On a conjecture by Campana-Peternell (about Kodaira dimension)

Christian Schnell | Stony Brook University

The abundance conjecture

Let X be a smoooth projective variety (over $\mathbb{C}$).

The Kodaira dimension $\kappa(X)$ measures the rate of growth of the spaces of pluricanonical divisors:

$$\dim H^0(X, mK_X) \sim m^{\kappa(X)}$$

The numerical Kodaira dimension $\nu(X)$ is defined as

$$\dim H^0(X, mK_X + A) \sim m^{\nu(X)},$$

where A is any sufficiently ample divisor on X.

Clearly, $\kappa(X) \leq \nu(X)$.

Abundance Conjecture

One always has $\kappa(X) = \nu(X)$.

The nonvanishing conjecture

The most important case of the abundance conjecture:

Nonvanishing Conjecture

If K_X is pseudo-effective (psef), then $\kappa(X) \geq 0$.

Abundance implies nonvanishing:

- Suppose that K_X is psef.
- Then $mK_X + A$ is effective for A sufficiently ample.
- Therefore $\kappa(X) = \nu(X) \geq 0$.

According to Boucksom-Demailly-Păun-Peternell,

$$K_X \text{ is psef} \quad \Longleftrightarrow \quad X \text{ is not uniruled}$$

Note. Hashizume has shown that this basic version implies the more technical version (for lc pairs).

The Campana-Peternell conjecture

In 2011, Campana and Peternell proposed the following variant of the nonvanishing conjecture.

Campana-Peternell Conjecture

Let D be an effective divisor on a smooth projective variety X. If $mK_X - D$ is psef for some $m \geq 1$, then $\kappa(X) \geq \kappa(X, D)$.

In other words, the Kodaira dimension of X should be at least as big as the Iitaka dimension of D:

$$\dim H^0(X, mD) \sim m^{\kappa(X,D)}$$

The Campana-Peternell conjecture

- contains the nonvanishing conjecture (for $D = 0$)
- is implied by the abundance conjecture.

Relation with abundance

Another candidate for the Kodaira dimension:

$$\mu(X) = \max\left\{ \kappa(X, D) \;\middle|\; mK_X - D \text{ psef for some } m \geq 1 \right\}$$

We have the following inequalities:

$$\kappa(X) \leq \mu(X) \leq \nu(X)$$

Proof of the second inequality:

- Suppose that $mK_X = D + E$, with E psef.
- For A sufficiently ample, this gives

$$mrK_X + A = rD + (rE + A),$$

  and $rE + A$ is effective for all $r \geq 1$.
- It follows that $\kappa(X, D) \leq \nu(X)$.

Goal of the talk

Split the Campana-Peternell conjecture into two parts:

- The nonvanishing conjecture (very hard)
- A new conjecture about certain algebraic fiber spaces (more tractable)

I am going to explain

- what the new conjecture is,
- how to prove it in certain cases.

The main tool is singular metrics.

Part I

Applications of the Campana-Peternell conjecture

Viehweg's hyperbolicity conjecture

The Campana-Peternell conjecture is used in the proof of Viehweg's hyperbolicity conjecture (by Viehweg-Zuo, Campana-Peternell, Campana-Păun, Popa-Schnell).

Theorem (in a special case)

Let $f: X \to Y$ be a smooth algebraic fiber space with fibers of general type. If Y is not uniruled, then the Campana-Peternell conjecture implies the inequality $\kappa(Y) \geq \mathrm{var}(f)$.

When f has maximal variation, Y is of general type (unconditionally).

Viehweg's hyperbolicity conjecture

In general, one needs the Campana-Peternell conjecture:

- Using Hodge theory, one gets an exact sequence

$$0 \to L \to (\Omega_Y^1)^{\otimes N} \to Q \to 0,$$

  with $\kappa(Y, L) \geq \mathrm{var}(f)$.
- $\det Q$ is psef (Campana-Peternell, Campana-Păun).
- Therefore $mK_Y \equiv L + \det Q$.
- The Campana-Peternell conjecture gives

$$\kappa(Y) \geq \kappa(Y, L) \geq \mathrm{var}(f).$$

Part II

A conjecture about fiber spaces

A conjecture about fiber spaces

From the Campana-Peternell conjecture, one can extract a part that is independent of the nonvanishing conjecture.

Conjecture A

Let $f\colon X \to Y$ be an algebraic fiber space with $\kappa(F) \geq 0$. Let H be an ample divisor on Y. If $mK_X - f^*H$ is psef for some $m \geq 1$, then $mK_X - f^*H$ becomes effective for $m \gg 0$.

What is the point?

- The nonvanishing conjecture is very hard.
- Conjecture A is the part of the Campana-Peternell conjecture that looks doable with existing techniques.

Deriving Conjecture A

Let me sketch the proof. Suppose that $mK_X - D$ is psef.

Step 1. We may assume that D is base-point free.

- Choose $n \gg 0$ so that $|nD|$ gives the Iitaka fibration.
- Let $\mu\colon X' \to X$ be a resolution of the linear system $|nD|$.
- Then $\mu^*|nD| = |G| + E$, with E effective and G free.
- It follows that

  $$mnK_{X'} - G \equiv n \cdot \mu^*(mK_X - D) + nmK_{X'/X} + E$$

  is still psef.
- The Campana-Peternell conjecture (for G) implies that

  $$\kappa(X) = \kappa(X') \geq \kappa(X', G) = \kappa(X, D).$$

Step 2. We may assume that $D = f^*H$, where $f\colon X \to Y$ is an algebraic fiber space, and H is ample on Y.

Deriving Conjecture A

Step 3. Let F be the general fiber of $f\colon X \to Y$. Now the Campana-Peternell conjecture actually predicts that

$$\kappa(X) = \kappa(F) + \dim Y.$$

Equivalently, $mK_X - f^*H$ is effective for $m \gg 0$ (Mori).

- From $mK_X - f^*H$ psef, we get K_F psef.
- The nonvanishing conjecture implies $\kappa(F) \geq 0$.
- Pick $r \geq 1$ so that rK_F has sections.
- Therefore $f_*\mathscr{O}_X(rK_X) \otimes \mathscr{O}_Y(\ell H)$ has sections for $\ell \gg 0$.
- Because $rK_X + f^*(\ell H) \geq f^*H$, we get

$$\kappa\Big(X, rK_X + f^*(\ell H)\Big) = \kappa(F) + \dim Y.$$

- But $(m\ell + r)K_X - \Big(rK_X + f^*(\ell H)\Big) = \ell(mK_X - f^*H)$ is psef, and so the Campana-Peternell conjecture implies

$$\kappa(X) \geq \kappa\Big(X, rK_X + f^*(\ell H)\Big) = \kappa(F) + \dim Y.$$

- The converse is the easy addition formula.

A conjecture about fiber spaces

In this way, we arrive at the following statement.

Conjecture A

Let $f\colon X \to Y$ be an algebraic fiber space with $\kappa(F) \geq 0$. Let H be an ample divisor on Y. If $mK_X - f^*H$ is psef for some $m \geq 1$, then $mK_X - f^*H$ becomes effective for $m \gg 0$.

In fact, the following two things are equivalent:

1. The Campana-Peternell conjecture
2. The nonvanishing conjecture and Conjecture A

Part III

Proof of the conjecture (in some cases)

The main result

Notation.

- $f: X \to Y$ is an algebraic fiber space with $\kappa(F) \geq 0$
- H is an ample divisor on Y.

I am going to sketch the proof of the following result.

Theorem B

Assume that Y is not uniruled. If $mK_X - f^*H$ is psef for some $m \geq 1$, then $mK_X - f^*H$ becomes effective for $m \gg 0$.

This proves Conjecture A under the assumption that

$$Y \text{ is not uniruled} \quad \iff \quad K_Y \text{ is psef.}$$

Two interesting aspects:

- Singular metrics on pluri-adjoint bundles
- How does adding multiples of K_X make things better?

Sketch of the proof

Here is the idea of the proof.

- Fix $r \geq 1$ such that rK_F has sections.
- Suppose that $m_0 K_X - f^*H$ is psef for some $m_0 \geq 1$.
- The divisor

$$L = (k + \ell + 1)(m_0 K_X - f^*H)$$

  is psef for every $k, \ell \geq 1$.
- By putting things together correctly, we get

$$f_*\mathscr{O}_X(mrK_X - f^*H) \cong f_*\mathscr{O}_X(K_X + L_n) \otimes \mathscr{O}_Y(kH) \otimes \mathscr{O}_Y(nK_Y + \ell H)$$

  where $n = mr - (k + \ell + 1)m_0 - 1$ and $L_n = nK_{X/Y} + L$.
- In fact, we have

$$K_X + L_n = mrK_X - f^*\Big(nK_Y - (k + \ell + 1)H\Big).$$

Sketch of the proof

- Because mrK_F is effective, the sheaf

$$f_*\mathscr{O}_X(K_X + L_n)$$

is torsion-free of generic rank $\dim H^0(F, mrK_F)$.
- By the work of Păun-Takayama, L_n has a singular metric h_n with semi-positive curvature.
- It induces a singular metric on

$$f_*\Big(\mathscr{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n)\Big),$$

with semi-positive curvature (in the sense of Griffiths).
- For $m \gg 0$, the inclusion

$$f_*\Big(\mathscr{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n)\Big) \subseteq f_*\mathscr{O}_X(K_X + L_n)$$

is generically an isomorphism.
- In particular, the sheaf on the left is nontrivial.

Sketch of the proof

- There is a Kollár-type vanishing theorem for the sheaf
$$f_*\Big(\mathscr{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n)\Big),$$
proved by Fujino-Matsumura.
- This leads to an effective nonvanishing theorem:
$$f_*\Big(\mathscr{O}_X(K_X + L_n) \otimes \mathcal{I}(h_n)\Big) \otimes \mathscr{O}_Y(kH)$$
has sections for some $1 \leq k \leq \dim Y + 1$
- Y is not uniruled, so K_Y is psef.
- For suitable $\ell \geq 1$, the divisor $nK_Y + \ell H$ is therefore effective for every $n \geq 1$.
- The conclusion is that
$$f_*\mathscr{O}_X(mrK_X - f^*H) \cong$$
$$f_*\mathscr{O}_X(K_X + L_n) \otimes \mathscr{O}_Y(kH) \otimes \mathscr{O}_Y(nK_Y + \ell H)$$
has sections for $m \gg 0$.

Singular hermitian metrics

Let L be a line bundle on a complex manifold X.

When X is projective, the following things are equivalent:

- L is psef ($=$ in the closure of the effective cone).
- L has a singular metric with semi-positive curvature.

This was proved by Demailly.

Singular hermitian metrics

Two ingredients for constructing singular metrics:

1. Global sections
2. Length function

A section $s \in H^0(X, L)$ induces a singular metric h on L:

- Declare that $|s|_h = 1$; singular where $s = 0$.
- In a local trivialization, $s = g\, s_0$, and $|s_0|_h^2 = |g|^{-2}$.
- The local weight function is plurisubharmonic (psh)

$$|s_0|_h^2 = e^{-\varphi} \quad \text{where} \quad \varphi = \log|g|^2.$$

- This is the definition of semi-positive curvature.

Note. The metric depends on the section (or sections).

Singular hermitian metrics

To get something intrinsic, we need a length function.

- Assume that X is compact.
- Consider $V = H^0(X, L)$.
- A continuous function $\ell \colon V \to [0, +\infty]$ such that

$$\ell(\lambda v) = |\lambda| \, \ell(v) \quad \text{and} \quad \ell(v) = 0 \Leftrightarrow v = 0$$

  is called a length function.

We can then define a singular metric h by the rule

$$|\xi|_{h,x} = \inf\left\{ \ell(v) \;\middle|\; v \in V \text{ satisfies } v(x) = \xi \right\}.$$

In a local trivialization, $v = g_v \, s_0$, with g_v holomorphic.

The local weight function is

$$\varphi = \hat{\sup}\left\{ \log|g_v|^2 \;\middle|\; v \in V \text{ satisfies } \ell(v) = 1 \right\}.$$

This is psh, with singularities along the base locus of V.

Singular hermitian metrics

When X is compact, a singular metric h on L gives a (singular) inner product on $H^0(X, K_X + L)$:

- Locally, $v = g\, s_0 \otimes (dx_1 \wedge \cdots \wedge dx_n)$ and $|s_0|_h^2 = e^{-\varphi}$.
- The formula for the inner product norm is

$$\|v\|^2 = \int_X |g|^2 e^{-\varphi} d\mu \in [0, +\infty].$$

- This is finite on the subspace

$$H^0(X, \mathscr{O}_X(K_X + L) \otimes \mathcal{I}(h)),$$

  where $\mathcal{I}(h)$ is the multiplier ideal sheaf.

Results by Păun-Takayama

Let L be a psef line bundle.

- L admits a singular metric h with semi-positive curvature.
- No control over the singularities!

Let $f\colon X \to Y$ be an algebraic fiber space.

- By the work of Păun-Takayama, $L_m = mK_{X/Y} + L$ has a singular metric h_m with semi-positive curvature.
- It induces a singular metric on

$$f_*\Big(\mathscr{O}_X(K_X + L_m) \otimes \mathcal{I}(h_m)\Big),$$

  with semi-positive curvature (in the sense of Griffiths).
- For $m \gg 0$, the inclusion

$$f_*\Big(\mathscr{O}_X(K_X + L_m) \otimes \mathcal{I}(h_m)\Big) \subseteq f_*\mathscr{O}_X(K_X + L_m)$$

  is generically an isomorphism.

Results by Păun-Takayama

This is a relative version of the following construction.

Consider $L_m = mK_X + L$.

If $H^0(X, L_m) \neq 0$, the line bundle L_m inherits a singular metric, called the generalized Narasimhan-Simha metric:

- Each $v \in H^0(X, L_m)$ has a length $\ell(v) \in [0, +\infty]$:

$$\ell(v)^{2/m} = \int_X |g|^{2/m} e^{-\varphi/m} d\mu$$

- Locally, $v = g\, s_0 \otimes (dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$ and $|s_0|_h^2 = e^{-\varphi}$.
- This length function puts a singular metric h_m on L_m.
- The metric $h_m^{(m-1)/m} h^{1/m}$ induces an inner product on

$$H^0(X, L_m) = H^0(X, K_X + (m-1)K_X + L).$$

- One has $\|v\| \leq \ell(v)$.

Key point. For $m \gg 0$, all sections have finite length. The reason is that $e^{-\varphi/m}$ becomes locally integrable for $m \gg 0$.

An open problem

We have proved Conjecture A when Y is not uniruled.

Conjecture A

Let $f\colon X \to Y$ be an algebraic fiber space with $\kappa(F) \geq 0$. Let H be an ample divisor on Y. If $mK_X - f^*H$ is psef for some $m \geq 1$, then $mK_X - f^*H$ becomes effective for $m \gg 0$.

What is left is the case where Y is rationally connected, using the maximally rationally connected (MRC) fibration.

Sadly, even the case $Y = \mathbb{P}^1$ is open!

Conjecture A over $\mathbb{P}^1$

Let $f\colon X \to \mathbb{P}^1$ be an algebraic fiber space with $\kappa(F) \geq 0$. If $mK_X - f^*\mathscr{O}(1)$ is psef for some $m \geq 1$, then $mK_X - f^*\mathscr{O}(1)$ becomes effective for $m \gg 0$.

Thank you!