alpha$ *and for some* $b \in \Omega'$*. Conversely, for each* $(a; b) \in \mathcal{H}_\alpha \times \Omega'$*,* $\mathscr{D} \cap (\mathbb{C} \times \{(a; b)\} = \mathfrak{c}(\ell \cap \Omega)$ *for some minimal rational curve* ℓ *passing through* $(i; 0; b) \in \Theta$*.*

The partial inverse Cayley transform on Ω via geometry

Denote by $\mathbf{O(2)}$, $\mathbf{O}(1)$ resp. $\mathbf{O}$ the restriction of $\mathcal{O}(2)$, $\mathcal{O}(1)$ resp. $\mathcal{O}$ to the minimal disk $\Lambda \cap \Omega \cong \Delta$, $T_\Omega\big|_{\Lambda\cap\Omega} = \mathbf{O}(2) \oplus \mathbf{O}(1)^p \oplus \mathbf{O}^q$.

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We have $(T_\Omega, g)\big|_\Lambda = (\mathbf{O}(2), h_2) \oplus (\mathbf{O}(1), h_1)^p \oplus (\mathbf{O}, h_0)^q$**, where** $(\mathbf{O}(2), h_2) \cong (\mathbf{O}(1), h_1)^{\otimes 2}$ **and** h_0 **is flat.**

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Geometric construction via dilatations

$$d\varphi_t(z_1; \mathbf{0}; \mathbf{0}) = \mathrm{diag}\left(\frac{1-t^2}{(1+itz_1)^2}; \frac{\sqrt{1-t^2}}{1+itz_1}, \cdots, \frac{\sqrt{1-t^2}}{1+itz_1}; 1, \cdots, 1\right);$$

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Denote by $\mathbf{O}(\mathbf{2})$, $\mathbf{O}(1)$ resp. $\mathbf{O}$ the restriction of $\mathcal{O}(2)$, $\mathcal{O}(1)$ resp. $\mathcal{O}$ to the minimal disk $\Lambda \cap \Omega \cong \Delta$, $T_\Omega\big|_{\Lambda\cap\Omega} = \mathbf{O}(2) \oplus \mathbf{O}(1)^p \oplus \mathbf{O}^q$.

We have $(T_\Omega, g)\big|_\Lambda = (\mathbf{O}(2), h_2) \oplus (\mathbf{O}(1), h_1)^p \oplus (\mathbf{O}, h_0)^q$**, where** $(\mathbf{O}(2), h_2) \cong (\mathbf{O}(1), h_1)^{\otimes 2}$ **and** h_0 **is flat.**

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We define now

$$\alpha_t(w) = \left(\frac{2(w_1+it)}{1-t^2} + it; \frac{2w_2}{\sqrt{1-t^2}}, \cdots \frac{2w_{p+1}}{\sqrt{1-t^2}}; 2w_{p+2}, \cdots, 2w_N\right);$$

$$\Phi_t(z) = \alpha_t(\varphi_t(s))$$

The partial inverse Cayley transform on Ω via geometry

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For each $t \in (0,1)$, $\Phi_t(\Lambda_\xi)$ is a minimal rational curve Λ_ξ^t passing through $c_t := \Phi_t(c) = \left(\frac{2i}{1-t}; 0, \cdots, 0; 0, \cdots, 0\right)$ such that $T_{c_t}\left(\Lambda_\xi^t\right) = \mathbb{C} d\Phi_t(c)(\alpha_\xi)$.

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$$
\begin{aligned}
d\Phi_t(c)(\alpha_\xi) &= \frac{2\alpha}{(1-t)^2} + \frac{2\xi}{1-t} + 2\mathfrak{q}(\xi) \\
&= \frac{2}{(1-t)^2}\left(\alpha + (1-t)\xi + (1-t)^2\mathfrak{q}(\xi)\right).
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$$= \frac{2}{(1-t)^2}\left(\alpha + (1-t)\xi + (1-t)^2\mathfrak{q}(\xi)\right).$$

Fix a point $x_\xi \neq c$ lying on the affine line $\Lambda_\xi \cap \mathbb{C}^N$. As $t \to 1$, $\Phi_t(z)$ converges to $\mathfrak{c}(z)$) and thus Λ_ξ^t converges to a minimal rational curve $\mathfrak{c}(\Lambda_\xi)$ passing through $\mathfrak{c}(x_\xi) \in \mathbb{C}^N$ such that $T_{\mathfrak{c}(x_\xi)} = \mathbb{C}\alpha$. In particular, all affine lines $\mathfrak{c}(\Lambda_\xi) \cap \mathbb{C}^N$ are parallel to Λ.

Write $\Omega \subset \widehat{\Omega}$ for the Borel embedding. Let $\pi : \Omega \to X_\Gamma := \Omega/\Gamma$ be the uniformization map, $Z \subset \Omega$ be an irreducible algebraic subset. Write $\mathscr{Z} \subset X_\Gamma$ for the Zariski closure of $\pi(Z)$ in X_Γ. **We have constructed $\mu_\Gamma : \mathcal{U}_\Gamma \to X_\Gamma$ arising from an irreducible component $\mathcal{U}$ universal family of the Chow scheme of the Hermitian symmetric space $\widehat{\Omega}$, by restriction to Ω and by taking quotients with respect to Γ.** (In the noncocompact case Mok-Zhong applies to prove quasi-projectivity of $\mathcal{U}_\Gamma$.)

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$(\mathcal{U}_\Gamma, \mathscr{F})$ is tautologically foliated. **Lift $\pi(Z)$ to $\mathcal{U}_\Gamma$, take its Zariski closure in $\mathcal{U}_\Gamma$, lift to $\mathcal{U}$ and project to Ω** to obtain an irreducible component $\widetilde{Z}$ of $\pi^{-1}(\mathscr{Z})$ containing Z. We have a multifoliated structure on some neighborhood of a general boundary point b on $\partial\widetilde{Z}$. Thus, there exists some open neighborhood U of b and a complex submanifold $S \subset U$ such that $\pi(S) \subset X_\Gamma$ contains a nonempty open subset of $\mathscr{Z}$ **in the complex topology**. For illustration consider the case where $\mathrm{rank}(\Omega) = 2$.

(1) Take a general boundary point $b \in \partial\widetilde{Z} \cap U$, $b \in \mathrm{Reg}(\partial\Omega)$. The point b lies on a unique boundary component Θ of rank 1 on $\mathrm{Reg}(\partial\Omega)$. Pick a minimal rational curve Λ passing through b such that $\Lambda \cap \Omega \neq \emptyset$, consider a one-parameter group $\{\varphi_t : -\infty < t < +\infty\}$ corresponding to a hyperbolic flow on the geodesic disk $D := \Lambda \cap \Omega$, fixing b (and any point on Θ) and pushing D to an opposite point $b' \in \Theta'$, an opposite boundary component on $\mathrm{Reg}(\partial\Omega)$.

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Rescaling Arguments for Ax-Lindemann-Weierstrass

(1) Take a general boundary point $b \in \partial\widetilde{Z} \cap U$, $b \in \mathrm{Reg}(\partial\Omega)$. The point b lies on a unique boundary component Θ of rank 1 on $\mathrm{Reg}(\partial\Omega)$. Pick a minimal rational curve Λ passing through b such that $\Lambda \cap \Omega \neq \emptyset$, consider a one-parameter group $\{\varphi_t : -\infty < t < +\infty\}$ corresponding to a hyperbolic flow on the geodesic disk $D := \Lambda \cap \Omega$, fixing b (and any point on Θ) and pushing D to an opposite point $b' \in \Theta'$, an opposite boundary component on $\mathrm{Reg}(\partial\Omega)$. Write $\check{\Gamma}$ for the image of $\pi_1(\mathcal{Z})$ in Γ. Applying estimates on intrinsic metrics, there exists $\gamma_n \in \check{\Gamma}$ such that $\gamma_n = \kappa_n \circ \varphi_n$ with $\{\kappa_n\}$ lying on a compact $Q \Subset G_0 := \mathrm{Aut}_0(\Omega)$. Then, $(\gamma_n)_*\widetilde{Z} = \widetilde{Z}$, $\kappa_{\sigma(n)} \to \kappa \in G_0$, and $\widetilde{Z} = (\gamma_n)_*\widetilde{Z} = (\kappa_n)_*(\varphi_n)_*\widetilde{Z}$ subconverges to $\kappa_* W$, where W is the limit of $(\varphi_n)_*\widetilde{Z}$. Near b, W decomposes into a disjoint union of holomorphic isometric copies of some $\mathbb{B}^m$.

(2) When $\widetilde{Z}$ is strictly pseudoconvex at b, rescaling gives a holomorphic isometric embedding of $\mathbb{B}^m$ into Ω, $m = \dim(\widetilde{Z})$. **By Mok (2012), $\widetilde{Z} \subset \Omega$ is algebraic, hence bi-algebraic, thus totally geodesic by Chan-Mok.** In general, $\widetilde{Z} = \kappa_* W$ decomposes into a disjoint union of images of holomorphic isometric embeddings of some $\mathbb{B}^m$.

(3) Recall that $\Theta \subset \mathrm{Reg}(\partial\Omega)$ is the boundary component passing through b, and $S \subset U$ analytically continues $\widetilde{Z}$ across $b \in \partial\widetilde{Z}$. In general S intersects Θ to give a complex analytic subvariety $E \subset \Theta \cap U$. We may assume that E is smooth at b and decompose $\widetilde{Z}$ near b into a disjoint union of nonsingular strictly pseudoconvex subsets $\widetilde{Z}_t$ parametrized holomorphically by $b_t \in E$.

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Sidney Frankel first made use of 1-parameter families of translations in the study of convex domains in $\mathbb{C}^N$ which cover compact complex manifolds.

The ALW Theorem on $X_\Gamma = \Omega/\Gamma$ for arbitrary cocompact Γ

Theorem (Ax-Lindemann-Weierstrass for cocompact Γ)

Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice. Write $X_\Gamma := \Omega/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map.

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Let $\widetilde{Z}$ be an irreducible component of $\pi^{-1}(Z)$. Assume wlog $0 \in \widetilde{Z}$. Denote by $\Omega' \subset \Omega$ the smallest totally geodesic complex submanifold containing $\widetilde{Z}$, $\Omega' \Subset \mathbb{C}^{N'}$ its Harish-Chandra realization. Starting with the real 1-parameter group $\Phi \subset G_0' := \mathrm{Aut}_0(\Omega')$ of translations and considering a maximal algebraic subgroup $H_0 \subset G_0'$ containing Φ, **we prove that $H_0 \subset G_0'$ is normal**. We claim that $H_0 = G_0'$, hence $\widetilde{Z} = \Omega'$, proving Thm. To this end we argue that $H_0 \neq G_0'$ would lead to a contradiction.

The assumption $H_0 \neq G_0'$ would allow us to enhance the dimension of leaves extending beyond $\partial\Omega'$ of some holomorphic foliation $\mathscr{F}$ defined on $U \cap \Omega'$ for some open neighborhood U of a good boundary point $p \in \partial\widetilde{Z} \subset \partial\Omega'$ on $\mathbb{C}^{N'}$. Applying the rescaling method for subvarieties at p, we would obtain an algebraic subgroup $H_0^\sharp \supsetneq H_0$ of G_0' contradicting the maximality of $H_0 \subset G_0'$ as an algebraic subgroup containing Φ.

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Strengthening of characterization of bialgebraic varieties replacing algebraicity on $Z \subset \Omega$ by an analytic condition

Theorem (Generalization of Chan-Mok (2022)) *Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization. Let $\Omega^\sharp \Supset \Omega$ be a bounded domain containing the topological closure $\overline{\Omega}$, $Z^\sharp \subset \Omega^\sharp$ be an irreducible complex-analytic subvariety, and $Z \subset \Omega$ be an irreducible component of $Z^\sharp \cap \Omega$.* **Suppose there exists a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}_0(\Omega)$ leaving Z invariant as a set such that $Y := Z/\check{\Gamma}$ is compact.** *Then, $Z \subset \Omega$ is a totally geodesic submanifold, hence $Y \hookrightarrow X_{\check{\Gamma}} := \Omega/\check{\Gamma}$ is a totally geodesic subset.*

Alles Gute zum Geburtstag,

Thomas!!