Very Ampleness Part of Fujita Conjecture and Multiplier Ideal Sheaves of Higher Order

by Yum-Tong Siu (Harvard)

Conference on Transcendental Aspects of Algebraic Geometry
from July 1 to 5, 2024 in Grand Hotel San Michele, Cetraro, Italy
to Celebrate the 70th Birthday of Thomas Peternell

Abstract: There are two parts in the Fujita conjecture. One is the freeness part which since 1995 has been proved with various weaker bounds.

The other is the very ampleness part, which remains open even with weaker bounds.

In this talk we will present a proof of the very ampleness part of the Fujita conjecture with a weaker bound by the method of multiplier ideal sheaves of higher order and other new techniques.

Statement of Fujita Conjecture (1987). Let X be a compact complex manifold of complex dimension n and L be a holomorphic line bundle which is positive (*i.e.,* ample) in the sense that there exists a smooth metric $e^{-\varphi}$ for the fibers of L with the complex Hessian $\partial\bar{\partial}\varphi$ strictly positive at every point of X.

Freeness Part. If $m \geq n+1$, then $mL+K_X$ is globally free on X in the sense that for every point P in X some global holomorphic section of $mL+K_X$ on X is nonzero at P.

Very Ampleness Part. If $m \geq n+2$, then $mL+K_X$ is very ample on X in the sense that a $\mathbb{C}$-basis of the section space $\Gamma(X, mL+K_X)$ can be used as homogeneous components for a holomorphic embedding of X into $\mathbb{P}_N$, where $N+1$ is the complex dimension of $\Gamma(X, mL+K_X)$.

One reason for the conjectured bounds is that the bad case is when K_X is negative, with the worst situation of $K_X = \mathcal{O}_{\mathbb{P}_n}(-(n+1))$ when $X = \mathbb{P}_n$ and $L = \mathcal{O}_{\mathbb{P}_n}(1)$, in which the smallest m is $n+1$ and $n+2$ in both parts.

Known Solution of Freeness Part with Weaker Bound. Over the years many mathematicians worked on the freeness part of the Fujita conjecture and obtained results for low dimensions.

For general dimension, Angehrn-Siu (1995) obtained freeness of $mL + K_X$ for $m \geq \frac{1}{2}\,n(n+1)+1$.

Heier (2002) improved the bound to $m \geq \left(e+\frac{1}{2}\right) n^{\frac{4}{3}} + \frac{1}{2}n^{\frac{2}{3}} + 1$.

Work on Very Ampleness Part. For the double adjoint situation, the very amplenss of $mL + 2K_X$ with an effective lower bound for m in terms of n was first done by Demailly, using the Monge-Ampère equation in 1992 and then redone by Siu a little later with vanishing theorems.

The very ampleness part of $mL + K_X$ (in original single adjoint situation) in the Fujita conjecture for general dimension, even for weaker bounds, is still open.

My paper in the 1999 Ohio Conference Proceedings, I outlined a method for very ampleness of $mL + K_X$ for some value of m depending only on n but gave the details only for the case $n = 2$. Here we present the proof of the case of general dimension.

We will start out with the general techniques for handling such problems by producing global sections by L^2 estimates.

Then we explain the hitherto-insurmountable difficulties of using such techniques to prove the very ampleness part.

Finally, we tell you the new ways to overcome the difficulties.

Vanishing Theorems from L^2 Estimates. The starting point is Cramer's rule for solving a finite number of linear equations with a finite number of unknowns. In matrix notations, to solve for the unknown x in $Tx = b$ with $Sb = 0$, the minimum solution is $x_{\min} = T^*(TT^* + S^*S)^{-1}b$, which is reduced to the usual Cramer's rule $x = T^{-1}b$ in the case of no compatibility condition $S = 0$. The key point is the invertibility of $TT^* + S^*S$.

For the case of a holomorphic line bundle L with smooth metric $e^{-\varphi_L}$ of strictly positive Hessian $\partial\bar{\partial}\varphi_L$ on a compact complex algebraic manifold X, the vanishing theorem for $H^1(X, L + K_X)$ comes from the L^2 estimate

$$\|T^*g\|^2_{\varphi_L} + \|Sg\|^2_{\varphi_L} \geq c\|g\|^2_{\varphi_L}$$

for some $c > 0$ (where T and S are densely defined $\bar\partial$ operators on test L-valued $(n,1)$-form g on X) to give the solvability for u in $Tu = f$ for given $\bar\partial$-closed L-valued $(n,1)$-form f with the estimate

$$\|u\|_{\varphi_L} \leq \frac{1}{\sqrt{c}}\,\|f\|_{\varphi_L}.$$

The metric $e^{-\varphi_L}$ for L defines the norm $\|g\|_{\varphi_L}$ for L-valued $(n,1)$-form g, which is a section of $L + K_X$. That is the reason why the canonical line bundle K_X comes in. The vanishing of $H^p(X, L+K_X)$ for $p \geq 1$ is Kodaira's vanishing theorem (1954).

When φ in the metric $e^{-\varphi}$ of L is only plurisubharmonic (and not smooth) with the complex Hessian $\partial\bar\partial\varphi$ dominating a smooth strictly positive $(1,1)$-form on X in the sense of generalized functions, the vanishing of

$$H^p(X, \mathcal{I}_\varphi(L+K_X)) \quad \text{for} \quad p \geq 1$$

holds, where $\mathcal{I}_\varphi$ is the multiplier ideal sheaf of $e^{-\varphi}$ defined as the set of all local holomorphic function germs f with $|f|^2 e^{-\varphi}$ locally integrable. This is known as Nadel's vanishing theorem (1989). In algebraic geometric setting it is known as the vanishing theorem of Kawamata-Viehweg (1982).

Global Sections from Vanishing Theorems for Metric with Desired Singularity. Let V_φ be the multiplier ideal subspace for $e^{-\varphi}$ whose structure sheaf is $\mathcal{O}_X / \mathcal{I}_\varphi$. Then the vanishing of $H^1(X, \mathcal{I}_\varphi(L+K_X))$ implies that any holomorphic section of $L+K_X$ defined on V_φ can be extended to a global holomorphic section of $L+K_X$ on X. In particular, if for every point P_0 there exists such a metric $e^{-\varphi_{P_0}}$ with $V_{\varphi_{P_0}}$ isolated at P_0, then $L+K_X$ is globally free.

The proof of Kodaira's embedding theorem locally constructs $e^{-\psi_{P_0,q}}$ with

$$\psi_{P_0,q} = \frac{1}{|z - z(P_0)|^{n+q}}$$

so that $\mathcal{I}_{\psi_{P_0},q} = \mathfrak{m}^q_{X,P_0}$ and then uses a partition of unity to get a metric $e^{-\psi}$ for mL with a sufficiently large, *noneffective* m which is equal to $\psi_{P_0,q}$ for prescribed points P_0 and $q = 1, 2$. This means that there are global sections of $mL + K_X$ assuming prescribed q-jets at points P_0 and $mL + K_X$ is very ample. Kodaira's original paper uses the blowing up of P_0 instead of

$$\psi_{P_0,q} = \frac{1}{|z - z(P_0)|^{n+q}}.$$

Metric from Multi-Valued Global Sections. For an ample line bundle L over X of complex dimension n, by the theorem of Riemann-Roch

$$\dim_{\mathbb{C}} \Gamma(X, mL) = \frac{c_1(L)^n}{n!} m^n + O(m^{n-1}) \geq \frac{m^n}{n!} + O(m^{n-1}) \quad \text{as} \quad m \to \infty.$$

Because of the lower order terms in $O(m^{n-1})$, we cannot get a global section with vanishing order $\geq n + q$ at P_0 for an *effective* m. However, for any rational $\varepsilon > 0$ and P_0 we can get an element s_N in $\Gamma(X, N(q + \varepsilon)L)$ with vanishing order $\geq Nq$. The *multivalued* global holomorphic section $(s_N)^{\frac{1}{N}}$ of $(q + \varepsilon)L$ would have vanishing order $\geq q$ at P_0. It defines a metric

$$\frac{1}{\left|(s_N)^{\frac{1}{N}}\right|^2}$$

for the $\mathbb{Q}$-bundle $(q + \varepsilon)L$.

In general, for multi-valued sections $s_1, \cdots, s_k$ of $\mathbb{Q}$-bundles $\alpha_1 L, \cdots, \alpha_k L$ respectively with (fractional) vanishing order $\geq q$, we have a metric

$$\frac{1}{\sum_{j=1}^k |\sigma_j s_j|^2}$$

of αL with $\alpha = \max(\alpha_1, \cdots, \alpha_k)$, where σ_j is a nowhere zero multi-valued section of $(\alpha - \alpha_j)L$.

To guarantee the condition of dominating a smooth positive $(1,1)$-form for the complex Hessian, for any $\delta > 0$ we can use the metric

$$\frac{e^{-\delta \varphi_L}}{\sum_{j=1}^k |\sigma_j s_j|^2}$$

of $(\alpha + \delta)L$ for some smooth metric

$$e^{-\varphi_L} = \frac{1}{\sum_\ell |\tau_\ell|^2}$$

of the ample line bundle L with strictly positive curvature (where τ_ℓ is a multivalued section of L).

We can also take the metric raised to a (fractional) β power to get a metric of $\beta(\alpha+\delta)L$. A technique of Shokurov (1985) of slightly perturbing the metric $e^{-\varphi}$ to make an appropriate component of V_φ irreducible.

For the freeness part, the goal is to make the multiplier subspace isolated at a prescribed point.

For the very ampleness part, the multiplier subspace is not only required to be isolated at prescribed points, but should also have higher-order multiplicity there.

Cutting Down Dimension of Multiplier Subspace by Extending Multi-Valued Sections from Subspace. For the metric

$$e^{-\varphi} = \frac{1}{\sum_j |s_j|^2}$$

of αL defined by multivalued sections s_j of αL, to cut down the positive dimension of V_φ at P_0, by using the theorem of Riemann-Roch on V_φ, one can use multivalued sections t_ℓ of βL on V_φ with a prescribed vanishing order at P_0 and extend t_ℓ to a multivalued section $\hat{t}_\ell$ of βL on X to form a new metric

$$\frac{1}{\sum_j |s_j|^{\frac{2\gamma}{\alpha}} + \sum_\ell |\hat{t}_\ell|^{\frac{2\gamma}{\beta}}}$$

of γL for some $\gamma \geq \max(\alpha, \beta)$.

Note that, unlike a global sections, a global multivalued section of an ample $\mathbb{Q}$-line bundle can always be extended to all of X by raising it first to a sufficiently high power.

In order to make β (and other similar numbers) effective, in the use of the theorem of Riemann-Roch on V_φ, one needs an effective bound on the multiplicity of V_φ at P_0.

Such a multiplicity is not under control, as one can see from the following computation of the multiplier ideal sheaf of a metric defined by global multi-valued sections, because the multiplicity of $\pi(E_j)$ given and defined below is not under control.

Multiplier Ideal Sheaf from Normal-Crossing Hypersurfaces after Monoidal Transformations. Let $\mathcal{J}$ be a coherent ideals sheaf on a local complex manifold X and $N \in \mathbb{N}$. Consider the metric

$$e^{-\varphi} = \frac{1}{|\mathcal{J}|^{\frac{2}{N}}},$$

where $|\mathcal{J}|^2$ means the sum of the absolute value squares of any set of local holomorphic generators of the ideal sheaf $\mathcal{J}$.

Let $\pi : \tilde{X} \to X$ be a holomorphic modification by a finite number of successive monoidal transformations with nonsingular centers and $\{E_\mu\}_\mu$ be a family of nonsingular complex hypersurfaces in $\tilde{X}$ with normal crossing such that

(i) $\pi^*\mathcal{J}$ is equal to the ideal sheaf $\sum_\mu r^*_\mu E_\mu$ for some nonnegative integer r^*_μ and

(ii) $K_{\tilde{X}} - \pi^* K_X = \sum_\mu b_\mu E_\mu$ for some nonnegative integer b_μ.

Let $r_\mu = \frac{r^*_\mu}{N}$. Let $\mathcal{I}d(W)$ denote the ideal sheaf of a subvariety W. For $r \in \mathbb{R}$ let $\lfloor r \rfloor$ be the largest integer not exceeding r. Then the multiplier ideal sheaf $\mathcal{I}_\varphi$ is

$$\bigcap_\mu \mathcal{I}d(\lfloor r_\mu - b_\mu \rfloor \ \pi(E_\mu)),$$

which means that $f \in \mathcal{O}_X$ belongs to the multiplier ideal sheaf $\mathcal{I}_h$ if and only if f vanishes at least to the order

$$\lfloor r_\mu - b_\mu \rfloor$$

across the irreducible subvariety E_μ for every μ.

Difficulty is the *noneffective* multiplicity of $\pi(E_\mu)$ at the point under consideration, as mentioned earlier, to be handled by the following semicontinuity argument for multiplier ideal sheaves.

Semicontinuity of Multiplier Ideal Sheaf from Ohsawa-Takigoshi-Type Extension. Let P_0 be the point of the multiplier subspace V_φ under consideration, where

$$e^{-\varphi} = \frac{1}{\sum_j |s_j|^2}$$

is a metric of αL defined by multivalued sections s_j of αL. At a regular point P of the multiplier subspace V_φ, choose a multivalued section s_P of βL on V_φ whose extension to X vanishes to order $\geq n+q$ at P, where m and q are effective. One adds s_P to form a new metric

$$e^{-\varphi_P} = \frac{1}{\sum_j |s_j|^{\frac{2\gamma}{\alpha}} + \sum_\ell |s_P|^{\frac{2\gamma}{\beta}}}$$

with $\gamma = \max(\alpha, \beta)$. As $P \to P_0$, one ends up with a metric

$$e^{-\varphi_{P_0}} = \frac{1}{\sum_j |s_j|^{\frac{2\gamma}{\alpha}} + \sum_\ell |s_{P_0}|^{\frac{2\gamma}{\beta}}},$$

whose multiplier subspace contains P_0 and is of lower dimension than V_φ.

For this step we need the semicontinuity of multiplier ideal sheaves from Ohsawa-Takigoshi-type extension (1987). For a natural local projection $\pi : \mathbb{C}^{n+p} \to \mathbb{C}^p$ with $\pi(P,\tau) = \tau$ and a multiplier ideal sheaf $\mathcal{I}_\psi$ on $\mathbb{C}^{n+p}$ as a germ at $(P,\tau) = (0,0)$, a section of $\mathcal{I}_{(\psi|_{\tau=0})}$ on the fiber $\mathbb{C}^n$ as a germ at $P = 0$ can be extended to a section of $\mathcal{I}_\psi$ on $\mathbb{C}^{n+p}$ as a germ at $(P,\tau) = (0,0)$.

This proves the freeness part of the Fujita conjecture with weaker bound. The reason for the weaker bound is that the dimension of V_φ has to be cut down one dimension at a time and for the step when $\dim_{\mathbb{C}} V_\varphi = k$, we need a multivalued section of $(k+\varepsilon)$ to end up finally with $m \geq (1+2+\cdots+n)+1$ with last 1 from contributions of $\varepsilon > 0$ of each step (in the 1995 result of Angehrn-Siu).

Theorem of Ohsawa-Takegoshi. Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinates $z_1, \cdots, z_n, w$. Let H be defined by $w = 0$.

Let φ be a smooth plurisubharmonic function on Ω. There exists a constant C_Ω depending only on Ω such that for any holomorphic function f on $\Omega \cap H$ with

$$\int_{H\cap\Omega} |f|^2 e^{-\varphi} < \infty$$

there exists a holomorphic function F on Ω extending f with the property that

$$\int_{\Omega} |F|^2 e^{-\varphi} \leq C_\Omega \int_{H\cap\Omega} |f|^2 e^{-\varphi}.$$

Remark. Recently there have been a lot of activities concerning the optimal constant C_Ω and its application.

Higher Multiplicity (Vanishing Order) Needed for Very Ampleness Part (Handled in Two New Techniques). For a multiplier subspace V_φ with higher multiplicity, the cutting of dimension has to be done on each stratum of multiplicity 1. The lower dimensional multiplier subspace in each stratum still needs higher multiplicity.

The cutting down of dimension has to be performed in lexicographical manner. A new technique of lexicographical multiplier ideal sheaves of higher order is needed.

For P in a regular part of V_φ with multiplicity, the multivalued section s_P on V_φ used to construct the next-step metric $e^{-\varphi_P}$ needs to have vanishing order $\geq n+q$ at P.

One trouble is the problem of semicontinuity of multiplier ideal sheaf for higher vanishing order. To solve the problem, another new technique of getting higher vanishing order in the limit of multiplier ideal sheaves is needed.

First Step in Construction of Lexicographical Multiplier Ideal Sheaves. The point of X under consideration is P_0 and the high multiplicity or vanishing order is an effective q.

We construct a metric from multivalued sections of an effective multiple of L whose multiplier ideal sheaf $\mathcal{M}_{J_1}$ is contained in $(\mathfrak{m}_{X,P_0})^q$.

By using fractional powers of the metric and Shokurov's technique of slight perturbation, we get a sequence of multiplier ideal sheaves

$$\mathcal{O}_X = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_{j_1} \supset \mathcal{M}_{j_1+1} \supset \cdots \supset \mathcal{M}_{J_1}$$

such that the ideal sheaf $\mathcal{M}_{j_1+1}$ at P_0 is the intersection of the ideal sheaf $\mathcal{M}_{j_1}$ at P_0 and the full ideal sheaf $\mathcal{I}_{Y_{j_1}}$ of some positive-dimensional irreducible subvariety Y_{j_1} in X with generic multiplicity 1.

Second Step in Construction of Lexicographical Multiplier Ideal Sheaves. For the second step, we fix $1 \leq j_1 < J_1$ and use Y_{j_1} to replace X to set a sequence of multiplier ideal sheaves

$$\mathcal{M}_{j_1} = \mathcal{M}_{j_1,0} \supset \mathcal{M}_{j_1,1} \supset \cdots \supset \mathcal{M}_{j_1,j_2} \supset \mathcal{M}_{j_1,j_2+1} \supset \cdots \supset \mathcal{M}_{j_1,J_{j_1}}$$

such that the ideal sheaf $\mathcal{M}_{j_1,j_2+1}$ at P_0 is the intersection of the ideal sheaf $\mathcal{M}_{j_1,j_2}$ at P_0 and the full ideal sheaf $\mathcal{I}_{Y_{j_1,j_2}}$ of some positive-dimensional irreducible subvariety Y_{j_1,j_2} in X with generic multiplicity 1 and

$$\mathcal{M}_{j_1,J_{j_1}} \subset (\mathfrak{m}_{X,P_0})^q + \mathcal{M}_{j_1+1}.$$

The term $\mathcal{M}_{j_1+1}$ is to provide the vanishing in the normal direction across Y_{j_1}.

Note that the vanishing on Y_{j_1} is added to $\mathcal{M}_{j_1}$ to yield $\mathcal{M}_{j_1+1}$, which means that Y_{j_1} is inside the variety Y_{j_1+1} of $\mathcal{M}_{j_1+1}$. The vanishing of the restriction of the multivalued section on Y_{j_1} plus the vanishing along the normal direction of Y_{j_1} yields $(\mathfrak{m}_{X,P_0})^q$ modulo $\mathcal{M}_{j_1+1}$.

After the first step, the problem of the noneffective multiplicity of Y_{j_1} at P_0 arises, which has to be solved by the new technique of getting higher vanishing order in the limit of multiplier ideal sheaves, because the argument of using Riemann-Roch to construct multivalued sections with effective vanishing order at a point works only when the multiplicity at the point is effective.

Now we discuss how lexicographical multiplier ideal sheaves help us get holomorphic sections of $\hat{m}L + K_X$ for an effective $\hat{m}$ to make $\hat{m}L + K_X$ very ample. So far we have seen only the first two steps in the construction of lexicographical multiplier ideal sheaves. We now write down the complete set of lexicographical multiplier ideal sheaves.

Formalism of Lexicographical Multiplier Ideal Sheaves. Suppose X is a compact complex algebraic manifold of complex dimension n and L is an ample line bundle over X. Let $P_0 \in X$. Suppose that for a sequence

$$j_1, j_2, \cdots, j_s$$

of nonnegative integers $0 \leq j_\ell \leq J_{j_1, \cdots, j_\ell}$ with $1 \leq s \leq S$ we have a coherent ideal sheaf

$$\mathcal{M}_{j_1, j_2, \cdots, j_s}$$

over X such that the following four conditions hold. When $\ell = 0$ and the set $\{j_1, \cdots, j_\ell\}$ is vacuous, we use the symbol $\hat{J}$ to denote $J_{j_1, \cdots, j_\ell}$.

(0) *Vanishing Cohomology.*

$$H^p\left(X, \mathcal{M}_{j_1, j_2, \cdots, j_s}\left(mL + K_X\right)\right) = 0 \text{ for } p \geq 1 \text{ and } m \geq m_0.$$

In addition we assume that we have the following properties.

(1) *Inclusion Relations.*

$$\mathcal{M}_{j_1, j_2, \cdots, j_s} \subset \mathcal{M}_{k_1, k_2, \cdots, k_t}$$

if $(k_1, k_2, \cdots, k_t)$ precedes $(j_1, j_2, \cdots, j_s)$ in lexicographical ordering.

(2) *Vanishing Order.*

$$\mathcal{M}_{j_1, \cdots, j_{s-2}, j_{s-1}, J_{j_1, \cdots, j_{s-1}}} \subset \mathfrak{m}_{X, P_0}^q + \mathcal{M}_{j_1, j_2, \cdots, j_{s-2}, j_{s-1}+1}$$

When $s = 1$, the condition reads

$$\mathcal{M}_{\hat{J}} \subset \mathfrak{m}_{X, P_0}^q.$$

(3) *Notational Convention.*

$$\mathcal{M}_{j_1, j_2, \cdots, j_{s-1}, 0} = \mathcal{M}_{j_1, j_2, \cdots, j_{s-1}},$$
$$\mathcal{M}_0 = \mathcal{O}_X.$$

(4) *Sheaf of Longest Word Agreeing With That of Second Longest Word Outside P_0.* The number $J_{j_1,\cdots,j_{S-1}}$ is equal to 1 and

$$\mathcal{M}_{j_1,\cdots,j_{S-2},j_{S-1},1} = \mathcal{M}_{j_1,\cdots,j_{S-2},j_{S-1}+1} \text{ on } X - P_0.$$

Descending induction on $1 \leq s \leq S$ yields the following.

Induction Statement. The cohomology group

$$H^p\left(X, \left(\left(\mathcal{M}_{j_1,\cdots,j_{s-1},j_s+1} + \mathfrak{m}^q_{X,P_0}\right) \cap \mathcal{M}_{j_1,\cdots,j_{s-1},j_s}\right)(mL + K_X)\right)$$

vanishes for $p \geq 1$ and $m \geq m_0$ and $1 \leq s \leq S$.

End Result of Induction. The cohomology group

$$H^p\left(X, \mathfrak{m}^q_{X,P_0}(mL + K_X)\right)$$

vanishes for $p \geq 1$ and $m \geq m_0$.

Multiplier Ideal Sheaves of Higher Order. The above induction argument needs the use of multiplier ideal sheaves of higher order. The vanishing theorem for multiplier ideal sheaves $\mathcal{I}$ gives us

$$H^p(X, \mathcal{I}(mL + K_X)) = 0 \text{ for } p \geq 1$$

when m is no less than some effective integer $m_0(\mathcal{I})$ associated to $\mathcal{I}$. A higher-order multiplier ideal sheaf $\mathcal{J}$ is defined so that there is an exact sequence of ideal sheaves

$$0 \to \mathcal{I}' \to \mathcal{J} \to \mathcal{I}'' \to 0,$$

where both $\mathcal{I}'$ and $\mathcal{I}''$ are multiplier ideal sheaves (of lower order) with associated positive integers $m_0(\mathcal{I}')$ and $m_0(\mathcal{I}'')$. Then for

$$m_0(\mathcal{J}) = \max(m_0(\mathcal{I}'), m_0(\mathcal{I}''))$$

we have the vanishing theorem

$$H^p(X, \mathcal{J}(mL + K_X)) = 0 \text{ for } p \geq 1$$

when $m \geq m_0(\mathcal{J})$.

Illustrative Simple Example to Handle Semicontinuity of Multiplier Ideal Sheaf of Higher Vanishing Order (Property (2) on Vanishing Order). On $\mathbb{C}^2$ with coordinates z, w, as a germ at $P_0 = 0$ let C be a complex curve defined by $z = \zeta^p$, $w = \zeta^r$ with $p < r$ relatively prime, both noneffectively larger than the effective given order q.

The curve C is like Y_j above which occurs as a stratum. Let $g_C = z^r - w^p$ be the defining equation for C. Consider the metric

$$\frac{1}{|g_C|^2}$$

which yields the full ideal sheaf $\mathcal{I}d(C)$ of C as its multiplier ideal sheaf.

To cut down the dimension of its multiplier subspace C, we use a multivalued holomorphic function germ f_P on $\mathbb{C}^2$ at 0 whose restriction to C vanishes to order more than an effective number a times q at a regular point P of C.

We construct the metric

$$e^{-\varphi_P} = \frac{1}{|g_C|^{2-\varepsilon}\left(|g_C|^{2\alpha} + |f_P|^{2\beta}\right)}$$

for some appropriate $\varepsilon > 0$ and $\alpha \geq 2$ and $\beta \geq 2$.

We hope that as $P \to P_0$, we end up with $e^{-\varphi_{P_0}}$ whose multiplier ideal sheaf satisfies

$$\mathcal{I}_{\varphi_{P_0}} \subset (\mathfrak{m}_{\mathbb{C}^2,0})^q + \mathcal{I}d(C).$$

The trouble is that in terms of ζ the condition of $(f_P)|_C$ vanishing at P to order $\geq aq$ only means

$$|((f_P)|_C)| \gtrsim |\zeta - \zeta(P)|^{aq}$$

near P. The limiting situation is

$$|((f_{P_0})|_C)| \gtrsim |\zeta|^{aq}$$

near $P_0 = 0$.

On the other hand, the restriction of a monomial $z^\mu w^\nu$ (with $\mu + \nu \geq 1$) to C is $\zeta^{\mu p + \nu r}$ with $\mu p + \nu r$ noneffective, far greater than aq, making the argument useless for the purpose of concluding

$$\mathcal{I}_{\varphi_{P_0}} \subset (\mathfrak{m}_{\mathbb{C}^2,0})^q + \mathcal{I}d(C).$$

The key new technique to handle the difficulty is to first observe that holomorphic function germs in $\mathcal{I}_{\varphi_{P_0}}$ are automatically convergent power series in z, w.

The condition we want can be formulated in terms of an effective number of linear equations. These linear equations are the vanishing of the coefficients of $\zeta^{\mu p+\nu r}$ with $\mu+\nu\leq q$.

Secondly, instead of f_P which vanishes to high order at P, we choose h_P whose restriction to C vanishes (without specification of multiplicity) at an appropriately chosen, effectively large number of points on C. The coefficient matrix for the system of linear equations is similar to the matrix of a Vandemonde determinant.

We now give more details of arithmetic computation of this key technique.

More Arithmetic Details of Key Argument. Suppose for ζ_σ we have a holomorphic function germ

$$h_\sigma(z,w)=\sum_{\mu,\nu}a^{(\sigma)}_{\mu,\nu}z^\mu w^\nu$$

on $\mathbb{C}^2$ at 0, which vanishes at $\zeta_\sigma e^{i\theta_{\sigma,1}},\cdots,\zeta_\sigma e^{i\theta_{\sigma,k}}$ with $0<\theta_{\sigma,1}<\cdots<\theta_{\sigma,k}<2\pi$ and k effective.

We assume that h_σ converges to the holomorphic function germ

$$h_\infty(z,w)=\sum_{\mu,\nu}a^{(\infty)}_{\mu,\nu}z^\mu w^\nu$$

on $\mathbb{C}^2$ at 0 as $\zeta_\sigma\to 0$ when $\sigma\to\infty$.

We would like to prove that $h_\infty\in(\mathfrak{m}_{\mathbb{C}^2,0})^{\hat{q}}$ when $0<\theta_{\sigma,1}<\cdots<\theta_{\sigma,k}<2\pi$ are appropriately chosen.

We have

$$\begin{aligned}h_\sigma\left(\left(\zeta_\sigma e^{i\theta_{\sigma,j}}\right)^p,\left(\zeta_\sigma e^{i\theta_{\sigma,j}}\right)^r\right)&=\sum_{\mu,\nu}a^{(\sigma)}_{\mu,\nu}\left(\zeta_\sigma e^{i\theta_{\sigma,j}}\right)^{p\mu}\left(\zeta_\sigma e^{i\theta_{\sigma,j}}\right)^{r\mu}\\&=\sum_\lambda\zeta_\sigma^\lambda\left(\sum_{p\mu+r\nu=\lambda}a^{(\sigma)}_{\mu,\nu}e^{i\lambda\theta_{\sigma,j}}\right).\end{aligned}$$

New Notations. We denote ζ_σ by X and denote λ by m_j, where $m_1<m_2<m_3<\cdots$ with $m_j=p\mu_j+r\nu_j$. Let

$$g_\sigma(X)=\sum_{j=1}^\infty a^{(\sigma)}_j X^{m_j}$$

such that as $\sigma \to \infty$, g_σ approaches

$$g_\infty(X) = \sum_{j=1}^{\infty} a_j^{(\infty)} X^{m_j}.$$

Fix ℓ. Choose integers $\alpha_j = j$ for $1 \le j \le \ell$ (or some other similar effective increasing sequence of integers of length ℓ). Choose nonzero X_σ with decreasing absolute value approaching 0 as $\sigma \to \infty$.

Lemma. If $g_\sigma(\alpha_j X_\sigma) = 0$ for every σ and every $1 \le j \le \ell$, then $a_j^{(\infty)} = 0$ for $1 \le j \le \ell$.

Proof. Fix σ and consider the system of ℓ equations $g_\sigma(\alpha_j X_\sigma) = 0$ for $1 \le j \le \ell$ in the ℓ unknowns $a_k^{(\sigma)} X_\sigma^{m_k}$ for $1 \le k \le \ell$. Rewrite ℓ equations $g_\sigma(\alpha_j X_\sigma) = 0$ as

$$\sum_{k=1}^{\ell} a_k^{(\sigma)} \alpha_j^{m_k} X_\sigma^{m_k} = - \sum_{p=\ell+1}^{\infty} a_p^{(\sigma)} \alpha_j^{m_p} X_\sigma^{m_p}$$

for $1 \le j \le \ell$. We solve this by Cramer's rule where the matrix of coefficients is

$$W(m_1, m_2, \cdots, m_\ell) = \begin{pmatrix} \alpha_1^{m_1} & \alpha_1^{m_2} & \cdots & \alpha_1^{m_\ell} \\ \alpha_2^{m_1} & \alpha_2^{m_2} & \cdots & \alpha_2^{m_\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_\ell^{m_1} & \alpha_\ell^{m_2} & \cdots & \alpha_\ell^{m_\ell} \end{pmatrix}.$$

Note that $m_1 < m_2 < \cdots < m_\ell$ are not effective. The determinant $W(m_1, \cdots, m_\ell)$ becomes the Vandemonde determinant

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^\ell \\ 1 & x_1 & x_1^2 & \cdots & x_1^\ell \\ 1 & x_2 & x_2^2 & \cdots & x_2^\ell \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_\ell^\ell \end{vmatrix} = \prod_{0 \le i < j \le \ell} (x_j - x_i)$$

when $x_0 = 0$ and $x_j = \alpha_j$ and $(m_1, \cdots, m_\ell) = (1, \cdots, \ell)$, and is therefore nonzero for generic α_j and m_j (which we assume to be the case).

From Cramer's rule, the unknowns $a_k^{(\sigma)} X_\sigma^{m_k}$ for $1 \leq k \leq \ell$ are now linear functions of the right-hand sides

$$-\sum_{p=\ell+1}^{\infty} a_p^{(\sigma)} \alpha_j^{m_p} X_\sigma^{m_p}$$

with coefficients which are the cofactors A_{jk} of the matrix of coefficients.

That is,

$$a_k^{(\sigma)} X_\sigma^{m_k} = -\sum_{j=1}^{\ell} A_{kj} \sum_{p=\ell+1}^{\infty} a_p^{(\sigma)} \alpha_j^{m_p} X_\sigma^{m_p}$$

for $1 \leq k \leq \ell$, which upon dividing by $X_\sigma^{m_k}$ yields

$$a_k^{(\sigma)} = -\sum_{j=1}^{\ell} A_{kj} \sum_{p=\ell+1}^{\infty} a_p^{(\sigma)} \alpha_j^{m_p} X_\sigma^{m_p - m_k}$$

for $1 \leq k \leq \ell$. By $\sigma \to \infty$, from $X_\sigma \to 0$ and $m_p > m_k$ for $k \leq \ell < \ell + 1 \leq p$, we get $a_k^{(\infty)} = 0$ for $1 \leq k \leq \ell$. Q.E.D.

Though $m_1 < m_2 < \cdots < m_\ell$ are not effective, we need to limit $\alpha_1, \cdots, \alpha_\ell$ to an effective range to make sure that $\alpha_k X_\sigma \to 0$ as $X_\sigma \to 0$ for $1 \leq k \leq \ell$.

Remark. In summary, the main point here is that the semicontinuity for higher order vanishing is not to use a sequence of higher order vanishing ideals at points which approach the limit point in question, but to use a well-situated collection of points with vanishing order only ≥ 1 each. A Vandermonde type determinant is then used for the argument.

The key is that though none of the terms in the finite sequence $m_1, m_2, \cdots, m_\ell$ is effective, the number of terms ℓ in the sequence is effective.

We only need to require the total vanishing order of the multi-valued sections at the ℓ regular points to be effective. This means that the multiple of the given line bundle required to yield such a multi-valued section is effective.

This is only happening on the curve C defined by $z = \zeta^p$, $w = \zeta^r$ with $p < q$ relatively prime, which means that the result is modulo the ideal sheaf of C, as formulated in Property (2) on Vanishing Order.

HAPPY BIRTHDAY, THOMAS!