Calabi-Yau 3-folds from Algebraic Dynamics & $C_2(X)$

$f \in \mathrm{Bir}\,V$, $\mathrm{Aut}\,V$ $f^{\sim}$

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CY3 $\left[\begin{array}{l} \text{smooth proj 3-fold}/\mathbb{C} \\ \mathcal{O}_X(K_X) \simeq \mathcal{O}_X \\ \pi_1(X^{an}) = \{1\} \end{array}\right.$

<u>Question</u> (Dinh-Sibony 2000~2010)

Find many <u>interesting</u> autom $g \in \mathrm{Aut}\,V$ of smooth proj var V of <u>positive entropy</u>

<u>positive entropy</u> <u>$\dim V = d$</u>

H very ample on V $g \in \mathrm{Aut}\,V$

$$d_p(g) = \lim_{n\to\infty}\left(\int \underbrace{(g^n)}_{}{}^*(H^p)\cdot H^{d-p}\right)^{\frac{1}{n}}$$

$$= \left\| g^* | H^{p,p}(V) \right\| \quad \begin{matrix} \\ N^p(V) \end{matrix}$$

$g \in \mathrm{Aut}$

$(g^n)^* = (g^*)^n$

$$= \rho(g^* | H^{p,p}(V)) \quad \begin{matrix} \\ N^p(V) \end{matrix}$$

spectral radius

$h_{top}(g) = \log \max d_p(g) \geq 0$

$> 0 \iff$ <u>$d_1(g) > 1$</u>

Interesting

Def (De-Qi Zhang)

$g \in \mathrm{Aut}(V)$

g: imprimitive $\Leftrightarrow$
$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, dashed, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, dashed, "\pi"] \\ W \arrow[r, dashed, "g_W"'] & W \end{tikzcd}$$

$0 < \dim W < \dim V$

primitive $\Leftrightarrow$ NOT imprim.

"interesting"?

Notion relative dyn. degree $d_k(g|\pi)$

$d_p(g) = \max(d_k(g|\pi)\, d_{p-k}(g_W))$

[Dinh-Nguyễn. Truong]

$\dim V = 2$

$d_1(f) > 1 \Rightarrow f$: primitive

$\Rightarrow V \underset{\text{birat}}{\sim} \boxed{\mathbb{P}^2, K3, \text{ab.surf}, \text{Enriques}}$

Cantat

$\dim V \geq 3$

Ex $S = (2,2,2) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ K3

generic

$$\begin{tikzcd} S \arrow[d, "2:1"'] \\ \mathbb{P}^1 \times \mathbb{P}^1,\ \mathbb{P}^1 \times \mathbb{P}^1,\ \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

$\leftarrow \iota_1, \iota_2, \iota_3$

$\mathrm{Aut}\, S = \langle \iota_1, \iota_2, \iota_3 \rangle = \mathbb{Z}_2^{*3}$

$X = (2, \cdots, 2) \subset \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{d+1}$ $d \geq 3$

general

$\Rightarrow \mathrm{Aut}\, X = \{id\}$

$\mathrm{Bir}\, X = \langle \iota_1, \cdots, \iota_{d+1} \rangle = \mathbb{Z}_2^{*d+1}$

[Cantat-O_]

Thm (O– & Truong)

$\exists\, X_3$ CY3 with $g \in \mathrm{Aut}\, X_3$ primitive & pos. entropy.

$$\begin{tikzcd} X_3 \arrow[r] & \overline{X_3} & A_3 = E_{\zeta_3}^3 \arrow[l] \\ \text{crep. resol} \arrow[u, bend left] \arrow[ur, ""] & A_3/\langle g_3 \rangle \arrow[u, phantom, "\parallel"] & \end{tikzcd}$$

$g_3 = \begin{pmatrix} \zeta_3 & & 0 \\ & \zeta_3 & \\ 0 & & \zeta_3 \end{pmatrix}$

$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3a^2 & 0 \end{pmatrix}$ $a \in \mathbb{Z}_{>1}$

$d_1(g) \neq d_2(g) \Rightarrow$ primitive

prod form

So far only one ex for CY3

Thm (O–)

$$\begin{tikzcd} X_7 \arrow[r] & \overline{X_7} & A_7 \arrow[l] \\ \text{crep resol} \arrow[ur] & A_7/\langle g_7 \rangle \arrow[u, phantom, "\parallel"] & \mathrm{Jac}(XY^3+YZ^3+ZX^3=0) \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

$\circlearrowright\, g_7 = \begin{pmatrix} \zeta_7 & & 0 \\ & \zeta_7^2 & \\ 0 & & \zeta_7^4 \end{pmatrix}$

$3\mathbb{F}_2$

$\dot{\ }\frac{1}{7}(1,2,4)$

$1+g_7 \in \mathrm{End}(A)$

$\underbrace{1+g_7}_{= g \,\curvearrowright\, X_7} \in \mathrm{Aut}(A_7)$

$d_1(g) \neq d_2(g)$

The role of $c_2(X)$ in Aut(CY3)

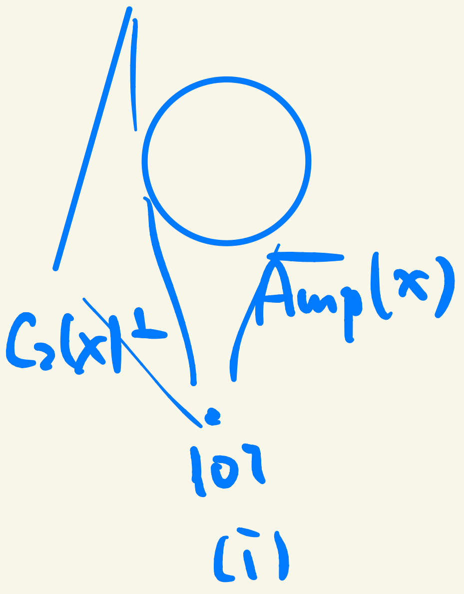
Thm (Miyaoka-Yau)

X CY3

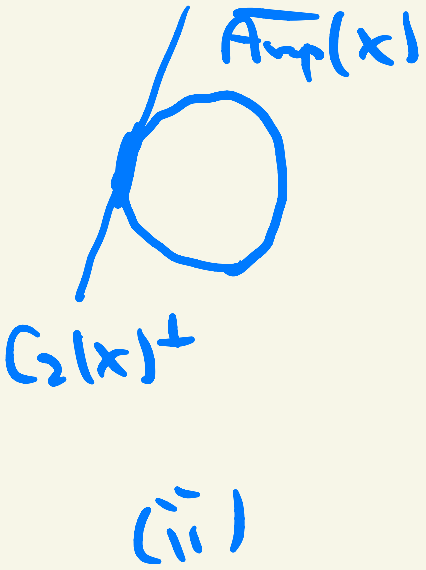
$\Rightarrow (c_2(X).*) > 0$ on $\mathrm{Amp}(X)$

$(c_2(X).*) \geq 0$ on $\overline{\mathrm{Amp}}(X)$

$N^1(X)_{\mathbb{R}}$



(i)



(ii)

Prop (Wilson)

If $|\mathrm{Aut}(X)| = \infty$

$\Rightarrow$ (ii)

Def (O- & Sakurai) $\dim W \geq 1$

X CY3 $\quad X \xrightarrow{f} W \leftarrow$ proj normal

c_2-contr $\Leftrightarrow (c_2(X).f^*N^1(W)) = 0$

$\Leftrightarrow c_2(X).f^*H_W = 0$

$\exists H_W$ ample on W

(W: $\mathbb{Q}$-fact.)

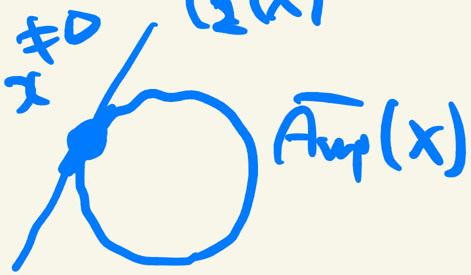
$\Rightarrow \exists! \; X \xrightarrow{f} W$ max c_2-contr.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\forall f'"'] & W \arrow[dl, "\exists!"] \\ W' & \end{tikzcd}$$

$W' \leftarrow c_2$-contr

Speculation

(ii) happens



⇒ ?? $\exists\, \underset{0}{\underset{\#}{x}} \in C_2(X)^\perp \cap \overline{\mathrm{Amp}}(X)(\mathbb{Q})$

true if $\rho \leq 3$
(Lazic-Peternell-O)

⇒ x : semi-ample ↑ $|\mathrm{Aut}| = \infty$

non-vanishing conj

⇒ $X \longrightarrow W_0$ $\exists\, C_2$-contr.

⇒

$$\begin{tikzcd} X \arrow[r] & W \\ \circlearrowleft & \circlearrowleft \\ \mathrm{Aut}X & \mathrm{Aut}X \end{tikzcd}$$

max C_2-contr.

⇒ non-prim <u>unless</u> $\dim W = 3$

Thm^{A} (O- & Sakurai)

$X \xrightarrow[f]{} W$ C_2-contr.

⇒ 1) $\dim W = 1$ $W = \mathbb{P}^1$ & abelian fib.

2) $\dim W = 2$

$\exists\, S$ K3 or ab. $G \curvearrowright S \times \overset{\text{ell}}{\overset{\downarrow}{E}}$

$\exists\, E$ ell. curve Gorenstein

$G \curvearrowright \bar{S} \leftarrow S$ $\exists$ contr

s.t. $W = \bar{S}/G$ &

$$\begin{tikzcd} X \arrow[r, "\sim"', "\text{bir}"] \arrow[dd] & G\text{-Hilb}(S \times E) \arrow[d] \\ & S \times E/G \arrow[d] \\ W \arrow[r, phantom, "="] & \bar{S}/G \end{tikzcd}$$

3) $\dim W = 3$

$$\Rightarrow (X \to W) \cong (X_3 \to \overline{X_3})$$

or

$$(X_7 \to \overline{X_7})$$

$\swarrow$ CY3

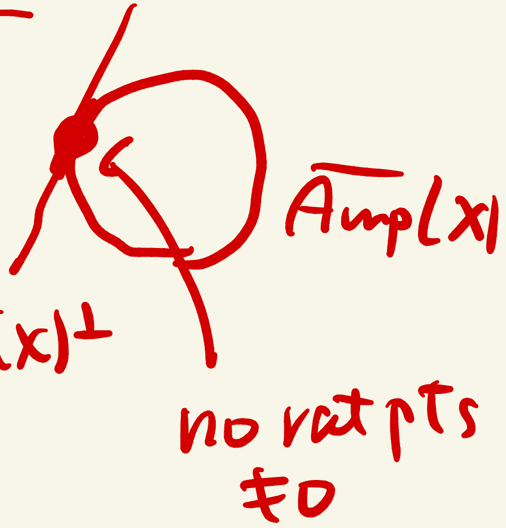
Conj X has prim. autom of pos. entropy

$\Rightarrow X \cong X_3$ or X_7 ?

Conj not True

$\Rightarrow$ either

$\exists$



or

$\exists$ non-vanishing

conj is not True

"Imprimitive one"

<u>Thm</u> (O– 2023) X $c\geq 3$

1) $X \xrightarrow{f} \mathbb{P}^1$ ab. fibered

$$\begin{tikzcd} X \arrow[r, "\exists g"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[r, "g_{\mathbb{P}^1}"'] & \mathbb{P}^1 \end{tikzcd}$$

$d_1(g) > 1$

$\Rightarrow (X \xrightarrow{f} \mathbb{P}^1) \cong (X_3 \xrightarrow{pr_3} E^3/\langle \zeta \rangle)$

$$E^3/\langle\zeta\rangle \;\|\; \mathbb{P}^1$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

2) $X \xrightarrow{f} W$ ell. fibered

$$\begin{tikzcd} X \arrow[r, "\exists g"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ W \arrow[r, "g_W"'] & W \end{tikzcd}$$

$d_1(g) > 1$

$$\Rightarrow \underbrace{X \to W \to \overline{W}}_{\overline{f}}$$

$\overline{f}$ is as in Thm A(2)

<u>Q1</u> How about $X \xrightarrow{f} W$ K3 fibered?

<u>Q2</u> $A_7 / \left\langle \begin{pmatrix} \zeta_7 & & 0 \\ & \zeta_7^2 & \\ 0 & & \zeta_7^4 \end{pmatrix} \right\rangle$ (in rat conn.) rational?