On a question of Borel and Haefliger

Claire Voisin

CNRS, IMJ-PRG

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Defi. $\mathrm{CH}_d(X)_{\mathrm{smooth}} \subset \mathrm{CH}_d(X)$ *subgroup of "smoothable" cycles, i.e. generated by classes of smooth subvarieties* $Z_i \subset X$.

Remark. Not to be confused with smoothability in the sense of deformation theory. Some singular subvarieties cannot be smoothed, even locally analytically (Thom).

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Answer

(a) Answer is yes if $c = 1$. By Serre, any divisor D can be written as $D = A - A'$ with A, A' very ample divisors. Then Bertini provides smooth divisors in $|A|$ and $|A'|$.

(b) $2 \leq c \leq d$ (or $2d \geq n$). There exist counterexamples for all pairs (c, d) with c satisfying a mysterious arithmetic condition. Some history:

- $c = 2$ Hartshorne-Rees-Thomas (1974).
- $c = 2$ Debarre (1995).
- Examples with $c = d$ (Benoist 2022).

(c) $d < c$. The "Whitney condition", with reference to the easy Whitney embedding theorem. Any compact real manifold of dimension d can be embedded in any real manifold of dimension $\geq 2d + 1$.

Theorem A. (Kollár-Voisin 2023) *One has* $\mathrm{CH}_d(X) = \mathrm{CH}_d(X)_{\mathrm{smooth}}$ *if* $d < c$.

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More on negative results

• Let $G = G(k,n)$ be the Grassmannian of vector subspaces $V_k \subset \mathbb{C}^n$. Then $\mathrm{CH}^2(G) = H^4(G,\mathbb{Z}) = \mathbb{Z}c_1^2 + \mathbb{Z}c_2$, where $c_i = c_i(\mathcal{E}_{\mathrm{taut}})$.

Thm. (Hartshorne-Rees-Thomas 1974) *Assume $k \geq 3$, $n-k \geq 3$. Then if $Z \subset G$ is a real oriented submanifold of codimension 4, $[Z] = ac_1^2 + bc_2$, with b even. So c_2 is not smoothable.*

• Let $C \subset J(C)$ be a smooth projective curve of genus g. Consider the singular subvariety $W_{g-2} \subset J(C)$ defined as the image of the sum map $C^{(g-2)} \to J(C)$.

Thm. (Debarre 1995) *Assume $g \geq 7$ and C very general. Then for any smooth subvariety $W \subset J(C)$, one has $[W] = a\theta^2$ with $a \in \mathbb{N}$. So the cycle of W_{g-2} is not smoothable (even cohomologically), since the class $[W_{g-2}] \in H^4(J(C),\mathbb{Z})$ equals $\theta^2/2$.*

Remark. Debarre's example is very different because it is not topological. Specialize $J(C)$ to $J_0 := E_1 \times \ldots \times E_g$. Then θ specializes to $\theta_1 + \ldots + \theta_g \in H^2(J_0,\mathbb{Z})$ so θ^2 specializes to $(\theta_1 + \ldots + \theta_g)^2 = 2\sum_{i>j}\theta_i\theta_j \in H^2(J_0,\mathbb{Z})$, and $\theta^2/2$ becomes smoothable on J_0.

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• Chow moving lemma applied to $\widetilde{Z} \Rightarrow \widetilde{Z} \equiv Z'$, with Z' in general position (wrt p). Then apply :

Prop. A *Let $f : Y \to X$ be a smooth projective morphism, and $Z' \subset Y$ be smooth* **in general position** *wrt f (eg: Z'= general complete intersection of very ample hypersurfaces). Then, if $2\dim Z' < \dim X$, $f_{|Z'} : Z' \to f(Z')$ is an isomorphism and $f(Z')$ is smooth.*

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Indeed, Chow moving lemma is obtained by liaison starting from $\widetilde{Z}$. This produces singularities in codimension 4 along $\widetilde{Z}$.

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Thm. (Hironaka 1968) *Cycles of dimension $d \leq 3$ are smoothable on X if $2d < n = \dim X$.*

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• The proof has two steps. Introduce the subring $\mathrm{CH}^*(X)_{\mathrm{Ch}} \subset \mathrm{CH}^*(X)$ generated by Chern classes $c_i(E)$, $E \to X$ algebraic vector bundle.

Step A. Formula $c_c(\mathcal{O}_Z) = (-1)^{c-1}(c-1)![Z]$ in $\mathrm{CH}^c(X) \Rightarrow$ $(c-1)!\mathrm{CH}_d(X) \subset \mathrm{CH}_d(X)_{\mathrm{Ch}}$.

Step B. Prove that $\mathrm{CH}_d(X)_{\mathrm{Ch}} \subset \mathrm{CH}_d(X)_{\mathrm{smooth}}$ if $2d < n$. For this, use Segre classes $s_i(E)$ instead of Chern classes. They also generate the ring $\mathrm{CH}^*(X)_{\mathrm{Ch}}$. We have

(*) $s_{i_1}(E_1) \ldots s_{i_l}(E_l) = \pi_*(h_1^{r_1-1+i_1} \ldots h_l^{r_r-1+i_l})$, where $E_j \to X$ has rank r_j, $\pi : \mathbb{P}(E_1) \times_X \ldots \times_X \mathbb{P}(E_l) \to X$, and $h_i := \mathrm{pr}_i^* c_1(\mathcal{O}_{\mathbb{P}(E_i)}(1))$.

• In (*), π is smooth projective, and the class $h_1^{r_1-1+i_1} \ldots h_l^{r_r-1+i_l}$ is a product of divisor classes, that can be expressed as a combination of classes of general complete intersections, hence smooth in general position. Then apply Prop. A. **qed**

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The starting point for the proof of Thm A is a variant of Proposition A.

Prop. B. *Let $f : Y \to X$ be a flat projective morphism with Y, X smooth, and $Z' \subset Y$ be smooth in general position (wrt f). Then if $2\dim Z' < \dim X$, $f_{|Z'} : Z' \to f(Z')$ is an isomorphism and $f(Z')$ is smooth.*

- Recall that flat $\Leftrightarrow$ equidimensional fibers since Y, X smooth.

Remark. We will apply this to general complete intersections of very ample hypersurfaces.

Defi. *Let $\mathrm{CH}(X)_{\mathrm{fl}_*\mathrm{Ch}} \subset \mathrm{CH}(X)$ be generated by cycles of the form f_*z, with $f : Y \to X$ flat projective, Y smooth, and $z \in \mathrm{CH}(Y)_{\mathrm{Ch}}$ (or z=product of divisor classes on Y).*

Corollary. *Cycles in $\mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$ are smoothable if $2d < n = \dim X$.*

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Sketch of proof of Thm B

- Obvious facts concerning $\mathrm{CH}(X)_{\mathrm{fl}_*\mathrm{Ch}}$:

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$\phi : Y \to X$ a flat projective morphism, then $\phi_*(\mathrm{CH}_d(Y)_{\mathrm{fl}_*\mathrm{Ch}}) \subset \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$.

Main Proposition. *Let X be smooth projective and $j : Y \to X$ be the inclusion of a smooth hypersurface. Then* $j_*(\mathrm{CH}_d(Y)_{\mathrm{fl}_*\mathrm{Ch}}) \subset \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$.

Defi. *We say that $W \subset X$ is a complete bundle section (cbs) if W is the zero locus of a transverse section of a vector bundle on X.*

Coro. 1 *Let X be smooth projective and $j : W \to X$ be the inclusion of a smooth cbs. Then* $j_*(\mathrm{CH}_d(W)_{\mathrm{fl}_*\mathrm{Ch}}) \subset \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$.

Coro. 2 *Let X be smooth projective and $j : W \to X$ be the inclusion of a smooth cbs. Let $\tau : X' := Bl_W(X) \to X$ be the blow-up of W. Then* $\tau_*(\mathrm{CH}_d(X')_{\mathrm{fl}_*\mathrm{Ch}}) \subset \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$.

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Coro. 3 *Let $Z \subset X$ be a connected component of a smooth cbs W. Then $[Z] \in \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$, $d = \dim Z$.*

Proof. Let $\tau : X' = \mathrm{Bl}_W(X) \to X$ be the blowup of W. Let $E_Z \to W$ be the exceptional divisor over Z. Then $E_Z^c \in \mathrm{CH}_d(X')_{\mathrm{Ch}}$ and $\tau_* E_Z^c = \pm[Z]$ in $\mathrm{CH}_d(X)$. Hence $[Z] \in \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$ by Coro 2. **qed**

Finally one proves

Thm C. *Let $Z \subset X$ be smooth, with $\dim X \geq 4 \dim Z$. Then there exists a sequence $X_N \xrightarrow{\tau_N} \dots X_1 \xrightarrow{\tau_1} X_0 = X$ of blow-ups along smooth cbs such that the proper transform of Z is a connected component of a smooth cbs.*

Coro 4. *Same assumptions on Z, $X \Rightarrow [Z] \in \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$.*

Proof. By Coro. 3 one gets that $\widetilde{Z} \in \mathrm{CH}_d(X_N)_{\mathrm{fl}_*\mathrm{Ch}}$. By Coro. 2, one gets : $[Z] = \tau_*[\widetilde{Z}] \in \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$ (with $\tau = \tau_1 \circ \ldots \circ \tau_N$). **qed**

Proof that Thm C $\Rightarrow$ Thm B. Indeed, let $Z \subset X$, with $\dim Z = d$. Desingularize Z, $\widetilde{Z} \to Z$, and embed $\widetilde{Z}$ in $X \times \mathbb{P}^r$, with r large. Apply Corollary 4 to $\widetilde{Z} \subset X \times \mathbb{P}^r$ and then pr_{X*} . **qed**

Coro. 3 *Let $Z \subset X$ be a connected component of a smooth cbs W. Then $[Z] \in \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$, $d = \dim Z$.*

Proof. Let $\tau : X' = \mathrm{Bl}_W(X) \to X$ be the blowup of W. Let $E_Z \to W$ be the exceptional divisor over Z. Then $E_Z^c \in \mathrm{CH}_d(X')_{\mathrm{Ch}}$ and $\tau_* E_Z^c = \pm[Z]$ in $\mathrm{CH}_d(X)$. Hence $[Z] \in \mathrm{CH}_d(X)_{\mathrm{fl}_*\mathrm{Ch}}$ by Coro 2. **qed**

Finally one proves

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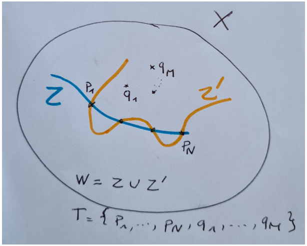
We sketch the proof when $\dim Z = 1$. Choose a general complete intersection curve W containing Z as a component.

Then $W = Z \cup Z'$ where Z' is smooth, and the intersection $Z \cap Z'$ is transverse at finitely many points $p_1, \ldots, p_N$.
Choose a smooth 0-dimensional complete intersection
$T = \{p_1, \ldots, p_N, q_1, \ldots, q_M\}$ containing all points p_i, with $q_i \notin W$.



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Idea of the proof of Thm C

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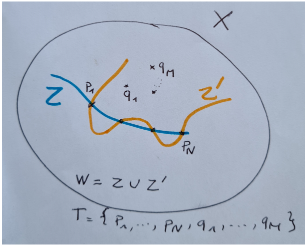

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• Open if $X = G(k,n)$, $k \geq 3$, $n-k \geq 3$. In particular:

Question. *Let $X = G(k,n)$, $k \geq 3$, $n-k \geq 3$. Is the class c_2 smoothable with rational coefficients ?*

• If $k \geq 3$, $n-k \geq 4$, the Barth-van de Ven Lefschetz type theorem combined with Serre's construction $\Rightarrow$ smooth codim. 2 subvarieties of $G(k,n)$ are 0-sets of sections of rank 2 vector bundles on $G(k,n)$.

Recall Hartshorne's conjecture on rank 2 vector bundles on $\mathbb{P}^N$.

Conj. *There exists n_0 such that rank 2 vector bundles on $\mathbb{P}^N$, with $N \geq n_0$ are split.*

Thm. (Benoist-Voisin 2024) *Hartshorme's conjecture implies that any rank 2 vector bundle on $G(k,n)$, $k \geq n_0$, $n-k \geq n_0$ is split, hence that $\mathrm{CH}^2(G(k,n))_{\mathrm{smooth}} = \mathbb{Z}c_1^2$. so that $\mathrm{CH}^2(G(k,n))_{\mathrm{smooth},\mathbb{Q}} \neq \mathrm{CH}^2(G(k,n))_{\mathbb{Q}}$.*

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The case of abelian varieties

Defi. *Let* $\mathrm{CH}(X)_{\mathrm{sm}_*\mathrm{Ch}} \subset \mathrm{CH}(X)$ *be generated by cycles of the form* f_*z*, with* $f: Y \to X$ *flat projective,* Y *smooth, and* $z \in \mathrm{CH}(Y)_{\mathrm{Ch}}$ *(or* z*=product of divisor classes on* Y*).*

Question. *Is* $\mathrm{CH}_d(X)_{\mathrm{sm}_*\mathrm{Ch}} = \mathrm{CH}_d(X)$ *for any smooth projective* X*, any* d*?*

Thm. *Yes if* X *is an abelian variety.*

Proof. Let $\mu : X \times X \to X$ be the sum map. For any $Z \subset X$ of dimension d, let $\tau : \widetilde{Z} \to X$ be a desingularization of Z. Smooth morphism $\phi := \mu \circ (\tau, Id_X) : \widetilde{Z} \times X \to X$ such that

(*) $\phi_*(\widetilde{Z} \times 0) = \phi_*(\mathrm{pr}_X^* 0) = Z$ in $\mathrm{CH}_d(X)$.

If $0 \in \mathrm{CH}_0(X)_{\mathrm{Ch}}$, this is finished. In general, $0 \notin \mathrm{CH}_0(X)_{\mathrm{Ch}}$ (Debarre 2008). However, if $j : C \subset X$ is a smooth complete intersection curve, $j_* : J(C) \to \mathrm{Alb}(X) = X$ is a smooth morphism. Furthermore it is known that $0 \in \mathrm{CH}_0(J(C))_{\mathrm{Ch}}$ (Mattuck). Then replace ϕ by $\phi' = \phi \circ (Id_{\widetilde{Z}}, j_*) : \widetilde{Z} \times J(C) \to X$ in (*). **qed**

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