FINITENESS RESULTS FOR SPECIAL SUBVARIETIES: HODGE THEORY, O-MINIMALITY, DYNAMICS

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1. Introduction

Many manifolds come equipped with a distinguished class of subvarieties: their *special subvarieties*. For instance:

- locally symmetric spaces come with their totally geodesic subspaces.

- abelian varieties come with the (translates by torsion points of) their abelian subvarieties, (mixed) Shimura varieties with their Shimura subvarieties. More generally, complex quasi projective varieties endowed with a variation of (mixed) Hodge structures come with the irreducible components of their Hodge loci.

- strata of abelian differentials come with their affine invariant submanifolds.

Recent years saw a lot of activity around the following unifying principle: there is a natural dichotomy *typical/atypical* among special subvarieties; the typical special subvarieties should be numerous, while the atypical ones should be rare (Zilber-Pink philosophy). In this workshop we will consider three incarnations of this principle:

Theorem 1.1. [BFMS21, Theor.1.1], [BFMS, Theor.1.1] *Let S be a finite volume real or complex hyperbolic manifold. If S contains infinitely many maximal totally geodesic subspaces of real dimension at least 2 then S is arithmetic.*

Theorem 1.1 is proven in [BFMS21] and [BFMS] by restating it purely in terms of homogeneous dynamics and proving a superigidity statement. In the complex hyperbolic case, if we restrict the statement of Theorem 1.1 to *complex* totally geodesic subspaces, then it is predicted by the Zilber-Pink heuristic in Hodge theory [K17]. In this context, it admits a completely different proof [BU, Theor.1.2.1] based on functional transcendence. Another instance of this Zilber-Pink heuristic is the following general finiteness statement (non-experts in Hodge theory need not worry, this statement will be explained in detail and widely illustrated in the lectures!):

Theorem 1.2. [BKU, Theor.1.5] *Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth connected complex quasi-projective variety S. If $\mathbb{V}$ is of level at least 3 and its adjoint generic Mumford-Tate group is simple then its Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ of positive period dimension is algebraic in S.*

Finally one has the following finiteness result in Teichmüller dynamics:

Theorem 1.3. [EFW18, Theor 1.5] *In each stratum of abelian differentials, all but finitely many affine invariant submanifolds have rank 1 and degree at most 2. In each genus g there is a finite union of rank 2 degree 1 affine invariant submanifolds $\mathcal{M}$ such that all but finitely many of the affine invariant submanifolds of rank 1 and degree 2 are a codimension 2 subvariety of one of these $\mathcal{M}$.*

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Again, strata of abelian differentials and period coordinates on them are defined via Hodge theory and Theorem 1.3 can be predicted by a Zilber-Pink heuristic.

All these finiteness results build on various tools and ideas: Hodge theory, bi-algebraic geometry and functional transcendence, o-minimality, homogeneous and non-homogeneous dynamics. The goal of this workshop is to present these tools and results to a mixed audience, to identify the many similarities and to explore further these new fascinating interconnected phenomena.

General references for a preparation to the workshop:
- on the Hodge theory side: https://www.youtube.com/watch?v=v1GY9IXCgo4 and the introduction of [BKU] to get an idea, [KlinglerICM] for a (technically heavier) survey lacking examples, https://www.youtube.com/watch?v=H0kmHwYxFLk&list=PLx5f8IelFRgGSPbT_vURSCqBcxdHbHHBr for a longer introduction to this circle of ideas (but predating [BKU]), [KUY18] for an older survey focusing on Shimura varieties.
- on the Teichmüller dynamics side: [Fsurvey], [Wr15], [Z06]
- on the Lie groups and symmetric spaces side:
a five lecture over view of the proof from a school in Jerusalem:
https://mathematics.huji.ac.il/node/3035906
a longer sequence of lectures on many key ingredients from IHES:
https://www.youtube.com/playlist?list=PLx5f8IelFRgHXD2W2xcVEikxmEALdVe-x
a lecture emphasizing the first step in the proof and analogous steps in related problems:
https://youtu.be/hiF0b3M6F6Y
a lecture emphasizing history of arithmeticity:
https://www.youtube.com/watch?v=uSoCkH7H0gs
Two related surveys, that are perhaps more relevant to motivation than to proofs [FisherMargulisBook, FisherICM].

2. Monday

9.30.am - 10.50 am - Lecture 1: Hodge theory. [Klingler]
Motivation for Hodge theory: linearization of (family of) algebraic varieties.
Pure Hodge structures, mixed Hodge structures, polarization, k-split Hodge structures for $k = \mathbb{Q}, \mathbb{R}$. Insist on mixed weight 1 (see Teichmüller dynamics). Mumford-Tate group, level of a Hodge structure as refinement of the weight.
Real multiplication, complex multiplication.
Variations of Hodge structures, period maps. Insist on Griffiths transversality, which is crucial in general, although it plays no role in level 1. Special case where we have a subvariety of the total space (see Teichmüller dynamics). Monodromy group and generic Mumford-Tate groups.
Hodge locus, Cattani-Deligne-Kaplan theorem, special subvarieties. Examples in weight 1, weight 2, weight 3, families of hypersurfaces...
Special subvarieties as intersection loci, the dichotomy typical/atypical.

10.50am - 11.30am - Coffee break.

11.30am - 12.50am - Lecture 2: Teichmüller dynamics: overview. [Filip]
Motivation: billiards, translation surfaces, dynamics of foliations on surfaces
Strata: developing map to H^1_{rel}, aka period coordinates.

Relation between the developing map and the period map of Lecture 1.
The "bialgebraic" point of view, to be developed in full generality in Lecture 5.
$\mathbf{GL}(2,\mathbb{R})^+$-action on strata. Not holomorphic!
Invariant subvarieties:

(1) their definition in terms of linear equations in period coordinates
(2) their invariants: field of affine definition, rank, relative rank

Orbit closures are invariant subvarieties ([EMM15] and [Fil16b])

Statement of Theorem 1.3.

1pm - 3pm: Lunch break.

3pm - 4.20pm - Lecture 3: Totally geodesic subspaces of locally symmetric manifolds. [Fisher]

Real and complex hyperbolic manifolds. Arithmeticity versus non-arithmeticity. Examples: first type arithmetic manifolds for $\mathrm{SO}(n,1)$ and Gromov Piatetski-Shapiro for non-arithemtic examples. Mention second type construction but no details. First type complex hyperbolic described, mention much more diverse examples, paucity of non-arithmetic examples.

Statement of Theorem 1.1 in [BFMS21] and [BFMS]. Point out that [BU, Theor.1.2.1] is a particular case of [BFMS], but with a completely different proof.

Reduction of Theorem 1.1 to a superrigidity statement:

- recall Margulis' strategy for deducing arithmeticity from superrigidity ([BFMS21, Section 3.2]).
- Introduce briefly the *compatibility* condition [BFMS21, Section 3.4] (use Definition 5.1 of strongly compatible below)
- state the two relevant superrigidity statements [BFMS21, Theor. 1.6] and [BFMS, Theor.1.3].

4.20pm - 5pm - Coffee break.

5pm-6.20pm - Lecture 4: The Zilber-Pink conjecture. [Daw]

This lecture is purely expository and contains no proof. The goal is to explain the Zilber-Pink conjecture, the results, and to illustrate them at a quiet pace!

Statement of the Zilber-Pink conjecture for the atypical Hodge locus [BKU, Conjecture 2.5] (generalizing [K17, Conjecture 1.9]), its counterpart for the typical Hodge locus [BKU, Conjecture 2.7], and their corollary [BKU, Conjecture 2.8].

Mention that the Zilber-Pink conjecture implies the special case of Theorem 1.1 where one considers only complex totally geodesic submanifolds: [BU, Section 6.1]. This will be treated in details in Lecture 13.

Results for the atypical locus: state [BKU, Theor. 3.1], which is the geometric version of[BKU, Conjecture 2.5]. As applications, discuss [BKU, Section 3.5.1] and [BKU, Section 3.5.2] related to arithmeticity.

Criterion for the Hodge locus to be atypical: [BKU, Theor. 3.3]. Sketch the proof, which is a Lie theory exercise once the role of Griffiths' transversality is understood [BKU, Section 7].

As a corollary of [BKU, Theor. 3.1] and [BKU, Theor. 3.3], one obtains Theorem 1.2 ([BKU, Theor.1.5]).

Discuss [BKU, Corollary 1.6] as an illustration of Theorem 1.2.

Results for the typical locus: [BKU, Theor. 3.9] and applications.

Mention [BKU, Remark 3.16], introducing Lecture 16.

3. TUESDAY

9.30.am-10.50 am - Lecture 5: Linear invariant submanifolds/Invariant subvarieties. [Apisa]

Fast recap of mixed Hodge theory in weight 1, already explained in Lecture 1.

Characterization of Invariant Subvarieties:

(1) Real Multiplication [Fil16a, Thm. 1.6]: Compatibility of Hodge decomposition with $\mathrm{GL}_2(\mathbb{R})$-invariant bundles; eigenform property of the holomorphic 1-form.
(2) Torsion [Fil16b, Thm. 1.3]: Explain the "twisted" torsion condition, relate this to k-split mixed Hodge structures, where k is the field of affine definition of the variety.

Explain how these two conditions give linear equations in period coordinates (plus some further equations, leading to the atypical subvarieties).

Examples of Invariant Subvarieties:

(1) Teichmüller curves: square-tiled/torus covers; genus 2 (Calta; McMullen); Bouw-Moeller family.
(2) Higher rank (time permitting): besides covering constructions, some selection from [EMMW20].

10.50am - 11.30am - Coffee break.

11.30am - 12.50am - Lecture 6: Homogeneous dynamics and totally geodesic submanifolds. [Mozes]

See the lecture by Mozes at `https://mathematics.huji.ac.il/node/3035906` for a possible model.

State Howe-Moore. Possibly explain strengthening in terms of metric ergodicity (see [BF22, Section 3].)

State version of Ratner for measures [R91]. State Mozes-Shah [MS95, Theorem 1.1] (can simplify a bit). Mention that to prove Mozes-Shah, we the linearization technique from Dani-Margulis and some statements on equidistribution [DM93]. Include details per your choice as time permits following Mozes above.

Discuss dictionary between totally geodesic submanifolds and simple group invariant measures/orbit closures in more detail than in lecture 3 if time permits. [BFMS21, BFMS]

Discuss how representations discussed in lecture 3 allow one to witness non-arithmeticity by lifting invariant measures to projective bundles.

Explain how limit of measures gives rise to $W \times \Gamma$ equivariant map from $\Psi : G \to \mathcal{M}(P(V))$ where V is a vector space [BFMS21, Section 3.3].

1pm - 3pm: Lunch break.

3pm - 4.20pm - Lecture 7: Tame geometry and Hodge theory. [Cantat]

The goal of this lecture is to present tame geometry, with the idea it might be useful to most participants in other contexts. This should cover most of [KlinglerICM, Section 3], with the exception of [KlinglerICM, Subsection 3.6.2] which is not useful for our purpose.

The importance of [KlinglerICM, Theor. 3.8] cannot be overestimated, as it is used in most results applying tame geometry.

Depending on time, the speaker can replace the proof of [BKT20, Theor.3.14] by the proof that the uniformisation map $\pi : \mathcal{F} \to S = \Gamma \backslash D$ for a Shimura variety S is definable [KUY16], starting with the easy case of the modular curve.

4.20pm - 5pm - Coffee break.

5pm-6.20pm - Lecture 8: Algebraic hulls. [Bader]

Follow [EFW18, §2] and [Fsurvey, §?].

Definition of algebraic hulls for a general dynamical system. Emphasize role of regularity of the invariant subbundles. Mention Tannakian point of view, analogy with Mumford–Tate groups and monodromy.

Basic results: Chevalley Theorem (algebraic subgroups described as stabilizers); Borel–Serre and Zimmer theorems (tameness of algebraic action on algebraic varieties and measures on them, see also [EFW18, Prop. 5.5]). Easy implication: continuity of invariant bundles implies containment of algebraic hull under containment of dynamical system.

Rigidity of algebraic hulls in the Teichmüller setting. State [Fil16a, Thm. 1.4]: the measurable and the "polynomial" algebraic hulls coincide . Proof is a game with expanding/contracting action of $\mathrm{GL}_2(\mathbb{R})$, present as much of proof as time allows, following §7 of [Fil16a].

Optional Examples: A, P, $G = \mathrm{GL}_2(\mathbb{R})$ groups acting on strata of translation surfaces, and their algebraic hulls. Compare measurable vs continuous hulls.

4. Wednesday

9.30.am - 10.50 am - Lecture 9 Algebraic representations: definitions and first steps. [Tholozan]

Recall the map Ψ from Lecture 6. Describe briefly why this map can be viewed as a map $\Psi : G \to H/L$ where H is an algebraic group and $L < H$ an algebraic subgroup. [BFMS21, Section 4.1]

Define algebraic representations and T-algebraic representations following Bader-Furman [BF22]. Point back to the map Ψ as an example.

Cover sections 3 and 4 of Bader-Furman with special emphasis on content from Theorem 4.3 to Theorem 4.6. Try to include the (quite soft) proofs for these parts.

Include [BFMS21, Lemma 4.4] and the preceding definition, both of which are implicit in [BF22].

Also include [BF22, Lemma 5.1]. Describe as motivation for why what we want in the end are algebraic representations where the stabilizer is trivial.

Youtube lectures by Bader in Jerusalem/IHES mentioned at the end of the introduction may be helpful in planning this lecture.

10.50am - 11.30am - Coffee break.

11.30am - 12.50am - Lecture 10: Bi-algebraic geometry and functional transcendence. [Brunebarbe]

This lecture should cover [KlinglerICM, Sections 4.1 and 4.2].

Present the bi-algebraic format. In addition to the examples in [KlinglerICM, Section 4.1] and the references therein, present the setup of [KL].

The lecture should focus on the Ax-Schanuel theorem. The special case of the Ax-Lindemann theorem should be presented first, as it is much easier to grasp. Both might be presented first in the simple case of Abelian varieties, then Shimura varieties, then for general $\mathbb{Z}$-variations of Hodge structure [KlinglerICM, Theor. 4.4] of Bakker-Tsimerman (the mixed version of Chiu and Gao-Klingler might be mentioned, but the emphasis should be on the pure case). Sketch the proof, following [BT20, Sections 6-8]. It uses the content of Lecture 7.

1pm - 3pm: Lunch break.

Free afternoon.

8pm - Social dinner.

5. THURSDAY

9.30.am - 10.50am - Lecture 11: Preparations for Theorem 1.3 following [EFW18]. [Lerer]

Rigidity of algebraic hull/invariant bundles: Prove [Fil16a, Thm. 1.4] that the measurable and the "polynomial" algebraic hulls coincide, following §7 of [Fil16a]. Three separate steps:

(1) on unstables: invariant bundles are real-analytic
(2) on unstables: real analytic bundles are polynomial
(3) joint polynomiality

For step (1), state Ledrappier invariance principle [Fil16a, Prop. 7.1] and emphasize crucial use of Hodge-orthogonality to get the Oseledets filtrations. For step (2), explain the game of expanding/contracting action of diagonal group. For step (3), note the many other results of the flavor: a function of two variables is "good" in each variable separately, then it is jointly "good".

Computation of Algebraic Hull: [EFW18, §3]. Give the statement, cover as much as time permits the proof based on the assertions in loc. cit. Crucial input is the polynomiality of invariant bundles, stated as [EFW18, Thm. 3.3]. Emphasize [EFW18, Prop. 3.7] and the description of the Grassmannian of 2-planes as $\mathrm{Gr}^{\circ}(2, T\mathcal{M}) = G/G_{\tau}$ where G is the (semi-direct product) symplectic group and G_{τ} is the stabilizer of the 2-plane.

10.50am - 11.30am- Coffee break.

11.30am - 12.50am - Lecture 12: First run through proofs of BFMS theorems: the real hyperbolic case. [Einsiedler]

Give the proof of [BFMS21, Theorem 1.1] as in section 4.3 of that paper, but use first the simpler definition of compatibility.

Definition 5.1. *Let G be $SO(n,1)$ or $SU(n,1)$ and $P < G$ a parabolic subgroup and U the unipotent radical of P. Let k be a local field and H a k-algebraic group. We say that H is* strongly compatible *with G if for every non-trivial k-algebraic subgroup $J < H$ and every homomorphism $P \to N_H(J)/J(k)$ the image of U is trivial.*

Explain that more complicated definition is only about H being split $G = SO(3,1)$, no details.

It is fine to not prove compatiblity, but only to sketch how it is easily obstructed for e.g. $H = \mathrm{SO}(p,q)$ if $\min p, q > 3$ using compact subgroups of P "not fitting in" H.

Point out that $SU(n,1)$ fails to be compatible with itself because of the center of the Heisenberg group. And that the inclusion $SU(n,1) < SL(n+1,\mathbb{C})$ shows that the $SL(n+1,\mathbb{C})$ is also not compatible.

Define incompatiblity datum motivated by this [BFMS, Section 3].

1pm - 3pm: Lunch break.

3pm - 4.30pm - Lecture 13: Proof of Theorem 1.2 and of [BU, Theor. 1.2.1]**.** This lecture is divided into two parts.

3pm - 3.45pm - Lecture 13a: Proof of Theorem 1.2. [Tayou] This lecture is 45 minutes long.

Recall [BKU, Theor.3.1], already stated in Lecture 4. Sketch the proof, following [BKU, Section 6]. The main ingredient is the Ax-Schanuel theorem explained in Lecture 10.

Together with [BKU, Theor 3.3] whose proof was (hopefully) sketched in Lecture 4, this implies Theorem 1.2: see [BKU, Section 8].

3.45pm - 4.30pm - Lecture 13b: Proof of [BU, Theor. 1.2.1]. [Stover] This lecture is 45 minutes long.

Recall the statement of [BU, Theor.1.2.1], already given in Lecture 3.

The goal is to reduce the proof to [BKU, Theor.3.1]. To do so:

- construction of a $\mathbb{Z}$-variation of Hodge structure: [BU, Theor. 1.3.1]. Sketch the proof: [BU, Section 3 and 4].

- identify the totally geodesic subvarieties as the special ones: [BU, Theor. 1.3.2]. Sketch of the proof: [BU, Section 5].

- Show that they are atypical: [BU, Prop. 6.1.4];

- conclude (the proof of finiteness in [BU] is subsumed in [BKU, Theor.3.1]).

4.30pm - 5pm - Coffee break.

5pm - 6.20pm - Discussions.

6. Friday

9.30.am - 10.50 am - Lecture 14: Proof of Theorem 1.3 following [EFW18]**.** [Möller]

Structure of Monodromy: Outline briefly the results of [EFW18, §4] about the Zariski closure of monodromy over invariant subvarieties.

Algebraic Hulls along limits: Prove [EFW18, Thm. 5.2] (absolute case). First proof is closer in formulation to the techniques of [BFMS21], so do that version. Emphasize the role of equidistribution from [EMM15]. Time permitting, discuss the second proof in §5.3.

Finiteness and Abundance: Cover [EFW18, §6.1] on finiteness from knowledge of algebraic hulls and the proof of the main goal of this subseries: Theorem 1.3. Prove the abundance result [EFW18, Thm. 1.7] as in §6.2 in loc.cit. Mention that it is analogous to the abundance of typical special subvarieties, when they exist. Lecture 14 [-]

10.50 - 11.30 - Coffee break.

11.30am - 12.50am - Lecture 15: Proofs of BFMS results: the complex hyperbolic case. [Baldi]

Recall the definition of incompatibility datum.

State that the only cases in which incompatibility arises are $\mathbf{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n, 1)$.

Briefly say that results of Simpson eliminates the first one.

Focus on the case of $\mathrm{SU}(n, 1)$ and explain how the incompatibility datum becomes a chain preserving boundary map following Section 6 of [BFMS].

If time permits, explain some ideas in Pozzetti's proof of [BFMS, Theorem 6.5] see [P15, Sections 4 and 5].

1pm - 3pm: Lunch break and discussion.

3pm- 4.20pm - Lecture 16: Hodge loci in level 1 and 2, and equidistribution. [Ullmo]
Discuss [BKU, Theor. 3.9] and its proof.
Discuss [TaTh], as a powerful equidistribution upgrading of [BKU, Theor. 3.9].

4.20pm - 5pm - Coffee break.

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