ETALE COHOMOLOGY AND THE WEIL CONJECTURES

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1. What is the "shape" of a scheme?

Starting in 1895 Poincaré [Po95] associated natural invariants to any separated locally compact path-connected topological space X: its (co)homology groups $H^\bullet_{\mathrm{Betti}}(X, \mathbb{Q})$ and its fundamental group $\pi_1(X)$, the group of loops in X up to homotopy. While cohomology groups were originally defined in many different ways (simplicial cohomology, cellular cohomology, singular cohomology...) Eilenberg and Steenrod [ES52] axiomatized in 1952 the properties of any good cohomology theory, with the effect that all these definitions essentially coincide. From a modern point of view a unifying definition is via *sheaf theory*:

$$H^\bullet_{\mathrm{Betti}}(X, \mathbb{Q}) := H^\bullet(X, \mathbb{Q}_X) \ ,$$

where $\mathbb{Q}_X$ denotes the constant sheaf with value $\mathbb{Q}$ on X. These cohomology groups are invariant under homotopy equivalence for X. Nowadays the shape of X has to be understood as the class of X in the homotopy category of spaces.

Suppose now that X is a scheme. We would like to understand its shape, in particular its cohomology and its fundamental group. The underlying topological space $|X|$ with its Zariski topology is usually not separated, in a very strong sense. Recall the

Definition 1.0.1. *A topological space X is* irreducible *if any two non-empty open subsets of X have non-empty intersection. A scheme X is irreducible if $|X|$ is.*

If we define the cohomology groups of X as $H^\bullet_{\mathrm{Betti}}(|X|, \mathbb{Q})$, this definition makes sense but is of no interest:

Lemma 1.0.2. *(Grothendieck) If Y is an irreducible topological space then $H^\bullet(Y, \mathcal{F}) = 0$ for any constant sheaf $\mathcal{F}$ on X.*

Proof. Let $F := H^0(X, \mathcal{F})$ be the group of global sections of the sheaf $\mathcal{F}$ on X. As $\mathcal{F}$ is constant, it is the sheafification of the presheaf with value F on any connected open subset U of X. As Y is irreducible any open subset of Y is connected. Hence $\mathcal{F}(U) = F$ for any open subset U of Y and the sheaf $\mathcal{F}$ is flasque, in particular acyclic. □

As a consequence, any reasonable definition of the "shape of a scheme" will depend not only on the underlying topological space but also on the finer schematic structure.

1.1. **Characteristic zero.** In the case where X/k is a separated scheme of finite type over a field k of characteristic $\operatorname{char} k = 0$ which admits an embedding $\sigma : k \hookrightarrow \mathbb{C}$ (i.e. k has cardinality at most the continuum), one can try using the embedding σ to define invariants attached to X. Let us write

$$X^\sigma := X \times_{k,\sigma} \mathbb{C} \ (:= X \times_{\operatorname{Spec} k,\sigma} \operatorname{Spec} \mathbb{C}) \ .$$

A natural topological invariant associated to X/k and σ is then

$$H^\bullet_{\mathrm{Betti}}((X^\sigma)^{\mathrm{an}}, \mathbb{Q}) \ ,$$

where $(X^\sigma)^{\mathrm{an}}$ denotes the complex analytic space associated to X. How does it depend on σ?

Theorem 1.1.1 (Serre)**.** *Suppose that X is a smooth projective variety over k. The dimension $b_i(X) := \dim_{\mathbb{Q}} H^i_{\mathrm{Betti}}((X^\sigma)^{\mathrm{an}}, \mathbb{Q})$ is independent of σ. We call it the i-th Betti number of X.*

Proof.

$$\begin{aligned} H^i_{\mathrm{Betti}}((X^\sigma)^{\mathrm{an}}, \mathbb{C}) &\simeq \bigoplus_{p+q=i} H^p((X^\sigma)^{\mathrm{an}}, \Omega^q_{(X^\sigma)^{\mathrm{an}}}) \\ &\simeq \bigoplus_{p+q=i} H^p(X^\sigma, \Omega^q_{X^\sigma}) \ . \end{aligned}$$

The first isomorphism is the Hodge decomposition for the cohomology of the smooth complex projective variety X^σ (see [Vois07] for a reference on Hodge theory). The second one is Serre's GAGA theorem [Se56] for smooth complex projective varieties. We conclude by noticing that $H^p(X^\sigma, \Omega^q_{X^\sigma}) = H^p(X, \Omega^q_X) \otimes_{k,\sigma} \mathbb{C}$ has dimension independent of σ. $\square$

Remark 1.1.2. Serre's theorem can be extended to quasi-projective varieties using more Hodge theory (logarithmic complex).

Even if embedding k in $\mathbb{C}$ defines unambiguously the Betti numbers of the scheme X/k, it does not define its fundamental group: in 1964 indeed, Serre constructed a smooth projective X over a number field k and $\sigma, \tau : l \hookrightarrow \mathbb{C}$ two different infinite places of k such that

$$\pi_1((X^\sigma)^{\mathrm{an}}) \not\simeq \pi_1((X^\tau)^{\mathrm{an}}) \ .$$

In particular $(X^\sigma)^{\mathrm{an}}$ is not in general homotopy equivalent to $(X^\tau)^{\mathrm{an}}$.

1.2. **Positive characteristic.** The situation is worse for schemes over a field k of positive characteristic. What do we understand as the "shape" of such a scheme? The question is particularly relevant if k is a finite field $\mathbb{F}_q$ (finite field with $q = p^n$ elements, p prime number). The basic theme of the Weil conjectures is that the shape of a separated scheme X of finite type over $\mathbb{F}_q$ is, in first approximation, described by counting points of X.

2. The Weil conjectures, first version

2.1. **Reminder on finite fields.** Let k be a field and consider the natural ring homomorphism $\mathbb{Z} \to k$ associating $n \cdot 1_k$ to $n \in \mathbb{Z}$. Its kernel is a prime ideal of $\mathbb{Z}$ hence of the form $p\mathbb{Z}$, p prime number, called the characteristic of k.

Suppose now that k is a finite field, hence necessarily of positive characteristic p. In particular $\mathbb{F}_p \hookrightarrow k$ and k is a finite dimensional $\mathbb{F}_p$-vector-space, so $|k| = p^n$ for some $n \in \mathbb{N}^*$. Fix an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$.

Theorem 2.1.1. *Let p be a prime number. For any $n \in \mathbb{N}^*$, there exists a unique field $\mathbb{F}_q$ of cardinal $q = p^n$ (up to isomorphism). If $\mathrm{Fr}_p : \overline{\mathbb{F}_p} \to \overline{\mathbb{F}_p}$ is the* arithmetic Frobenius *associating to x its p-power $\mathrm{Fr}_p(x) := x^p$, the field $\mathbb{F}_q$ is the fixed field $\overline{\mathbb{F}_p}^{\mathrm{Fr}_p^n=1}$.*

The field $\mathbb{F}_{p^n}$ is nothing else than the splitting field of $X^{p^n} - X$, in particular it is Galois over $\mathbb{F}_p$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ generated by Fr_q and

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \varprojlim_n \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \varprojlim \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$$

topologically generated by Fr_p.

Examples 2.1.2.

$$\begin{aligned}
\mathbb{F}_4 &= \mathbb{F}_2[X]/(X^2 + X + 1)\\
\mathbb{F}_8 &= \mathbb{F}_2[X]/(X^3 + X + 1)\\
\mathbb{F}_9 &= \mathbb{F}_3[X]/(X^2 + 1)
\end{aligned}$$

2.2. **Schematic points.** Which points of X do we want to count? Recall that we have two different notions of points for schemes.

The first notion of point for a scheme X is the obvious one: an element $x \in |X|$. We call such a point a *schematic point* of X. Define by

$$Z(x) := \overline{\{x\}}$$

the associated closed subscheme of X. One obtains natural partitions of $|X| = \coprod_{r\in\mathbb{N}} X^{(r)} = \coprod_{r\in\mathbb{N}} X_{(r)}$, where

$$\begin{aligned}
X^{(r)} &= \{x \in |X|, \mathrm{codim}_X Z(x) = r\}\\
X_{(r)} &= \{x \in |X|, \dim Z(x) = r\}
\end{aligned}$$

Here dimension and codimension are understood in their topological sense: the dimension $\dim Z$ of a scheme Z is the maximum length n of a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ of non-empty closed irreducible subsets of Z; the codimension $\mathrm{codim}_X Y$ of a closed irreducible subscheme $Y \subset X$ is the maximal length n for a chain $Z_0 = Y \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ of closed irreducible subsets of X.

Remark 2.2.1. Recall that $\mathrm{codim}_X Y$ is not necessarily equal to $\dim X - \dim Y$, even for X irreducible: take $X = \mathrm{Spec}\, k[[t]][u]$, $Y = V(tu-1)$. Then $\dim Y$ is the Krull dimension of the field $k[[t]][u]/(tu-1) = k[[t]][\frac{1}{t}]$, hence zero. On the other hand $\dim X = \dim k[[t]] + 1 = 2$ and $\mathrm{codim}_X Y$ is the height of the ideal $(tu-1)$, hence 1.

The set $|X|$ is usually infinite. To count schematic points, we will need a notion of "size" which guarantees that there are only finitely many schematic points of given size. This can be conveniently done for any scheme X of finite type over $\mathbb{Z}$ if one restrains oneself to the atomization $X_{(0)}$ of X.

The following lemma is straightforward:

Lemma 2.2.2. *Let X be a scheme of finite type over $\mathbb{Z}$ and let $x \in |X|$. The following properties are equivalent:*

(a) $x \in X_{(0)}$.
(b) *the residue field $k(x)$ is finite.*

If we define the norm *of $x \in X_{(0)}$ as $N(x) = |k(x)|$, there are only finitely many $x \in X_{(0)}$ with given norm.*

Our vague question about counting points can thus be precisely rephrased as:

Problem 2.2.3. *Given a scheme X of finite type over $\mathbb{Z}$, compute the number of points of $X_{(0)}$ of given norm.*

This problem looks even more natural if one extends it a little bit. Let us consider the case $X = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ denotes the ring of integers of a number field K. We want to count not only prime ideals of $\mathcal{O}_K$ (i.e. points of $X_{(0)}$) but *all* ideals of $\mathcal{O}_K$. As $\mathcal{O}_K$ is a Dedekind ring we are in fact counting *effective* zero-cycles on X in the sense of the following:

Definition 2.2.4. *Let X be a scheme. The group of algebraic cycles on X is the free abelian group $Z(X) :=< |X| >$ generated by the points of X. Hence an element $\alpha \in Z(X)$ is a linear combination $\alpha = \sum_{i=1}^{r} n_i \cdot x_i$, $n_i \in \mathbb{Z}$, $x_i \in |X|$. The cycle α is said to be* effective *if all n_i are positive.*

The group $Z(X)$ is naturally graded: $Z^p(X) =< X^{(p)} >$ or $Z_p(X) =< X_{(p)} >$. If X is of finite type over $\mathbb{Z}$ the norm N on $X_{(0)}$ extends to $N : Z_0(X) \to \mathbb{Q}$ by

$$N(\sum_{i=1}^{r} n_i \cdot x_i) = \prod_{i=1}^{r} N(x_i)^{n_i} \quad .$$

Once more there are only finitely many *effective* zero-cycles of given norm and Problem 2.2.3 can be extended to:

Problem 2.2.5. *Given a scheme X of finite type over $\mathbb{Z}$, compute the number of effective zero-cycles on X of given norm.*

2.3. **Scheme-valued points.** The second notion of points come from the interpretation of a scheme as a functor.

Definition 2.3.1. *Let S be a scheme and X, T two S-schemes. One defines the set of T-points of X as:*

$$X(T)_S := \operatorname{Hom}_S(T, X) \quad .$$

We denote $X(T) := X(T)_{\mathbb{Z}}$.

We are interested in the case $T = \operatorname{Spec} K$, K a field.

Proposition 2.3.2. *Let K be a field. Then*

$$X(K) = \{(x,i), x \in |X|, i : k(x) \to K \text{ field homomorphism}\} \ .$$

More generally if $k \subset K$ is a field extension:

$$X(K)_k = \coprod_{x \in |X|} \mathrm{Hom}_{k-\mathrm{alg}}(k(x), K) \ .$$

Proof. Let $s : \mathrm{Spec}\, K \to X$ be a morphism. It is entirely defined by the continuous map $|s| : |\mathrm{Spec}\, K| \to |X|$, i.e. the point x image of $|\mathrm{Spec}\, K|$, plus the morphism of sheaves of rings $s^{-1}\mathcal{O}_X \to \mathcal{O}_{\mathrm{Spec}\, K}$ over $\mathrm{Spec}\, K$, i.e. the ring morphism

$$\Gamma(\mathrm{Spec}\, K, s^{-1}\mathcal{O}_X) =: \mathcal{O}_{X,x} \to \Gamma(\mathrm{Spec}\, K, \mathcal{O}_{\mathrm{Spec}\, K}) = K \ .$$

Notice that the ring morphism $\mathcal{O}_{X,x} \to k$ uniquely factorizes through $k(x)$.

The proof of the generalization is similar. □

Exercice 2.3.3. Show that $X(T)_{\mathbb{F}_p} = X(T)$ and $X(T)_{\mathbb{Q}} = X(T)$ but that $X(T)_k \neq X(T)$ for a general field k.

Let X be a scheme of finite type over $\mathbb{F}_q$. Counting points of X can also be understood as:

Problem 2.3.4. *Given a scheme X of finite type over $\mathbb{F}_q$, compute the number $|X(\mathbb{F}_{q^r})_{\mathbb{F}_q}|$ for all positive integers r.*

2.4. **Counting points for schemes of finite type over $\mathbb{F}_q$.** Problem 2.2.3 and Problem 2.3.4 are essentially equivalent for schemes of finite type over $\mathbb{F}_q$:

Lemma 2.4.1. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then*

$$|X(\mathbb{F}_{q^r})_{\mathbb{F}_q}| = \sum_{e|r} e \cdot \left|\{x \in X_{(0)} / \deg(x) (:= [k(x) : \mathbb{F}_q]) = e\}\right| \ .$$

Proof. By Proposition 2.3.2:

$$X(\mathbb{F}_{q^r})_{\mathbb{F}_q} = \coprod_{x \in |X|} \mathrm{Hom}_{\mathbb{F}_q}(k(x), \mathbb{F}_{q^r}) \ .$$

In particular if $\mathrm{Hom}_{\mathbb{F}_q}(k(x), \mathbb{F}_{q^r}) \neq 0$, the field $k(x)$ is finite hence x belongs to $X_{(0)}$ thanks to Lemma 2.2.2. Moreover $\deg(x)|r$. Hence:

$$X(\mathbb{F}_{q^r})_{\mathbb{F}_q} = \coprod_{\deg(x)|r} \mathrm{Hom}_{\mathbb{F}_q}(k(x), \mathbb{F}_{q^r}) \ .$$

Now $\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q) \simeq \mathbb{Z}/r\mathbb{Z}$ acts transitively on $\mathrm{Hom}_{\mathbb{F}_q}(k(x), \mathbb{F}_{q^r})$, with stabilizer $\mathrm{Gal}(\mathbb{F}_{q^r}/k(x)) \simeq \mathbb{Z}/(\frac{r}{\deg(x)}) \cdot \mathbb{Z}$. Thus:

$$|\mathrm{Hom}_{\mathbb{F}_q}(k(x), \mathbb{F}_{q^r})| = \deg(x) \ .$$

□

Counting points of a scheme X of finite type over $\mathbb{F}_q$ is usually a hard problem. The following immediate corollary of Lemma 2.4.1 enable us to do it in simple cases.

Corollary 2.4.2. *If a scheme X of finite type over $\mathbb{F}_q$ satisfies $X_{(0)} = \coprod (X_i)_{(0)}$ for a family (X_i) of subschemes (i.e. closed subscheme of open subscheme) of X then*

$$X(\mathbb{F}_{q^r})_{\mathbb{F}_q} = \coprod X_i(\mathbb{F}_{q^r})_{\mathbb{F}_q} \quad .$$

Examples 2.4.3. (1) $X = \mathbb{A}^n_{\mathbb{F}_q}$. As $\mathbb{A}^n_{\mathbb{F}_q}(\mathbb{F}_{q^r})_{\mathbb{F}_q} = \mathrm{Hom}_{\mathbb{F}_q-\mathrm{alg}}(\mathbb{F}_q[T_1, \cdots, T_n], \mathbb{F}_{q^r}) \simeq (\mathbb{F}^q)^{\oplus n}$ one obtains

$$|\mathbb{A}^n_{\mathbb{F}_q}(\mathbb{F}_{q^r})_{\mathbb{F}_q}| = q^{nr} \quad .$$

(2) $X = \mathbf{P}^n_{\mathbb{F}_q}$. As $(\mathbf{P}^n_{\mathbb{F}_q})_{(0)} = \coprod_{i=0}^n (\mathbb{A}^i_{\mathbb{F}_q})_{(0)}$ one obtains

$$|\mathbf{P}^n_{\mathbb{F}_q}(\mathbb{F}_{q^r})_{\mathbb{F}_q}| = 1 + q^r + q^{2r} + \cdots + q^{nr} \quad .$$

(3) Let us give an example which shows that counting points is usually difficult. Let

$$X = \{y^2 = x^3 + x,\ y \neq 0 \subset \mathbb{A}^2_{\mathbb{F}_q}\} \subset \overline{X} = \{y^2 z = x^3 + xz^2 \subset \mathbf{P}^2_{\mathbb{F}_q}\} \quad .$$

The variety X is an affine curve, its closure $\overline{X}$ in $\mathbf{P}^2_{\mathbb{F}_q}$ is an elliptic curve. Hence:

$$\begin{aligned} |\overline{X}(\mathbb{F}_q)_{\mathbb{F}_q}| &= |X(\mathbb{F}_q)_{\mathbb{F}_q}| + |\{u \in \mathbb{F}_q, u^3 + u = 0\}| + |\{u \in \mathbb{F}_q, u^3 = 0\}| \\ &= |X(\mathbb{F}_q)_{\mathbb{F}_q}| + \begin{cases} 1+1 & \text{if } \sqrt{-1} \notin \mathbb{F}_q \quad (i.e.\ q = -1 \mod 4) \\ 3+1 & \text{if } \sqrt{-1} \in \mathbb{F}_q \quad (i.e.\ q = 1 \mod 4) \end{cases} \end{aligned}$$

Recall that there are exactly $(q-1)/2$ squares in $\mathbb{F}_q^*$.

Let us assume that $q = -1 \mod 4$. As -1 is not a square, if $c \in \mathbb{F}_q^*$ then either c or $-c$ is a square but not both. Hence $u^3 + u$ or $-(u^3 + u) = ((-u)^3 + (-u))$ is a square but not both. Hence $a^3 + a = b^2$ for exactly $(q-1)/2$ values of a, with two choices for b each time. Hence

$$|X(\mathbb{F}_q)_{\mathbb{F}_q}| = 2 \times \frac{q-1}{2} = q - 1 \quad \text{and} \quad |\overline{X}(\mathbb{F}_q)_{\mathbb{F}_q}| = (q-1) + 2 = q + 1 \quad .$$

I don't know of any general procedure for $q = 1 \mod 4$. For $q = 5$ writing the table of all possibilities one obtains $|X(\mathbb{F}_5)_{\mathbb{F}_5}| = 0$, hence $|\overline{X}(\mathbb{F}_5)_{\mathbb{F}_5}| = 4$.

2.5. **Weil conjectures, first version.** In [We49] Weil proposed a general set of conjectures describing the number of points of any scheme of finite type over $\mathbb{F}_q$ (we will come back later to the history of these conjectures). Recall first:

Definition 2.5.1. *A q-Weil number of weight $m \in \mathbb{N}$ is an algebraic number whose Archimedean valuations are all $q^{m/2}$.*

Remark 2.5.2. In the literature Weil numbers are sometimes assumed to be algebraic integers.

Example 2.5.3. $1 \pm 2i$ is a 5-Weil number of weight 1.

Conjecture 2.5.4 (Weil)**.** *Let X be a scheme of finite type over $\mathbb{F}_q$.*

1. *(Rationality) There exists a finite set of algebraic integers α_i, β_j such that:*

$$\forall r \in \mathbb{N}, \quad |X(\mathbb{F}_{q^r})_{\mathbb{F}_q}| = \sum \alpha_i^r - \sum \beta_j^r \quad .$$

2. *(Functional equation) If X is smooth and proper of pure dimension d then $\gamma \mapsto \frac{q^d}{\gamma}$ induces a permutation of the α_i's and a permutation of the β_j's.*

3. *(Purity) If X has dimension d, the α_i's and β_j's are Weil q-numbers of weights in* $[0, 2d]$.

   If moreover X is smooth and proper the weights of the α_i's are even while the weights of the β_j's are odd.
4. *(link with topology) Suppose that $X/\mathbb{F}_q$ is the smooth and proper special fiber of a smooth and proper $\mathcal{X}/R$, $\mathbb{F}_q \twoheadleftarrow R \hookrightarrow \mathbb{C}$. Then*

$$\dim_{\mathbb{C}} H^m((\mathcal{X}_{\mathbb{C}})^{\mathrm{an}}, \mathbb{C}) = \begin{cases} |\{\alpha_i, \mathrm{weight}(\alpha_i) = m\}| & \text{if } m \text{ is even,} \\ |\{\beta_j, \mathrm{weight}(\beta_j) = m\}| & \text{if } m \text{ is odd.} \end{cases}$$

3. Zeta functions

We will refine our understanding of points for a scheme X of finite type over $\mathbb{Z}$ by constructing a generating function encoding the numbers $N(x)$, $x \in X_{(0)}$. How do we construct such a generating function? There is no general recipe. We can only learn through experiment, starting with Euler and Riemann.

3.1. Riemann zeta function.

The Riemann zeta function is the well-known function of one complex variable s:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad .$$

It encodes the "counting" of points of $\operatorname{Spec} \mathbb{Z}$, i.e. of prime numbers. It was first studied by Euler (around 1735) for s real, then by Riemann for s complex (1859). This function serves as a model for any other zeta or L-function. Let us prove its basic properties.

3.1.1. *Convergence.*

Proposition 3.1.1 (Riemann)**.** *The function $\zeta(s)$ converges absolutely (uniformly on compacts) on the domain $\mathrm{Re}(s) > 1$, where it defines a holomorphic function. It diverges for $s = 1$.*

Proof. Write $s = u + iv$, $u, v \in \mathbb{R}$. Then $|n^{-s}| = n^{-u}$. To prove absolute convergence we can thus assume that s belongs to $\mathbb{R}$.

For $s \in \mathbb{R}$ the function $t \mapsto t^{-s}$ is decreasing. Thus the series $\sum_1^\infty n^{-s}$ converges if and only if the integral $\int_1^\infty t^{-s} dt$ converges. Hence the convergence for $s > 1$ and the divergence at $s = 1$.

The previous comparison yields:

$$|\sum_{N}^{\infty} n^{-s}| \leq \int_{N-1}^{\infty} t^{-\mathrm{Re}(s)} dt = \frac{(N-1)^{1-\mathrm{Re}(s)}}{\mathrm{Re}(s) - 1} \quad ,$$

which proves the uniform convergence on compacts.

The function $\zeta(s)$ is a limit, uniform on compacts, of holomorphic functions. Hence it is holomorphic. □

3.1.2. *Euler product.*

Proposition 3.1.2 (Euler)**.**

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \quad ,$$

where the product on the right is absolutely convergent for $\mathrm{Re}(s) > 1$*.*

To prove Proposition 3.1.2, we introduce the notion of a completely multiplicative function:

Definition 3.1.3. *A function* $a : \mathbb{N}^* \to \mathbb{C}$ *is* completely multiplicative *if* $a(mn) = a(m)a(n)$ *for all* $m, n \in \mathbb{N}^*$*.*

Proposition 3.1.2 follows immediately from the following lemma:

Lemma 3.1.4. *Let* $a : \mathbb{N}^* \to \mathbb{C}$ *be completely multiplicative. The following are equivalent:*

(i) $\sum_1^\infty |a(n)| < +\infty$.
(ii) $\prod_p \frac{1}{1-a(p)} < +\infty$.

If one of these equivalent conditions is satisfied then

$$\sum_1^\infty a(n) = \prod_p \frac{1}{1-a(p)} \quad .$$

Proof. Assume (i). Thus for any prime p the sum $\sum_m a(p^m)$ converges absolutely, with sum $\frac{1}{1-a(p)}$. Let $E(x) \subset \mathbb{N}^*$ be the set of integers whose prime factors are smaller than x. As

$$\sum_{n \in E(x)} a(n) = \prod_{p<x} \sum_m a(p^m) = \prod_{p<x} \frac{1}{1-a(p)} \quad ,$$

one obtains

$$\left| \sum_1^\infty a(n) - \prod_{p<x} \frac{1}{1-a(p)} \right| = \left| \sum_{n \notin E(x)} a(n) \right| \leq \sum_{n \geq x} |a(n)| \quad .$$

Hence $\prod_p \frac{1}{1-a(p)}$ converges to $\sum_1^\infty a(n)$, absolutely (replacing a by $|a|$).

Conversely assume (ii). Then

$$\sum_{n<x} |a(n)| \leq \sum_{n \in E(x)} |a(n)| = \prod_{p<x} \frac{1}{1-a(p)}$$

hence (i). □

3.1.3. *Formal Dirichlet series.* It will be convenient to first work with formal series, without convergence questions:

Definition 3.1.5. *A formal Dirichlet series is* $f = \sum_1^\infty \frac{a_n}{n^s}$ *where* $n \in \mathbb{N}^*$, $a_n \in \mathbb{C}$. *Given another formal Dirichlet series* $g = \sum_1^\infty \frac{b_n}{n^s}$ *one defines*

$$f + g = \sum_{n\geq 1} \frac{a_n + b_n}{n^s} \quad ;$$

$$f \cdot g = \sum_{n\geq 1} \frac{c_n}{n^s} \quad \text{with} \quad c_n = \sum_{pq=n} a_p b_q \quad .$$

Formal Dirichlet series form a commutative ring $\mathrm{Dir}(\mathbb{C})$, where we can perform formal computations.

Definition 3.1.6. *Let* $f = \sum_1^\infty \frac{a_n}{n^s}$ *be a formal Dirichlet series. If* $f \neq 0$ *one defines the order* $\omega(f)$ *as the smallest integer* n *with* $a_n \neq 0$ *(if* $f = 0$ *one puts* $\omega(f) = +\infty$*).*

Notice that the subsets $\{f \mid \omega(f) \geq N\}$ are ideals of $\mathrm{Dir}(\mathbb{C})$. They define a topology on $\mathrm{Dir}(\mathbb{C})$ making $\mathrm{Dir}(\mathbb{C})$ a complete topological ring. Hence:

Corollary 3.1.7. *A sequence* $(f_n)_{n\in\mathbb{N}}$ *of* $\mathrm{Dir}(\mathbb{C})$ *is summable if and only if* $\lim_{n\to+\infty} \omega(f_n) = +\infty$*; a sequence* $(1 + f_n)_{n\in\mathbb{N}}$*, with* $\omega(f_n) > 1$ *for all* n*, is multipliable in* $\mathrm{Dir}(\mathbb{C})$ *if and only if* $\lim_{n\to+\infty} \omega(f_n) = +\infty$.

Lemma 3.1.4 implies immediately the following:

Proposition 3.1.8. *Let* $a : \mathbb{N}^* \to \mathbb{C}$ *be a completely multiplicative function. Consider the formal identity*

$$\sum_{n\geq 1} \frac{a(n)}{n^s} = \prod_p \left(1 - \frac{a(p)}{p^s}\right)^{-1} \quad .$$

Given a real number α*, the left hand side converges absolutely for* $\mathrm{Re}(s) > \alpha$ *if and only if the right hand side converges absolutely for* $\mathrm{Re}(s) > \alpha$*.*

3.1.4. *Extension to* $\mathrm{Re}(s) > 0$.

Proposition 3.1.9 (Riemann)**.** *The function* $\zeta(s)$ *extends meromorphically to* $\mathrm{Re}(s) > 0$ *with a unique simple pole at* $s = 1$ *and residue* 1.

Proof. Define $\zeta_2(s) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}$. Recall:

Lemma 3.1.10 (Abel)**.** *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be two sequences of complex numbers. Then*

$$\sum_{n=m}^{m'} a_n b_n = \sum_{n=m}^{m'-1} \left(\sum_{i=m}^{n} a_i\right)(b_n - b_{n+1}) + \left(\sum_{n=m}^{m'} a_n\right) b_{m'} \quad .$$

In particular if there exists $\varepsilon > 0$ such that $|\sum_{i=m}^n a_i| \leq \varepsilon$ for all $m \leq n \leq m'$ and if the sequence $(b_n)_{n\in\mathbb{N}}$ is real and decreasing then

$$\left|\sum_{n=m}^{m'} a_n b_n\right| \leq \varepsilon b_m \quad .$$

Using Abel's lemma for $a_n = (-1)^n$ and $b_n = n^{-s}$ one checks that $\zeta_2(s)$ converges (not absolutely!) for $\mathrm{Re}(s) > 0$. Notice that

$$\zeta(s) - \zeta_2(s) = \sum_n \frac{1}{n^s}(1-(-1)^{n+1}) = \sum_{n=2k} \frac{1}{2^s k^s} \cdot 2 = 2^{1-s}\zeta(s)$$

hence $\zeta_2(s) = (1-2^{1-s})\zeta(s)$. So $(s-1)\zeta(s)$ extends meromorphically to $\mathrm{Re}(s) > 0$.

As $\zeta_2(1) = \log 2$ one obtains $\lim_{s\to 1}(s-1)\zeta(s) = 1$.

More generally for $r \in \mathbb{N} \setminus \{0,1\}$ we define

$$\zeta_r(s) = \frac{1}{1^s} + \cdots + \frac{1}{(r-1)^s} - \frac{r-1}{r^s} + \frac{1}{(r+1)^s} + \cdots$$

Once more: $\zeta_r(s)$ is analytic for $\mathrm{Re}(s) > 0$ and

$$\left(1 - \frac{1}{r^{s-1}}\right)\zeta(s) = \zeta_r(s) \quad .$$

Suppose that $s \neq 1$ is a pole of $\zeta(s)$.

- for $r = 2$ one obtains $2^{s-1} = 1$ hence $s = \frac{2\pi\sqrt{-1}k}{\log 2} + 1$ for some $k \in \mathbb{N}^*$.
- for $r = 3$ one obtains similarly $s = \frac{2\pi\sqrt{-1}l}{\log 3} + 1$ for some $l \in \mathbb{N}^*$.

Hence $3^k = 2^l$: contradiction.

□

3.1.5. *Extension to $\mathbb{C}$ and functional equation.* Recall the Γ function:

$$\Gamma(s) = \int_0^{+\infty} t^s e^{-t} \frac{dt}{t} \quad ,$$

which converges for $\mathrm{Re}(s) > 0$. It satisfies $\Gamma(1) = 1$ and the functional equation $\Gamma(s+1) = s\Gamma(s)$. Hence $\Gamma(s)$ extends meromorphically to $\mathbb{C}$ with a simple pole at $s = -n$, $n \in \mathbb{N}$, with residue $(-1)^n/n$.

Theorem 3.1.11 (Riemann)**.** *The function $\zeta(s)$ extends to a function on $\mathbb{C}$, holomorphic except for a single pole at $s = 1$. If $\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ then away from 0 and 1 the function $\Lambda(s)$ is bounded in any vertical strip and satisfies $\Lambda(1-s) = \Lambda(s)$.*

In particular $\zeta(s)$ does not vanish for $\mathrm{Re}(s) > 1$. In $\mathrm{Re}(s) < 0$ it has simple zeroes at -2, -4, -6, etc... All the other zeroes are inside the "critical strip" $0 \leq \mathrm{Re}(s) \leq 1$.

Proof. For $\mathrm{Re}(s) > 1$:

$$\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \sum_{n\geq 1}\int_0^\infty e^{-t}t^{\frac{s}{2}}\pi^{-\frac{s}{2}}n^{-s}\frac{dt}{t}$$
$$= \int_0^\infty \left(\sum_{n\geq 1} e^{-\pi u n^2}\right) u^{\frac{s}{2}}\frac{du}{u} = \int_0^\infty \tilde{\theta}(u)\frac{u^{\frac{s}{2}}du}{u}$$

where we made the change of variables $t = \pi u n^2$ and defined $\theta(u) := \sum_{n\in\mathbb{Z}} e^{-\pi u n^2}$ and $\tilde{\theta}(u) := \sum_{n\geq 1} e^{-\pi u n^2} = \dfrac{\theta(u)-1}{2}$.

Recall that the Fourier transform of a real integrable function f is $\hat{f}(y) = \int_{\mathbb{R}} f(x)\exp(2\pi i x y)dx$ and that the Poisson formula states the equality:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \quad .$$

Considering $f(x) = e^{-\pi u x^2}$ one obtains $\hat{f}(y) = \dfrac{e^{-\frac{\pi y^2}{u}}}{\sqrt{u}}$ and the Poisson formula reads:

$$\theta(\frac{1}{u}) = \sqrt{u}\theta(u) \tag{1}$$

(in other words: the theta function θ is a modular form of half integer weight). Equation (1) implies

$$\tilde{\theta}(\frac{1}{u}) = \sqrt{u}\tilde{\theta}(u) + \frac{1}{2}(\sqrt{u}-1) \quad .$$

Since $\int_1^\infty t^{-s}dt = \frac{1}{s-1}$ one obtains:

$$\begin{aligned}\Lambda(s) &= \int_0^1 \tilde{\theta}(u)\frac{u^{\frac{s}{2}}du}{u} + \int_1^\infty \tilde{\theta}(u)\frac{u^{\frac{s}{2}}du}{u}\\ &= \int_1^\infty \tilde{\theta}(\frac{1}{u})\frac{u^{-\frac{s}{2}}du}{u} + \int_1^\infty \tilde{\theta}(u)\frac{u^{\frac{s}{2}}du}{u}\\ &= \int_1^\infty \left(\sqrt{u}\tilde{\theta}(u) + \frac{1}{2}(\sqrt{u}-1)\right)\frac{u^{-\frac{s}{2}}du}{u} + \int_1^\infty \tilde{\theta}(u)\frac{u^{\frac{s}{2}}du}{u}\\ &= \int_1^\infty \tilde{\theta}(u)\cdot\left(u^{\frac{s}{2}} + u^{\frac{1-s}{2}}\right)\frac{du}{u} + \frac{1}{s-1} - \frac{1}{s} \quad .\end{aligned}$$

The right hand side integral is a priori defined only for $\mathrm{Re}(s) > 1$ but using $\tilde{\theta}(u) = O(e^{-\pi u})$ one easily checks it is entire. Moreover it is clearly bounded in every vertical strip. Finally the right hand side is symmetric with respect to $s \mapsto 1-s$.

□

Theorem 3.1.12 (Hadamard- De La Vallée-Poussin, 1896)**.** *$\zeta(s)$ does not vanish on* $\mathrm{Re}(s) = 1$.

Proof. $\log \zeta(s) = \sum_{m\geq 1,p} \frac{p^{-ms}}{m}$ hence

$$\mathrm{Re}\left(3\log\zeta(u) + 4\log\zeta(u+iv) + \log\zeta(u+2iv)\right) = \sum_{p,m} \frac{p^{-mu}}{m}\mathrm{Re}(3+4p^{-miv}+p^{-2miv}) \ .$$

Using that $3 + 4\cos t + \cos(2t) = 2(1+\cos t)^2 \geq 0$ one obtains:

$$\zeta(u)^3|\zeta(u+iv)|^4|\zeta(u+2iv)| \geq 1 \ .$$

The left hand side is equivalent to $c(u-1)^{k+4h-3}$ as $u \to 1$, where c denotes a positive constant and h, k denote the order of $\zeta(x)$ at $s = u+iv$ and $u+2iuv$ respectively. Hence $k + 4h - 3 \leq 0$ hence $h = 0$ as $h, k \geq 0$. □

This result is enough to show the *prime number theorem* (cf. [El75, chap.2]):

Theorem 3.1.13 (Hadamard- De La Vallée-Poussin, 1896)**.** *Define $\pi(x)$ as the number of primes smaller than x. Then*

$$\pi(x) \sim \frac{x}{\log x} \ .$$

Theorem 3.1.11 enables to state the famous

Conjecture 3.1.14 (Riemann hypothesis)**.** *All the zeroes of $\zeta(s)$ inside the critical strip lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

We refer to [El75] for a nice survey on the relation between the Riemann hypothesis and the distribution of prime numbers.

3.2. Zeta functions for schemes of finite type over $\mathbb{Z}$.

3.2.1. *Arithmetic zeta function for schemes of finite type over $\mathbb{Z}$.* The definition of $\zeta(\mathrm{Spec}\,\mathbb{Z}, s) := \zeta(s)$ as an Euler product generalizes naturally to any scheme X of finite type over $\mathbb{Z}$:

Definition 3.2.1. *Let X be a scheme of finite type over $\mathbb{Z}$. One defines*

$$\zeta(X, s) = \prod_{x\in X_{(0)}} \frac{1}{1-N(x)^{-s}} \ .$$

Remarks 3.2.2.
(a) By Proposition 2.3.2 there are only finitely many points $x \in X_{(0)}$ of given norm. Hence Corollary 3.1.7 implies that $\zeta(X, s)$ is a formal Dirichlet series.
(b) Notice that $\zeta(X, s)$ depends only on $X_{(0)}$.

Developping the product $\prod_{x\in X_{(0)}} \frac{1}{1-N(x)^{-s}}$ one obtains as in the case of the zeta function:

Lemma 3.2.3. *Let X be a scheme of finite type over $\mathbb{Z}$. Then*

$$\zeta(X, s) = \sum_{c\in Z_0(X)^+} \frac{1}{N(c)^s}$$

in $\mathrm{Dir}(\mathbb{C})$*, where $Z_r(X)^+$ denotes the monoïd of effective r-cycles of X.*

Moreover one immediately obtains from the definition:

Lemma 3.2.4. *Let X be a scheme of finite type over $\mathbb{Z}$. If X satisfies $X_{(0)} = \coprod (X_i)_{(0)}$ for a family (X_i) of subschemes of X then*

$$\zeta(X,s) = \prod_{i=1}^{\infty} \zeta(X_i, s) \quad .$$

In particular

$$\zeta(X,s) = \prod_{p} \zeta(X_p, s) \quad ,$$

where X_p is the fiber of $X \to \operatorname{Spec} \mathbb{Z}$ over p.

3.3. **Geometric zeta function for a scheme of finite type over $\mathbb{F}_q$.** If X is a scheme of finite type over $\mathbb{F}_q$, one can introduce a more natural generating series encoding the points of X: its *geometric zeta function*.

Definition 3.3.1. *Let X be a scheme of finite type over $\mathbb{F}_q$. Its geometric zeta function is defined as:*

$$Z(X/\mathbb{F}_q, t) := \exp\left(\sum_{n=1}^{\infty} \left| X(\mathbb{F}_{q^n})_{\mathbb{F}_q} \right| \frac{t^n}{n} \right) = \sum_{c \in Z_0(X)^+} t^{\deg c} \quad .$$

Here the degree of a zero-cycle $c = \sum_i n_i \cdot x_i \in Z_0(X)$ is defined as $\deg c = \sum_i n_i [k(x) : \mathbb{F}_q]$.

Remark 3.3.2. Let X be a scheme of finite type over $\mathbb{F}_q$. Notice that the degree $\sum_i n_i [k(x) : \mathbb{F}_q]$ of a zero-cycle $c = \sum_i n_i \cdot x_i \in Z_0(X)$ depends on the base field $\mathbb{F}_q$, while $N(c)$ does not. As a corollary the geometric zeta function $Z(X/\mathbb{F}_q, t)$ really depends on the base field: if X is defined over $\mathbb{F}_{q^r}$ then $Z(X/\mathbb{F}_q, t) = Z(X/\mathbb{F}_{q^r}, t^r)$. On the other hand $\zeta(X,s)$ is an absolute notion.

The following obvious formula is the basis for any calculation with the geometric zeta function:

Lemma 3.3.3. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then*

$$t \cdot \frac{d}{dt} \log Z(X/\mathbb{F}_q, t) = \sum_{n=1}^{\infty} \left| X(\mathbb{F}_{q^n})_{\mathbb{F}_q} \right| t^n \quad .$$

The relation between $\zeta(X,s)$ and $Z(X/\mathbb{F}_q, t)$ is given by the following:

Lemma 3.3.4. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then $\zeta(X,s) = Z(X/\mathbb{F}_q, q^{-s})$.*

Proof.

$$\log \zeta(X,s) = \sum_{x \in X_{(0)}} -\log(1 - N(x)^{-s}) = \sum_{x \in X_{(0)}} \sum_{m=1}^{\infty} \frac{N(x)^{-ms}}{m}$$

$$= \sum_{m=1}^{\infty} \sum_{x \in X_{(0)}} \frac{N(x)^{-ms}}{m} = \sum_{m=1}^{\infty} \sum_{x \in X_{(0)}} \frac{q^{-m \deg(x) s}}{m} = \sum_{n=1}^{\infty} \left(\sum_{\substack{x \in X_{(0)} \\ \deg(x) | n}} \deg(x) \right) \frac{q^{-ns}}{n}$$

By Lemma 2.4.1 $|X(\mathbb{F}_{q^n})_{\mathbb{F}_q}| = \sum_{\substack{x\in X_{(0)}\\ \deg(x)|n}} \deg(x)$ hence the result. □

3.4. **Properties of zeta functions.**

Lemma 3.4.1. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then $\zeta(\mathbb{A}^1_X, s) = \zeta(X, s-1)$.*

Proof. Applying Lemma 3.2.4 one obtains

$$\zeta(\mathbb{A}^1_X, s) = \prod_{x\in X_{(0)}} \zeta(\mathbb{A}^1_x, s) \quad .$$

Hence we are reduced to show that $\zeta(\mathbb{A}^1_x, s) = \zeta(x, s-1)$. Applying Lemma 3.3.4 and writing $k(x) = \mathbb{F}_q$ one gets

$$\begin{aligned}\zeta(\mathbb{A}^1_x, s) &= \exp\left(\sum_{n=1}^{\infty} |\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^n})_{\mathbb{F}_q}| \frac{q^{-ns}}{n}\right)\\ &= \exp\left(\sum_{n=1}^{\infty} q^n \cdot \frac{q^{-ns}}{n}\right) = \frac{1}{1-q^{1-s}}\\ &= \zeta(x, s-1) \quad .\end{aligned}$$

□

Theorem 3.4.2. *Let X be a scheme of finite type over $\mathbb{Z}$. Then $\zeta(X,s)$ converges for $s > \dim X$.*

Proof. Step 1: One can assume that X is irreducible. Indeed suppose that $X = X_1 \cup X_2$, $X_i \subset X$ closed subscheme, $i = 1, 2$. It follows from proposition 3.2.4 that

$$\zeta(X, s) = \frac{\zeta(X_1, s)\cdot\zeta(X_2, s)}{\zeta(X_1\cap X_2, s)} \quad ,$$

where $X_1 \cap X_2$ denotes the schematic intersection $X_1 \times_X X_2 \hookrightarrow X$. Hence the statement for X_1 and X_2 implies the statement for X by induction on the dimension.

Step 2. One can assume that X is affine (and integral). Indeed if $Z \hookrightarrow X \hookleftarrow U$ one similarly obtains

$$\zeta(X, s) = \zeta(U, s)\cdot\zeta(Z, s) \quad .$$

Hence the statements for U and X are equivalent by induction on the dimension.

Step 3. If $f : X \to Y$ is a finite morphism and if the statement holds for Y then it holds for X. Indeed it follows from Lemma 3.2.4 that:

$$\zeta(X, s) = \prod_{y\in Y_{(0)}} \zeta(X_y, s) \quad .$$

Let d be the degree of f, the fiber X_y has at most d closed points. If $x \in (X_y)_{(0)}$ is such a point then $N(x)$ is a power of $N(y)$ hence for $\mathrm{Re}(s) > 0$:

$$\left|\frac{1}{1-N(x)^{-s}}\right| \leq \left|\frac{1}{1-N(y)^{-s}}\right|$$

This implies:

$$|\zeta(X_y, s)| \leq |\zeta(y, s)|^d \ .$$

It follows that $|\zeta(X, s)| \leq |\zeta(Y, s)|^d$ and the result.

Step 4. We can assume that $X = \mathbb{A}^d_{\mathbb{Z}}$ or $X = \mathbb{A}^d_{\mathbb{F}_p}$. Indeed let $X \to \operatorname{Spec} \mathbb{Z}$ be affine and integral. Recall the

Lemma 3.4.3. *(Noether normalization lemma) For any field k and any finitely generated commutative k-algebra A, there exists a nonnegative integer d and algebraically independent elements $x_1, \cdots, x_d$ in A such that A is a finitely generated module over the polynomial ring $k[x_1, \cdots, x_d]$*

Equivalently: every affine k-scheme of finite type is finite over an affine d-dimensional space.

- if $X \to \operatorname{Spec} \mathbb{Z}$ is dominant, it follows from Noether normalisation lemma applied to $X_{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ that there exists a finite flat morphism $X_{\mathbb{Q}} \to \mathbb{A}^d_{\mathbb{Q}}$. It extends to a finite flat morphism $f : X_U \to \mathbb{A}^d_U$ for some open subset $U \subset \operatorname{Spec} \mathbb{Z}$. We can assume that $X = X_U$ by step 2, then $X = \mathbb{A}^d_U$ by step 3, then $X = \mathbb{A}^d_{\mathbb{Z}}$ by step 2 again.
- otherwise there exists some prime p so that X is of finite type over $\mathbb{F}_p$. Applying Noether normalization lemma to $X \to \operatorname{Spec} \mathbb{F}_p$ we are reduced to $X = \mathbb{A}^d_{\mathbb{F}_p}$.

If $X = \mathbb{A}^d_{\mathbb{Z}}$ then Lemma 3.4.1 gives $\zeta(\mathbb{A}^d_{\mathbb{Z}}, s) = \zeta(s - d)$, which converges absolutely for $\operatorname{Re}(s) > d + 1 = \dim X$.

If $X = \mathbb{A}^d_{\mathbb{F}_p}$ then $\zeta(\mathbb{A}^d_{\mathbb{F}_p}, s) = \dfrac{1}{1 - p^{d-s}}$, which converges absolutely for $\operatorname{Re}(s) > d = \dim X$.

□

3.5. Some examples.

Example 3.5.1. $X = \mathbb{A}^n_{\mathbb{F}_q}$. We computed $\left|\mathbb{A}^n_{\mathbb{F}_q}(\mathbb{F}_{q^r})_{\mathbb{F}_q}\right| = q^{nr}$ hence

$$Z(\mathbb{A}^n_{\mathbb{F}_q}, s) = \exp\left(\sum_{m=1}^{\infty} q^{mn} \frac{t^m}{m}\right) = \frac{1}{1 - q^n t} \ .$$

Example 3.5.2. $X = \mathbf{P}^n_{\mathbb{F}_q}$. We computed $\left|\mathbf{P}^n_{\mathbb{F}_q}(\mathbb{F}_{q^r})_{\mathbb{F}_q}\right| = 1 + q^r + \cdots + q^{nr}$ hence

$$Z(\mathbf{P}^n_{\mathbb{F}_q}, s) = \exp\left(\sum_{m=1}^{\infty} (1 + q^m + \cdots + q^{nm}) \frac{t^m}{m}\right) = \frac{1}{(1-t)\cdot(1-qt)\cdots(1-q^n t)} \ .$$

3.6. Some questions and conjectures.

Question 3.6.1. *Let X be a scheme of finite type over $\mathbb{Z}$. Suppose that we know $\zeta(X, s)$. What can we say of X?*

- If $X = \operatorname{Spec} \mathcal{O}_K$ is the ring of integers of a number field K then $\zeta(\mathcal{O}_K, s) = \zeta_K(s)$ is the Dedekind zeta function of K. One shows that

$$\operatorname{ord}_{s=0} \zeta_K(s) = r_1 + r_2 - 1 =: r \quad \text{and} \quad \lim_{s \mapsto 0} s^{-r} \zeta_K(s) = -\frac{h_K \cdot R_K}{\omega_K} \ ,$$

where r_1 is the number of real places $\sigma_1, \cdots, \sigma_{r_1}$ of K, r_2 is the number of complex places $\sigma'_1, \cdots, \sigma'_{r_2}$ not conjugate two by two, $h_K = |\mathrm{Pic}\,\mathcal{O}_K|$ is the class number of K, ω_K is the number of roots of unity of K and R_K is its regulator i.e. the covolume of the lattice $\mathcal{O}_K^*/torsion$ in $\mathbb{R}^r$ under the regulator map

$$\mathrm{reg} : \mathcal{O}_K^* \longrightarrow \mathbb{R}^r = \{\textstyle\sum_{i=1}^{r+1} x_i = 0\} \subset \mathbb{R}^{r_1+r_2}$$

$$u \longrightarrow \left(\log\sigma_1(u), \ldots, \log\sigma_{r_1}(u), 2\log\sigma'_1(u), \ldots, 2\log\sigma'_{r_2}(u)\right) \quad .$$

A theorem of Mihaly Bauer (1903) says that if K, L are two number fields which are Galois over $\mathbb{Q}$ then $K \simeq L$ is equivalent to $\zeta_K = \zeta_L$. On the other hand Gassmann (1936) showed that there do exist non-isomorphic number fields K, L (in fact $h_K \neq h_L$) with $\zeta_K = \zeta_L$. The example of smallest degree occur in degree 7 over $\mathbb{Q}$.

- If X is a smooth projective curve over a finite field $\mathbb{F}_q$, the curve X is not determined by its zeta function. However Tate (1966) and Turner (1978) proved that two curves X, Y over $\mathbb{F}_q$ satisfy $\zeta(X, s) = \zeta(Y, s)$ if and only if their Jacobians are isogeneous.

Conjecture 3.6.2. *Let X be a scheme of finite type over $\mathbb{Z}$. The function $\zeta(X, s)$ extends meromorphically to all $\mathbb{C}$ and satisfies a functional equation with respect to $s \mapsto \dim X - s$.*

This is proved for $d_X = 1$, for some very particular cases for $d_X > 1$ when X is flat over $\mathbb{Z}$ and for all d_X when X is a scheme of finite type over $\mathbb{F}_p$ (the so called positive characteristic case).

It follows from the Weil conjectures that $\zeta(X, s)$ always has a meromorphic continuation to $\mathrm{Re}(s) > \dim X - 1/2$.

3.7. **Weil conjectures.** We now concentrate on the positive characteristic case.

Definition 3.7.1. *A q-Weil polynomial (resp.* pure of weight $m \in \mathbb{N}$*) is a polynomial*

$$P := \prod_{i=1}^{r} (1 - \gamma_i t) \in \mathbb{Z}[t]$$

where the γ_i's are q-Weil numbers (resp. of same weight m*).*

The Weil conjectures can be stated as follows:

Conjecture 3.7.2 (Weil)**.** *Let X be a scheme of finite type over $\mathbb{F}_q$ of dimension d.*

1. *(Rationality) $Z(X/\mathbb{F}_q, t) \in \mathbb{Q}(t)$.*
2. *(Functional equation) If X is smooth and proper of pure dimension d, let χ be the self-intersection of the diagonal in $X \times X$. Then*
$$Z(X/\mathbb{F}_q, \frac{1}{q^d t}) = \pm q^{\frac{d \cdot \chi}{2}} t^{\chi} Z(X, t) \quad .$$
3. *(Purity) If X has dimension d then*
$$Z(X/\mathbb{F}_q, t) = \prod_{i=0}^{2d} P_i(t)^{(-1)^{i+1}}$$

where the P_i's are q-Weil polynomials.

If moreover X is smooth and proper of pure dimension d then the P_i's are pure of weight i and

$$P_{2d-i}(t) = C_i\, t^{\deg P_i}\, P_i(\frac{1}{q^d t}) \quad \text{with} \quad C_i \in \mathbb{Z} \ .$$

4. *(link with topology)* $\chi = \sum_{i=0}^{2d}(-1)^i \deg P_i$. *If moreover $X/\mathbb{F}_q$ is the smooth and proper special fiber of a smooth and proper $\mathcal{X}/R$, R finitely generated $\mathbb{Z}$-algebra, $\mathbb{F}_q \twoheadleftarrow R \hookrightarrow \mathbb{C}$, then*
$$\deg P_i = b_i((\mathcal{X}_{\mathbb{C}})^{\mathrm{an}}) \ .$$
In particular χ coïncides with the Euler characteristic $\chi(\mathcal{X}_{\mathbb{C}})^{\mathrm{an}}$.

Remark 3.7.3. The rationality of $Z(X/\mathbb{F}_q, s)$ is already a highly non-trivial result. It implies in particular that if we know $|X(\mathbb{F}_{q^r})|$ for sufficiently many (depending on X) values of $r \in \mathbb{N}$ then we know $|X(\mathbb{F}_{q^r})|$ for all $r \in \mathbb{N}$.

Remark 3.7.4. To prove rationality of zeta functions (Conjecture 3.7.2(1)), it is enough to prove it for X an irreducible hypersurface in $\mathbb{A}^n_{\mathbb{F}_q}$. Indeed arguing as in Lemma 3.2.4 and by induction on dimension we can assume that X is irreducible and affine. But then (using generic projections) X is birational over $\mathbb{F}_q$ with a hypersurface in an affine space and we are done by induction on dimension.

4. The Weil conjectures for curves

In this section we prove the Weil conjectures for a smooth projective, geometrically irreducible, curve C over a finite field $\mathbb{F}_q$. Recall first that the fundamental invariant of the curve C is its genus $g = h^0(C, \omega_C)$ where $\omega_C = \Omega^1_{C/\mathbb{F}_q}$ is the canonical line bundle of C. Second, it follows from [SGA1] (as we will see later) that the curve C is always the special fiber of some smooth projective curve $\mathcal{C}$ over a finitely generated $\mathbb{Z}$-algebra R, $\mathbb{F}_q \twoheadleftarrow R \hookrightarrow \mathbb{C}$ [SGA1], so that we are in the situation of Conjecture 3.7.2.4. Classically the smooth projective complex curve $\mathcal{C}_{\mathbb{C}}$ satisfies $b_0(\mathcal{C}_{\mathbb{C}}) = b_2(\mathcal{C}_{\mathbb{C}}) = 1$ and $b_1(\mathcal{C}_{\mathbb{C}}) = 2g$, hence $\chi(\mathcal{C}_{\mathbb{C}}) = 2 - 2g$. Hence the Weil conjectures for C are the following:

Theorem 4.0.1 (E.Artin, Schmidt, Hasse, Weil)**.** *Let C be a geometrically irreducible smooth projective curve of genus g over $\mathbb{F}_q$. Then:*

$$Z(C/\mathbb{F}_q, t) = \frac{P(t)}{(1-t)(1-qt)}$$

where $P(t) = \prod_{i=1}^{2g}(1-\alpha_i t) \in \mathbb{Z}[t]$ is a polynomial of degree $2g$ and constant term 1, with inverse roots α_i of absolute value $|\alpha_i| = \sqrt{q}$ for any embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$. Moreover it satisfies the functional equation:

$$Z(C/\mathbb{F}_q, \frac{1}{qt}) = q^{1-g} t^{2-2g} Z(C/\mathbb{F}_q, t) \ .$$

Corollary 4.0.2 (Riemann hypothesis for curves over finite fields)**.** *Let C be a geometrically irreducible smooth projective curve of genus g over $\mathbb{F}_q$. Then all the roots of $\zeta(C, s)$ lie on the line* $\mathrm{Re}(s) = 1/2$.

Proof. As $\zeta(C,s) = Z(C/\mathbb{F}_q, q^{-s})$, the roots of $\zeta(C,s)$ are the roots of $P(q^{-s}) = \prod_{i=1}^{2g}(1-\alpha_i q^{-s})$. The purity condition $|\gamma_i| = \sqrt{q}$ for any embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ is thus equivalent to saying that these roots lie on the line $\mathrm{Re}(s) = 1/2$. $\square$

Let us give some short historical comments (we refer to [Aud12] for more details). In his thesis E.Artin (1921) defined the zeta function of a quadratic extension of $\mathbb{F}_q((t))$ and proved its rationality (in p^{-s}). F.K. Schmidt [Sch31] generalized Artin's definition to any function field over $\mathbb{F}_q$ in one variable. He deduced the rationality and the functional equation for $Z(C/\mathbb{F}_q, t)$ from his proof of the Riemann-Roch theorem for C. In [Ha36] H.Hasse proved the Riemann Hypothesis for elliptic curves over finite field. A.Weil [We40] announced the proof of the Riemann Hypothesis for curves over finite fields and gave a complete proof eight years later after a complete refoundation of algebraic geometry.

We now indicate the general strategy for the proof of the Weil conjectures for curves (we refer to [Sil09, Chap.5] for an elementary proof in the case of elliptic curves). While it is difficult to understand 0-cycles on a general scheme, a zero cycle on the curve C is nothing else than a divisor. Counting points on C is thus equivalent to counting sections of line bundles on C. The Riemann-Roch formula provides a complete answer to this problem for line bundles of degree big enough. The rationality of $Z(C/\mathbb{F}_q, t)$ follows immediately. The functional equation for $Z(C/\mathbb{F}_q, t)$ is then a shadow of Serre duality for the cohomology of line bundles on curves. As is the case in higher dimension, the most delicate part of Theorem 4.0.1 is purity, equivalently the Riemann Hypothesis. It is easily seen to be equivalent to proving the bounds

$$\big| |C(\mathbb{F}_{q^n})_{\mathbb{F}_q}| - q^n - 1 \big| \leq 2g\sqrt{q^n} \quad . \tag{2}$$

For proving these bounds, we introduce one of the main player of this entire course: the *geometric Frobenius* $\mathrm{Fr}_{X,q}$, a canonically defined endomorphism of any scheme X over $\mathbb{F}_q$. The set $C(\mathbb{F}_{q^n})_{\mathbb{F}_q}$ can be interpreted as the intersection in $(C \times C)_{\overline{\mathbb{F}_q}}$ of the graph of $\mathrm{Fr}^n_{\overline{C},q}$ with the diagonal Δ. The bounds eq. (2) then follow from the Hodge index theorem on the surface $(C \times C)_{\overline{\mathbb{F}_q}}$.

4.1. **Heuristics.**

4.2. **The Riemann-Roch's formula.**

Let k be a field and C be a smooth projective curve over k. We denote by $\overline{C}$ its base change to an algebraic closure $\overline{k}$ of k. If $\pi : \overline{C} \to C$ is the natural projection then $H^i(C, \mathcal{F}) \otimes_k \overline{k} \simeq H^i(\overline{C}, \pi^*\mathcal{F})$ for any quasicoherent $\mathcal{O}_C$-module $\mathcal{F}$. We will assume that C is geometrically irreducible, i.e. $\overline{C}$ is irreducible.

The group $Z_0(C)$ coincide with the group of Weil divisors $Z^1(C)$ on C. As C is smooth (in particular integral, separated and locally factorial) the group of Weil divisor coincide with the group $H^0(C, F_C^*/\mathcal{O}_C^*)$ of Cartier divisors on C. Here F denotes the function field of C and F_C the associated constant sheaf on C (as C is integral it coïncides with the sheaf of rational functions on C). Moreover principal Weil divisors and principal Cartier divisors do coincide.

To any divisor D, seen as a Cartier divisor (U_i, f_i), we associate the line bundle $\mathcal{O}(D) \subset F_C$ on C generated as an $\mathcal{O}_C$-module by f_i^{-1} on U_i. We denote by $h^0(\mathcal{O}(D))$

the k-dimension of its space of global sections

$$H^0(C, \mathcal{O}(D)) = \{f \in F^* \mid D + (f) \in Z_0(C)^+\} \ .$$

This defines an isomorphism between the group $Z_0(C)/\sim$, where two divisors are rationally equivalent if their difference is principal, with the group $\mathrm{Pic}\,(C)$ of isomorphism classes of line bundles on C. The degree morphism $\deg : Z_0(C) \to \mathbb{Z}$ descends to $\deg : \mathrm{Pic}\,(C) \to \mathbb{Z}$. Moreover one has a short exact sequence:

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}\,(C) \stackrel{\deg}{\to} \mathbb{Z} \ .$$

A priori the degree map is not surjective:

Definition 4.2.1. *We denote by $\delta > 0$ the* index *of the curve C: the unique positive integer such that*

$$\deg(\mathrm{Pic}\,(C)) = \delta\mathbb{Z} \ .$$

Remark 4.2.2. Notice that $\delta | 2g - 2 = \deg \omega_C$. For curves over $\mathbb{F}_q$ we will show that $\delta = 1$.

Given a line bundle $\mathcal{L}$ on C the set of effective divisors D on C with $\mathcal{O}(D) \simeq \mathcal{L}$ is in bijection with the quotient $H^0(C, \mathcal{L}) \setminus 0$ by the action of $H^0(C, \mathcal{O}_C^*)$ via multiplication. As $\overline{C}$ is irreducible and projective we obtain $H^0(\overline{C}, \mathcal{O}_{\overline{C}}) = \overline{k}$ hence $H^0(C, \mathcal{O}_C) = k$.

The Riemann-Roch formula states that for any line bundle $\mathcal{L}$ on C one has:

$$h^0(\mathcal{L}) - h^0(\omega_C \otimes \mathcal{L}^{-1}) = \deg(\mathcal{L}) + 1 - g \ . \tag{3}$$

As a corollary:

$$\text{If } \deg(\mathcal{L}) > 2g - 2 \text{ then } h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g \ . \tag{4}$$

4.3. **Rationality.**

Proposition 4.3.1. *Let C be a smooth projective, geometrically irreducible, curve over $\mathbb{F}_q$. Then $Z(C/\mathbb{F}_q, t)$ is a rational function.*

Proof.

$$Z(C, t) = \sum_{D \in Z_0(C)^+} t^{\deg(D)} \ .$$

It follows from our discussion of the relation between line bundles and effective divisors that:

$$\begin{aligned}
Z(C/\mathbb{F}_q, t) &= \sum_{\substack{\mathcal{L} \in \mathrm{Pic}\, C \\ \deg \mathcal{L} \geq 0}} \left| \mathbf{P} H^0(C, \mathcal{L}) \right| \cdot t^{\deg \mathcal{L}} = \sum_{\substack{\mathcal{L} \in \mathrm{Pic}\, C \\ \deg \mathcal{L} \geq 0}} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg \mathcal{L}} \\
&= \sum_{0 \leq \deg \mathcal{L} \leq 2g-2} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg \mathcal{L}} + \sum_{2g-2 < \deg \mathcal{L}} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg \mathcal{L}} \\
&= \sum_{0 \leq \deg \mathcal{L} \leq 2g-2} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg \mathcal{L}} + \sum_{2g-2 < \deg \mathcal{L}} \frac{q^{\deg \mathcal{L} + 1 - g} - 1}{q - 1} \cdot t^{\deg \mathcal{L}} \ .
\end{aligned}$$

Lemma 4.3.2. *The group $\mathrm{Pic}^0(C)$ is finite.*

Proof. Fix $n > 2g$ a multiple of δ. Then any divisor D of degree n satisfies $h^0(\mathcal{O}(D)) = n + 1 - g > 0$ hence is effective. Thus the group $\mathrm{Pic}^0(C)$ has a (free) orbit in $\mathrm{Pic}(C)$ consisting precisely of the rational equivalence classes of effective divisors of degree n. As we already saw (see the discussion after Definition 2.2.4) that the number of effective divisors of degree n on C is finite the result follows. □

Remark 4.3.3. Of course the "correct proof" is as follows: for any k-variety X there exists a k-variety $\mathrm{Pic}^0 X$ whose set of k'-points is the group $\mathrm{Pic}^0(X \times_k k')$ for any field extension k' of k. In our case $\mathrm{Pic}^0(C) = (\mathrm{Pic}^0 C)(\mathbb{F}_q)$ hence is necessarily finite.

Our computation of $Z(C/\mathbb{F}_q, s)$ continues as:

$$Z(C/\mathbb{F}_q, t) = \sum_{0 \leq \deg \mathcal{L} \leq 2g-2} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg \mathcal{L}} + |\mathrm{Pic}^0(C)| \cdot \sum_{\rho := \frac{2g-2}{\delta} < n} \frac{q^{n\delta + 1 - g} - 1}{q - 1} \cdot t^{n\delta} \ .$$

Notice that the first term

$$f_1(t) := \sum_{0 \leq \deg \mathcal{L} \leq 2g-2} \frac{q^{h^0(\mathcal{L})} - 1}{q - 1} \cdot t^{\deg \mathcal{L}}$$

is a polynomial in t^δ of degree at most $\rho = (2g-2)/\delta$. On the other hand one computes the second term

(5)
$$f_2(t) := |\mathrm{Pic}^0(C)| \cdot \sum_{\rho < n} \frac{q^{n\delta + 1 - g} - 1}{q - 1} \cdot t^{n\delta} = \frac{|\mathrm{Pic}^0(C)|}{q - 1} \cdot \left(q^{1-g} \cdot \frac{(qt)^{\delta(\rho+1)}}{1 - (qt)^\delta} - \frac{t^{\delta(\rho+1)}}{1 - t^\delta} \right) \ .$$

One concludes that one might write

$$Z(C/\mathbb{F}_q, t) = \frac{P(t^\delta)}{(1 - t^\delta)(1 - (qt)^\delta)} \ , \tag{6}$$

where P is a polynomial with rational coefficients, of degree less than $\rho + 2$.

Since $Z(C/\mathbb{F}_q, t)$ has integer coefficients one obtains that P has integer coefficients as well.

This shows that $Z(C/\mathbb{F}_q, t)$ is a rational function.

□

Proposition 4.3.4. *The curve C has index $\delta = 1$: it admits a divisor of degree 1.*

Proof. Looking at the expression 5 for f_2 one obtains:

$$\lim_{t \to 1} (t - 1) Z(C/\mathbb{F}_q, t) = -\frac{|\mathrm{Pic}^0(C)|}{q - 1} \cdot \lim_{t \to 1} \frac{t - 1}{1 - t^\delta} = \frac{|\mathrm{Pic}^0(C)|}{\delta(q - 1)} \ .$$

In particular $Z(C/\mathbb{F}_q, t)$ has a pole of order one at $t = 1$.

Lemma 4.3.5. *Let X be a variety over $\mathbb{F}_q$. Then*

$$Z(X \times_{\mathbb{F}_q} \mathbb{F}_{q^r} / \mathbb{F}_{q^r}, t^r) = \prod_{i=1}^{r} Z(X/\mathbb{F}_q, \xi^i t) \ ,$$

where ξ denotes a primitive root of order r of 1.

Proof.

$$\begin{aligned}
\log Z((X \times_{\mathbb{F}_q} \mathbb{F}_{q^r})/\mathbb{F}_{q^r}, t^r) &= \sum_{m=1}^{\infty} \left|(X \times_{\mathbb{F}_q} \mathbb{F}_{q^r})(\mathbb{F}_{q^{mr}})_{\mathbb{F}_{q^r}}\right| \cdot \frac{t^{mr}}{m} \\
&= \sum_{m=1}^{\infty} \left|X(\mathbb{F}_{q^{mr}})_{\mathbb{F}_q}\right| \cdot \frac{t^{mr}}{m} \quad \text{as} \quad (X \times_{\mathbb{F}_q} \mathbb{F}_{q^r})(\mathbb{F}_{q^{mr}})_{\mathbb{F}_{q^r}} = X(\mathbb{F}_{q^{mr}})_{\mathbb{F}_q} \\
&= \sum_{l=1}^{\infty} |X(\mathbb{F}_{q^l})_{\mathbb{F}_q}| \cdot (\sum_{i=1}^{r} \xi^{il}) \cdot \frac{t^l}{l} \quad \text{as} \quad \sum_{i=1}^{r} \xi^{il} = \delta_{r,l} \cdot r \\
&= \sum_{i=1}^{r} \sum_{l=1}^{\infty} |X(\mathbb{F}_{q^l})_{\mathbb{F}_q}| \cdot \frac{(\xi^i t)^l}{l} \\
&= \sum_{i=1}^{r} \log Z(X/\mathbb{F}_q, \xi^i t) \ .
\end{aligned}$$

□

It follows from Lemma 4.3.5 and the Formula eq. (6) that $Z((C \times_{\mathbb{F}_q} \mathbb{F}_{q^\delta})/\mathbb{F}_{q^\delta}, t^\delta) = Z(C/\mathbb{F}_q, t)^\delta$. On the other hand we can apply our results so far to $C \times_{\mathbb{F}_q} \mathbb{F}_{q^\delta}$: the function $Z((C \times_{\mathbb{F}_q} \mathbb{F}_{q^\delta})/\mathbb{F}_{q^\delta}, t)$ has a pole of order one at 1, hence also $Z((C \times_{\mathbb{F}_q} \mathbb{F}_{q^\delta})/\mathbb{F}_{q^\delta}, t^\delta)$. Thus $\delta = 1$. This finishes the proof of Proposition 4.3.4. □

Remark 4.3.6. Even if C admits a divisor of degree 1 it does not necessarily admits an $\mathbb{F}_q$-point. Consider for example the genus 2 curve on $\mathbb{F}_3$ with affine equation

$$y^2 = -(x^3 - x)^2 - 1 \ .$$

This curve does not have any $\mathbb{F}_3$-point. However if y_1 and y_2 are the two roots of $y^2 = -1$ the divisor $D_1 := (0, y_1) + (0, y_2)$ is defined over $\mathbb{F}_3$. Similarly the divisor $D_2 := (x_1, 1) + (x_2, 1) + (x_3, 1)$ is defined over $\mathbb{F}_3$, where x_i, $1 \leq i \leq 3$, are the roots of $x^3 - x = -1$. Then $D := D_2 - D_1$ is a divisor of degree 1 defined over $\mathbb{F}_3$.

Corollary 4.3.7.

$$Z(C/\mathbb{F}_q, t) = \frac{P(t)}{(1-t)(1-qt)} \ ,$$

where $P \in \mathbb{Z}[t]$ is a polynomial of degree at most $2g$ and constant term 1.

4.4. **Functional equation.**

Proposition 4.4.1.

$$Z(C/\mathbb{F}_q, \frac{1}{qt}) = q^{1-g} t^{2-2g} Z(C/\mathbb{F}_q, t) \ .$$

Proof. We come back to our expression $Z(C/\mathbb{F}_q, t)$ obtained in the proof of Proposition 4.3.1. Rearranging this expression a little bit we write $Z(C/\mathbb{F}_q, t) = g_1(t) + g_2(t)$ with (as $\delta = 1$):

$$g_1(t) = \sum_{0 \leq \deg \mathcal{L} \leq 2g-2} \frac{q^{h^0(\mathcal{L})}}{q-1} \cdot t^{\deg \mathcal{L}} \quad \text{and} \quad g_2(t) = \frac{|\mathrm{Pic}^0(C)|}{q-1} \cdot \left(q^{1-g} \cdot \frac{(qt)^{2g-1}}{1-qt} - \frac{1}{1-t} \right)$$

A direct computation shows that $g_2(\frac{1}{qt}) = q^{1-g}t^{2-2g}f_2(t)$.

To deal with $g_1(t)$ notice that

$$\mathcal{L} \mapsto \omega_C \otimes \mathcal{L}^{-1}$$

defines an involution on the set of line bundles on C of degree in $[0, 2g-2]$. Hence:

$$\begin{aligned}
g_1(\frac{1}{qt}) &= \sum_{i=0}^{2g-2}\left(\sum_{\mathcal{L}\in \mathrm{Pic}^i(C)} \frac{q^{h^0(\mathcal{L})}}{q-1}\right)\cdot\left(\frac{1}{qt}\right)^i \\
&= \sum_{i=0}^{2g-2}\left(\sum_{\mathcal{L}\in \mathrm{Pic}^i(C)} \frac{q^{h^0(\omega_C\otimes\mathcal{L}^{-1})}}{q-1}\right)\cdot\left(\frac{1}{qt}\right)^{2g-2-i} \\
&= \sum_{i=0}^{2g-2}\left(\sum_{\mathcal{L}\in \mathrm{Pic}^i(C)} \frac{q^{h^0(\mathcal{L})}}{q-1}\right)\cdot (qt)^{i+2-2g} \quad \text{by the Riemann-Roch's formula} \\
&= q^{1-g}t^{2-2g}g_1(t) \quad .
\end{aligned}$$

□

Remark 4.4.2. Hidden in this proof is Serre duality: we identified $h^1(\mathcal{L})$ with $h^0(\omega_C \otimes \mathcal{L}^{-1})$ in the Riemann-Roch's formula.

Corollary 4.4.3. *The polynomial P is of degree exactly $2g$.*

Proof. This follows immediately from the functional equation. □

4.5. **The geometric Frobenius.** In this section we introduce the Frobenius endomorphism, whose role will be crucial in the proof of the Riemann Hypothesis for curves (the most delicate part of Theorem 4.0.1) and for this course in general.

Let X be a scheme of finite type over $\mathbb{F}_q$. Then one has the equality:

$$X(\mathbb{F}_{q^n})_{\mathbb{F}_q} = (X(\overline{\mathbb{F}_q})_{\mathbb{F}_q})^{\mathrm{Fr}_q^n} \quad .$$

There are however two conceptually different interpretations of the action of Fr_q on $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$.

(1) We already presented the first one. Consider Fr_q as a topological generator of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \simeq \hat{\mathbb{Z}}$. The action we are looking for is a special case of the natural action of $\mathrm{Gal}(k'/k)$ over $X(k')_k = \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k', X)$ via its natural action on $\mathrm{Spec}\, k'$ over k.

(2) On the other hand we can define a Frobenius endomorphism

$$\mathrm{Fr}_{X,q} : X \to X$$

as the morphism of local ringed spaces $(1_X, \mathrm{Fr}^\sharp_{X,q})$ where $\mathrm{Fr}^\sharp_{X,q} : \mathcal{O}_X \to \mathcal{O}_X$ maps $f \in \mathcal{O}_X(U)$ to $f^q \in \mathcal{O}_X(U)$. If $X = \mathrm{Spec}\, A$ is affine, with A a finitely generated $\mathbb{F}_q$-algebra, then $\mathrm{Fr}_{X,q}$ is just given by the algebra homomorphism $A \to A$ associating f^q to $f \in A$. The existence of this Frobenius endomorphism is what makes geometry over finite fields very different from geometry over any other field.

Now $\mathrm{Fr}_{X,q}$ acts on $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ as a particular case of the action of any endomorphism of X on $\mathrm{Hom}_{\mathrm{Spec}\,k}(\mathrm{Spec}\,k', X)$.

The morphism $\mathrm{Fr}_{X,q} : X \to X$ induces a morphism

$$\mathrm{Fr}_{\overline{X},q} = \mathrm{Fr}_{X,q} \times \mathrm{id} : \overline{X} \to \overline{X} \quad .$$

Lemma 4.5.1. *The actions of Fr_q on $X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ and $\mathrm{Fr}_{\overline{X},q}$ on $\overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}} = X(\overline{\mathbb{F}_q})_{\mathbb{F}_q}$ do coincide.*

In other words if Δ denotes the diagonal of $S = \overline{X} \times \overline{X}$ and Γ_r the graph of $\mathrm{Fr}^r_{\overline{X},q}$ then $X(\mathbb{F}_{q^r})_{\mathbb{F}_q}$ is in natural bijection with the closed points of $\Gamma_r \cap \Delta$.

4.6. The Riemann hypothesis for curves over finite fields.

Proposition 4.6.1. *Write $P(t) = \prod_{i=1}^{2g}(1 - \alpha_i t)$. Then every α_i is an algebraic integer and $|\alpha_i| = \sqrt{q}$ for any embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$.*

Proof. First notice that the functional equation implies:

$$\prod_{i=1}^{2g}(t - \frac{\alpha_i}{q}) = q^{-g}\prod_{i=1}^{2g}(1 - \alpha_i t) \quad .$$

As a consequence $\prod_{i=1}^{2g} \alpha_i = q^g$ and the multiset $\{\alpha_1, \cdots, \alpha_{2g}\}$ is invariant under the map $x \mapsto \frac{q}{x}$.

Hence it is enough to prove that $|\alpha_i| \leq \sqrt{q}$ for all i, $1 \leq i \leq 2g$: by the symmetry above one gets $|\alpha_i| \geq \sqrt{q}$ for all i, and the result.

For $n \in \mathbb{N}^*$ let us define $a_n := 1 + q^n - |C(\mathbb{F}_{q^n})_{\mathbb{F}_q}|$. Derivating the logarithm of the equality $P(t) = Z(C/\mathbb{F}_q, t)(1-t)(1-qt)$ and multiplying by t one obtains:

$$t \cdot \frac{d\log P(t)}{dt} = -\sum_{i=1}^{n} a_n t^n \quad . \tag{7}$$

Hence $a_n = \sum_{i=1}^{2g} \alpha_i^n$ for every $n \in \mathbb{N}^*$. One can then rephrase the Riemann Hypothesis for curves as an estimate for the a_n's:

Lemma 4.6.2. *One has $|\alpha_i| \leq \sqrt{q}$ for all i, $1 \leq i \leq 2g$, if and only if $|a_n| \leq 2g\sqrt{q^n}$ for every $n \geq 1$.*

Proof. As $a_n = \sum_{i=1}^{n} \alpha_i^n$ one implication is trivial. For the converse: if $|a_n| \leq 2gq^{n/2}$ then the series eq. (7) converges for $|t| < q^{-1/2}$. Hence $P(t)$ has no zeroes in this domain. By the functional equation $P(t)$ has no zeroes in $|t| > q^{-1/2}$. Finally all the zeroes of $P(t)$ have absolute value $q^{-1/2}$. □

Hence we are reduced to prove that $|a_n| \leq 2g\sqrt{q^n}$ for every $n \geq 1$. Notice it is enough to show that

$$|a_1| \leq 2g\sqrt{q} \quad .$$

Indeed applying this result to $C \times_{\mathbb{F}_q} \mathbb{F}_{q^n}$ yields the required inequality for $|a_n|$.

We will use intersection theory on surfaces. Consider the smooth projective surface $S := \overline{C} \times \overline{C}$ over $\overline{\mathbb{F}_q}$. We know that

$$N_1 := |C(\mathbb{F}_q)| = (\Gamma \cdot \Delta) \quad .$$

Recall the Hodge index theorem for a smooth projective surface over an algebraically closed field (see for example [Har77, Chap V.1]): if E is a divisor on S such that $(E \cdot H) = 0$ for H ample then $(E^2) < 0$. Fix D any divisor on S and apply the Hodge index theorem to $H = L_1 + L_2$ (where $L_1 = \overline{C} \times \mathrm{pt}$ and $L_2 = \mathrm{pt} \times \overline{C}$) and $E = D - (D \cdot L_2)L_1 - (D \cdot L_1)L_2$: one obtains

$$(D^2) < 2(D \cdot L_1)(D \cdot L_2) \quad . \tag{8}$$

Let us compute the different intersection numbers. Notice that $(\Delta \cdot L_1) = (\Delta \cdot L_2) = 1$ and $(\Gamma \cdot L_1) = q$ while $(\Gamma \cdot L_2) = 1$. We still have to compute (Δ^2) and (Γ^2). As Γ and Δ are smooth curves of genus g one can apply the adjunction formula $K_Y = (K_S + Y)_{|Y}$ for a smooth divisor Y, noting that $K_S = (2g-2)(L_1 + L_2)$:

$$2g - 2 = (K_\Delta^2) = (\Delta \cdot (\Delta + K_S)) = (\Delta^2) + 2(2g-2),$$
$$2g - 2 = (K_\Gamma^2) = (\Gamma \cdot (\Gamma + K_S)) = (\Gamma^2) + (q+1)(2g-2).$$

Therefore $(\Delta^2) = -(2g-2)$ and $(\Gamma^2) = -q(2g-2)$.

We apply eq. (8) to $D = a\Delta + b\Gamma$, $a, b \in \mathbb{Z}$:

$$-a^2(2g-2) - qb^2(2g-2) + 2abN_1 \leq 2(a+bq)(a+b) \quad .$$

Hence:

$$ga^2 - ab(q+1-N_1) + gqb^2 \geq 0 \quad .$$

This holds for all $a, b \in \mathbb{Z}$ hence

$$(q+1-N_1)^2 \leq 4qg^2$$

and the result.

□

One deduces immediately from the estimates on a_1:

Corollary 4.6.3. *Let C be a smooth projective, geometrically irreducible, curve of genus g over $\mathbb{F}_q$. Then*

$$1 + q - 2g\sqrt{q} \leq |C(\mathbb{F}_q)_{\mathbb{F}_q}| \leq 1 + q + 2g\sqrt{q} \quad .$$

In particular C admits a $\mathbb{F}_q$-point as soon as $q \geq 4g^2$.

5. Transition to étale cohomology

5.1. Heuristic for the Weil conjectures: about the Lefschetz trace formula.

This section is borrowed from lectures of Beilinson on the Weil conjectures [Bei07].

How can one guess the Weil conjectures, for example the rationality of the zeta function

$$Z(X,t) = \exp\left(\sum_{i=1}^{\infty} \left|X(\mathbb{F}_q^n)_{\mathbb{F}_q}\right| \cdot \frac{t^n}{n}\right)$$

for a scheme X of finite type over $\mathbb{F}_q$?

As $X(\mathbb{F}_q^n)_{\mathbb{F}_q} = \overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}}^{\mathrm{Fr}^n_{\overline{X},q}}$, a more general question is the following: let $\phi(= \mathrm{Fr}_{\overline{X},q})$ be an automorphism of a set $S(= \overline{X}(\overline{\mathbb{F}_q})_{\overline{\mathbb{F}_q}})$ such that for all $n \in \mathbb{N}^*$, the set $S^{\phi^n = 1}$ is finite.

Can we compute $|S^{\phi=1}|$ and more generally

$$Z((S,\phi),t)M = \exp \sum_{i=1}^{\infty} |S^{\phi^n=1}| \cdot \frac{t^n}{n} \quad ?$$

5.1.1. *The finite case.* Suppose for simplicity that the set S is finite. Notice that even in this case the rationality of $Z((S,f),t)$ is not a priori obvious. Let $\mathbb{Q}[S]$ be the $\mathbb{Q}$-vector space generated by S. The automorphism ϕ of S induces a linear action of ϕ on $\mathbb{Q}[S]$.

Lemma 5.1.1.

$$Z((S,\phi),t) = \det(1 - t\cdot\phi \mid \mathbb{Q}[S])^{-1} \quad . \tag{9}$$

Proof. Denote by $(\alpha_i)_{i\in I}$ the eigenvalues of ϕ acting on $\mathbb{Q}[S]$. Hence

$$\det(1 - t\cdot\phi|\mathbb{Q}[S]) = \prod_{i\in I}(1-\alpha_i t) \quad .$$

Applying $t\cdot\frac{d\log}{dt}$ to both sides of eq. (9) one obtains:

$$\begin{aligned}\sum_{n\geq 1} |S^{\phi^n=1}|\cdot t^n &= \sum_{i\in I}\sum_{n\geq 1}(\alpha_i t)^n \\ &= \sum_{n\geq 1} \mathrm{tr}(\phi^n \mid \mathbb{Q}[S])\cdot t^n \quad .\end{aligned}$$

Hence we are reduced to proving that $\mathrm{tr}(\phi^n \mid \mathbb{Q}[S]) = |S^{\phi^n}|$, which is obvious for any permutation ϕ of the finite set S. □

How can we generalize this kind of arguments for S infinite?

5.1.2. *The differentiable case.* Suppose now that S is a closed $\mathcal{C}^\infty$ manifold and $\phi: S \to S$ is a diffeomorphism such that for all $n\in\mathbb{N}^*$, the set $S^{\phi^n=1}$ is finite. For simplicity we will assume:

(1) the manifold S is orientable.
(2) for any fixed point $s\in S$ of ϕ^n one has $\det(1-\phi^n \mid T_sS) > 0$.

Remark 5.1.2. The assumption $\det(1-\phi^n \mid T_sS) \neq 0$ means that the point $s\in S$ is non-degenerate for ϕ^n, i.e. that the diagonal Δ_S and the graph $\Gamma(\phi^n)$ are transverse at $s\in S$. The condition $\det(1-\phi^n \mid T_sS) > 0$ means moreover that the local index of f at s is positive.

In this situation, Lefschetz [Lef26] proved:

Theorem 5.1.3. *(Lefschetz trace formula)*

$$|S^{\phi^n=1}| = \sum_{i=0}^{\dim S} (-1)^i \mathrm{tr}(\phi^n \mid H^i(S,\mathbb{Q})). \tag{10}$$

Equivalently: $Z((S,\phi),t) = \prod_{i=0}^{\dim S} \det(1 - t\cdot\phi^* \mid H^i(S,\mathbb{Q}))^{(-1)^{i+1}}$.

Remark 5.1.4. If S is finite we have $H^0(X,\mathbb{Q}) = \mathbb{Q}[S]^*$ and the higher cohomologies vanish, hence we recover Lemma 5.1.1.

Weil's main idea is that the methods from algebraic topology should be applicable in characteristic $p > 0$: if one has a "sufficiently nice" cohomology theory for schemes over $\mathbb{F}_q$ then $Z(X,t)$ can be computed through the Lefschetz trace formula for $\mathrm{Fr}_{\overline{X},q}$ acting on the (compactly supported) cohomology of $\overline{X}$ and a good part of the Weil conjectures is "formal".

5.2. **Weil cohomologies.** We fix a base field k, and a coefficient field K *of characteristic zero*. We define axiomatically what a "nice" cohomology theory with coefficients in K should be, at least on the category $\mathrm{SmProj}(k)$ of smooth projective k-schemes (with k-morphisms). Let $\mathrm{Vect}_K^{\mathbb{Z}}$ be the category of graded K-vector spaces of finite dimension, with its graded tensor product.

Definition 5.2.1. *A pure Weil cohomology on k with coefficients in a field K of characteristic zero is a functor:*

$$H^\bullet : \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{Vect}_K^{\mathbb{Z}}$$

satisfying the following axioms:

(i) Dimension*: For any $X \in \mathrm{SmProj}(k)$ of dimension d_X, $H^i(X) = 0$ for $i \notin$* $[0, 2d_X]$.

(ii) Orientability*: $\dim_K H^2(\mathbf{P}^1_k) = 1$; we denote this space by $K(-1)$.*

(iii) Additivity*: For any $X, Y \in \mathrm{SmProj}(k)$ the canonical morphism*

$$H^\bullet(X \coprod Y) \to H^\bullet(X) \oplus H^\bullet(Y)$$

is an isomorphism.

(iv) Künneth formula*: For any $X, Y \in \mathrm{SmProj}(k)$ one has an isomorphism*

$$\kappa_{X,Y} : H^\bullet(X) \otimes_K H^\bullet(Y) \xrightarrow{\sim} H^\bullet(X \times_k Y)$$

natural in X, Y, satifying obvious compatibilities. In particular we require $H^\bullet(\mathrm{Spec}\, k) =$ K in degree 0 and $H^\bullet$ is monoidal.

(v) Trace and Poincaré duality*: For any $X \in \mathrm{SmProj}(k)$, purely of dimension d_X, one has a canonical morphism*

$$\mathrm{Tr}_X : H^{2d_X}(X) \to K(-d_X) := K(-1)^{\otimes d_X}$$

which is an isomorphism if X is geometrically connected, and such that $\mathrm{Tr}_{X\times_k Y} =$ $\mathrm{Tr}_X \otimes \mathrm{Tr}_Y$ modulo the Künneth formula. The Poincaré pairing

$$< .,. >_X : H^i(X) \otimes H^{2d_X - i}(X) \to H^{2d_X}(X \times_k X) \xrightarrow{\Delta_X^*} H^{2d_X}(X) \xrightarrow{\mathrm{Tr}_X} K(-d_X)$$

is perfect.

(vi) Cycle class*: For any $X \in \mathrm{SmProj}(k)$ and $i \in \mathbb{N}$ one has a homomorphism:*

$$\gamma_X : CH^i(X) \to H^{2i}(X)(i) := \mathrm{Hom}(K(-i), H^{2i})$$

where $CH^i(X) = Z^i(X)/\sim_{\mathrm{rat}}$ is the i-th Chow group, satisfying:

(a) *for any $f : X \to Y$, $\quad \gamma_X \circ f^* = f^* \circ \gamma_Y$.*

(b) *for any cycle α, β, one has $\gamma_{X\times_k X}(\alpha \times_k \beta) = \gamma_X(\alpha) \otimes \gamma_X(\beta)$ in $H^\bullet(X \times_k X) =$ $H^\bullet(X) \otimes_K H^\bullet(X)$.*

(c) *If X is geometrically connected of dimension d_X then for any $\alpha \in CH^{d_X}(X)$ one has:*

$$< 1, \gamma(\alpha) >= \deg(\alpha) \ .$$

Notice that any Weil cohomology is endowed with a natural ring structure, the cup product on $H^\bullet(X)$ being defined as:

$$\forall \alpha \in H^i(X),\ \beta \in H^j(X), \quad \alpha \cdot \beta = \Delta_X^*(\alpha \otimes \beta) \ .$$

5.2.1. *Digression on Chow groups.* At this point we don't want to review the theory of Chow groups. We just recall the basic definition (see [Ful98] for details). Let X be an arbitrary variety over k. The group $CH^r(X)$ is the quotient of $Z^r(X)$ by the rational equivalence relation $\sim_{\mathrm{rat}}$, where the equivalence relation $\sim_{\mathrm{rat}}$ is generated by forcing $\mathrm{Div}_Y(\varphi) = 0$ where Y is an irreducible closed subvariety of X of codimension $r-1$ and φ is a non-zero rational function on Y. We do not give the general definition of $\mathrm{Div}_Y(\varphi)$. For Y normal this is the usual definition of the principal divisor corresponding to a rational function. For any morphism $f : X \to Y$ between *smooth varieties* one defines non-trivially a pull-back $f^* : CH^\bullet(Y) \to CH^\bullet(X)$. In the case where Z is an irreducible subvariety of Y such that $f^{-1}(Z)$ has pure dimension $\dim Z + \dim X - \dim Y$ and f is flat in a neighbourhood of Z then $f^*[Z] := [f^{-1}Z] := \sum_W n_W W$, where W go through the irreducible components of Z^{red} and $n_W := l_{\mathcal{O}_{Z,W}}(\mathcal{O}_{Z,W})$ is the length of its generic point. The product on $CH^\bullet(X)$ is defined by $[Z_1] \cdot [Z_2] = \Delta_X^*([Z_1 \times Z_2])$.

5.2.2. *Basic properties of Weil cohomologies.* Let $f : X \to Y \in \mathrm{SmProj}(k)$. One defines the direct image

$$f_* : H^i(X) \to H^{i+2(d_Y - d_X)}(Y)(d_Y - d_X)$$

as the Poincaré dual of

$$f^* : H^{2d_X - i}(Y)(d_X) \to H^{2d_X - i}(X)(d_X) \ .$$

Hence $\mathrm{Tr}_X = a_{X*}$, where $a_X : X \to \mathrm{Spec}\, k$ is the structural morphism. One easily checks the projection formula: $f_*(x \cdot f^* y) = f_* x \cdot y$.

Lemma 5.2.2. *Let $X, Y \in \mathrm{SmProj}(k)$. One has a canonical isomorphism:*

$$\mathrm{Hom}^r(H^\bullet(X), H^\bullet(Y)) \simeq H^{2d_X + r}(X \times Y)(d_X) \ .$$

Proof.

$$\begin{aligned}
\mathrm{Hom}^r(H^\bullet(X), H^\bullet(Y)) &= \prod_{i \geq 0} \mathrm{Hom}(H^i(X), H^{i+r}(Y)) \\
&= \prod_{i \geq 0} H^i(X)^* \otimes H^{i+r}(Y) \\
&\simeq \prod_{i \geq 0} H^{2d_X - i}(X)(d_X) \otimes H^{i+r}(Y) \quad \text{(Poincaré)} \\
&= H^{2d_X + r}(X \times_k Y)(d_X) \quad \text{(Künneth)} \ .
\end{aligned}$$

□

Remark 5.2.3. Hence an element of $H^{2d_X + r}(X \times Y)(d_X)$ has to be thought as a covariant correspondance of degree r from $H^\bullet(X)$ to $H^\bullet(Y)$.

Poincaré duality defines (via transposition) an isomorphism

$$\mathrm{Hom}^r(H^\bullet(X), H^\bullet(Y)) \simeq \mathrm{Hom}^{2d_x-2d_Y+r}(H^\bullet(Y), H^\bullet(X))(d_x - d_Y)$$

or equivalently

$$H^{2d_X+r}(X \times Y)(d_X) \simeq H^{2d_X+r}(Y \times X)(d_X)$$

denoted $\varphi \mapsto {}^t\varphi$. One easily checks that ${}^t\varphi$ coincides with $\sigma^*_{X,Y}\varphi$, where $\sigma_{X,Y} : X \times Y \to Y \times X$ permutes the factors.

Example 5.2.4. Let $f : X \to Y$ be a morphism. Then $f^* \in H^{2d_Y}(Y \times X)(d_Y)$ and by definition ${}^tf^* = f_*$.

Lemma 5.2.5. *(Lefschetz trace formula) Let $H^\bullet : \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{Vect}_K^{\mathbb{Z}}$ a Weil cohomology. Then for any $X, Y \in \mathrm{SmProj}(k)$ pure of dimension d_X, d_Y respectively and any $\phi \in H^{2d_X+r}(X \times_k Y)(d_X)$, $\psi \in H^{2d_Y-r}(Y \times_k X)(d_Y)$ then*

$$< \phi, {}^t\psi >_{X\times_k Y} = \sum_{i=0}^{2d_X} (-1)^i \mathrm{tr}(\psi \circ \phi \,|\, H^i(X)) \quad .$$

Proof. By the Künneth formula and bilinearity one can assume that $\phi = v \otimes w$, $\psi = w' \otimes v'$ where $v \in H^{2d_X-i}(X)(d_X)$, $w \in H^{i+r}(Y)$, $w' \in H^{2d_Y-j-r}(Y)(d_Y)$ and $v' \in H^j(X)$. Then ϕ (resp. ψ) vanishes outside $H^i(X)$ (resp. $H^{j+r}(Y)$).

If $x \in H^i(X)$ and $y \in H^{j+r}(Y)$ then

$$\phi(x) = < x, v >_X \cdot w, \quad \psi(y) = < y, w' >_Y \cdot v' \quad .$$

Hence $\psi \circ \phi(x) = 0$ except if $i = j$, in which case

$$\psi \circ \phi(x) = < x, v >_X < w, w' >_Y \cdot v'$$

thus

$$\mathrm{tr}(\psi \circ \phi) = < v', v >_X < w, w' >_Y \quad .$$

On the other hand:

$$\begin{aligned}
< \phi, {}^t\psi >_{X\times_k Y} &= (-1)^{j(j+r)} < v \otimes w, v' \otimes w' >_{X\times_k Y} \\
&= (-1)^{j(j+r)} \mathrm{Tr}_{X\times_k Y}(v \otimes w \cdot v' \otimes w') \\
&= (-1)^{j(j+r)+j(i+r)} \mathrm{Tr}_{X\times_k Y}(v \cdot v' \otimes w \cdot w') \\
&= \delta_{ij} \mathrm{Tr}_X(v \cdot v') \cdot \mathrm{Tr}_Y(w \cdot w') = \delta_{ij} < v, v' >_X \cdot < w, w' >_Y \\
&= \delta_{ij} (-1)^{j(2d_x-i)} < v', v >_X < w, w' >_Y = \delta_{ij} (-1)^i \mathrm{tr}(\psi \circ \phi) \quad .
\end{aligned}$$

□

5.3. Applications to the Weil conjectures.

Corollary 5.3.1. *Suppose $H^\bullet : \mathrm{SmProj}(\overline{\mathbb{F}_q}) \to \mathrm{Vect}_K^{\mathbb{Z}}$ is a Weil cohomology. Then for any $X \in \mathrm{SmProj}(\mathbb{F}_q)$, geometrically irreducibe, one has:*

$$|X(\mathbb{F}_{q^n})| = \sum_{i=0}^{2n} (-1)^j \mathrm{tr}(\mathrm{Fr}^*_{\overline{X},q} \,|\, H^j(\overline{X})) \quad .$$

where $\mathrm{Fr}_{\overline{X},q} : \overline{X} \to \overline{X}$ is the Frobenius endomorphism.

Proof. We already saw in Lemma 4.5.1 that $X(\mathbb{F}_{q^n})_{\mathbb{F}_q}$ is in bijection with the closed points of $\Delta_{\overline{X}} \cap \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}$. More precisely:

$$\begin{aligned} \left|X(\mathbb{F}_{q^n})_{\mathbb{F}_q}\right| &= \deg\left(\Delta_{\overline{X}} \cdot \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}\right) \\ &=< 1, \gamma(\Delta_{\overline{X}} \cdot \Gamma_{\mathrm{Fr}^n_{\overline{X},q}}) >=< 1, \gamma(\Delta_{\overline{X}}) \otimes \gamma(\Gamma_{\mathrm{Fr}^n_{\overline{X},q}}) >, \end{aligned}$$

where γ denotes $\gamma_{\overline{X} \times_{\overline{\mathbb{F}_q}} \overline{X}}$. But $\gamma(\Delta_{\overline{X}}) = (\Delta_{\overline{X}})_*$ and $\gamma(\Gamma_{\mathrm{Fr}^n_{\overline{X},q}}) = (\mathrm{Fr}^n_{\overline{X},q})_* = {}^t(\mathrm{Fr}^n_{\overline{X},q})^*$ hence

$$\begin{aligned} \left|X(\mathbb{F}_{q^n})_{\mathbb{F}_q}\right| &=< \Delta^*_{\overline{X}}, {}^t(\mathrm{Fr}^n_{\overline{X},q})^* > \\ &= \sum_{j=0}^{2d_X} (-1)^j \mathrm{tr}(\mathrm{Fr}^n_{\overline{X},q}{}^* \,|\, H^j(\overline{X})) \quad \text{by the LFT} \ . \end{aligned}$$

Here on the first line the schemes $\Gamma_{\mathrm{Fr}^n_{\overline{X},q}}$ and $\Delta_{\overline{X}}$ are understood as elements of $CH^{d_X}(\overline{X} \times_{\overline{\mathbb{F}_q}} \overline{X})$ and their product is in $CH^\bullet(\overline{X} \times_{\mathbb{F}_q} \overline{X})$. The second equality follows from axioms $(vi)(c)$ and $(vi)(b)$ for Weil cohomologies, the last one from Lemma 5.2.5. □

Theorem 5.3.2. *Suppose that there exists a Weil cohomology* $H^\bullet : \mathrm{SmProj}(\overline{\mathbb{F}_q}) \to \mathrm{Vect}^{\mathbb{Z}}_K$. *Then for any* $X \in \mathrm{SmProj}(\mathbb{F}_q)$ *one has:*

$$Z(X,t) = \prod_{i=0}^{2d_X} \det(1 - t \cdot \mathrm{Fr}_{\overline{X},q}{}^* \,|\, H^i(\overline{X}))^{(-1)^{j+1}} \ .$$

In particular $Z(X,t)$ *is rational and has the expected functional equation.*

Proof. The computation of $Z(X,t)$ is the same as the one in the proof of Theorem 5.1.3. As a corollary $Z(X,t) \in K(t) \cap \mathbb{Q}[t] = \mathbb{Q}(t)$ [B, IV.5, Ex3].

The functional equation follows from Poincaré duality. Indeed as

$$< (\mathrm{Fr}_{\overline{X},q})_*(x), x' >=< x, \mathrm{Fr}^*_{\overline{X},q} x' > \ ,$$

one obtains that $(\mathrm{Fr}_{\overline{X},q})_*|H^j(\overline{X})$ et $(\mathrm{Fr}_{\overline{X},q})^*|H^{2d_X-j}(\overline{X})$ have the same eigenvalues. But $\mathrm{Fr}_{\overline{X},q*} \circ \mathrm{Fr}^*_{\overline{X},q} = q^{d_X}$ as $\mathrm{Fr}_{\overline{X},q} : \overline{X} \to \overline{X}$ is finite of degree q^{d_X}. Hence if $(\alpha_i)_{i \in I}$ are the eigenvalues of $\mathrm{Fr}^*_{\overline{X},q}$ on $H^{2d_X-j}(\overline{X})$, then $(\frac{q^{d_X}}{\alpha_i})_{i\in I}$ are the eigenvalues of $\mathrm{Fr}^*_{\overline{X},q}$ on $H^j(\overline{X})$. The functional equation follows. □

At this point it remains to construct such a Weil cohomology on $\mathrm{SmProj}(\overline{\mathbb{F}_q})$. In fact for any field k and for each prime $l \neq \mathrm{char} k$, Grothendieck and Artin construct a Weil cohomology on $\mathrm{SmProj}(k)$ with coefficients in $\mathbb{Q}_l$: the l-adic cohomology. In some sense we have now too many cohomologies. In particular for each l we obtain polynomials $P_{j,l} = \det(1 - t\mathrm{Fr}_{\overline{X},q} \,|\, H^j_{\text{ét}}(\overline{X})) \in \mathbb{Q}_l[t]$ which depends *a priori* from l. In some sense all these cohomologies can be compared, but not canonically. This problem gives birth to the notion of motives.

Exercice 5.3.3. Deduce from the Riemann hypothesis over finite fields (purity) that in fact $P_{j,l} = P_{j,l'} \in \mathbb{Q}[t]$ for $l \neq l'$.

6. Differential calculus

In this section we introduce étale morphisms, in the most geometric way. They naturally occur while studying differential calculus, namely properties of morphisms relatively to "infinitesimally closed points". Algebraic geometry (or more generally the geometry of locally ringed spaces) has a particularly nice format for such a calculus. Our presentation essentially follows [Il96], which summarizes [EGAIV, 16 and 17].

6.1. Thickenings.

Definition 6.1.1. *A morphism of schemes $i : X \to X'$ is a thickening if this is a closed immersion (recall this means that $|i|$ identifies X with a closed subspace of $|X'|$ and $i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X$ is surjective) such that $|X| \overset{|i|}{\simeq} |X'|$.*

It is a thickening of order 1 if moreover the quasi-coherent ideal sheaf

$$\mathcal{I} = \ker(i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X)$$

defining the closed subscheme X of X' has square zero: $\mathcal{I}^2 = 0$.

Remarks 6.1.2.
(i) This notion generalizes in an obvious way to the notion of *thickening over a base.*
(ii) the notion of morphism of thickenings over a base S is given by the usual commutative square over S.

Let $i : X \to X'$ be a thickening. Any local section of $\mathcal{I} = \ker i^\sharp$ is thus locally nilpotent. One says that $i : X \to X'$ is a *thickening of finite order n* if $\mathcal{I}$ is globally nilpotent of order n: $\mathcal{I}^n \neq 0$ and $\mathcal{I}^{n+1} = 0$. In this situation one has a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \cdots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

corresponding to a filtration

$$X = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n \subset X_{n+1} = X'$$

where each inclusion map $X_i \to X_{i+1}$ is a first order thickening. The study of finite order thickenings is thus reduced to the study of first-order ones.

6.2. First infinitesimal neighborhood.

Let $j : Z \to X$ be an immersion (i.e. j is an isomorphism of Z with a closed subscheme $j(Z)$ of an open subscheme U of X), of ideal $\mathcal{I}$ (i.e. $\mathcal{I}$ is the quasi-coherent sheaf of ideals of $\mathcal{O}_U$ defining $j(Z)$ in U).

Definition 6.2.1. *The first infinitesimal neighborhood of Z in X is the closed subscheme $Z' \overset{i}{\hookrightarrow} U$ defined by $\mathcal{I}^2$.*

Hence one has a factorization of j as

$$Z \overset{i}{\hookrightarrow} Z' \xrightarrow{j'} X \quad .$$

The morphism $Z \overset{i}{\hookrightarrow} Z'$ is a thickening of order 1 and one easily checks it satisfies the following:

Lemma 6.2.2. *Let $j : Z \to X$ be an immersion. The first infinitesimal neighborhood Z' of Z in X has the following universal property: for any solid commutative diagram*

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d] & Z \arrow[d] \\ T' \arrow[r, dashed, "a'"] \arrow[dr] & Z' \arrow[d] \\ & X \end{tikzcd}$$

where $T \to T'$ is a thickening of order 1 over X, there exists a unique morphism

$$(a', a) : (T \subset T') \to (Z \subset Z')$$

of thickenings over X factorizing the diagram.

6.3. **Conormal subsheaf of an immersion.** The nice formalism of infinitesimal neighborhoods in algebraic geometry makes it natural to first define the notion of conormal sheaf and cotangent sheaf and then the dual notion of normal sheaf and tangent sheaf (notice that in differential geometry one usually proceeds the other way round).

Let $Z \overset{i}{\hookrightarrow} X$ be a closed immersion of ideal $\mathcal{I} \subset \mathcal{O}_X$. The following short sequence of quasi-coherent sheaves on X is exact:

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0 \ \ .$$

Recall the following classical fact:

Lemma 6.3.1. *The functor*

$$i_* : \mathrm{QCoh}(\mathcal{O}_Z) \to \mathrm{QCoh}(\mathcal{O}_X)$$

is exact, fully faithful, with essential image the $\mathcal{O}_X$-quasi-coherent sheaves $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Hence the sheaf $\mathcal{I}/\mathcal{I}^2$, which is killed by $\mathcal{I}$, corresponds to a sheaf on Z: the *conormal sheaf* $\mathcal{C}_{Z/X}$ of Z in X.

We recover the classical "differential geometric" notion: the conormal sheaf of a $\mathcal{C}^\infty$ submanifold Z of a $\mathcal{C}^\infty$ manifold X defined by equations $f_1 = \ldots = f_r = 0$ is generated by the first order part of the f_i's: it is the subsheaf of $i^*\Omega^1_X$ annihilating the subsheaf TZ of the tangent bundle TX.

More generally if $i : Z \hookrightarrow X$ is an immersion we define $\mathcal{C}_{Z/X}$ as $\mathcal{C}_{Z/U}$, where U is the maximal open subscheme of X such that Z is a closed subscheme of U.

Remark 6.3.2. In [EGAIV] the conormal sheaf is denoted $\mathcal{N}_{Z/X}$ but we keep this notation for the *normal sheaf*

$$\mathcal{N}_{Z/X} := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) \ \ .$$

Here we assume that $\mathcal{C}_{Z/X}$ has finite presentation otherwise $\mathcal{N}_{Z/X}$ is not even quasi-coherent.

Lemma 6.3.3. *Let*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Z' \arrow[r, hook, "i'"] & X' \end{tikzcd}$$

be a commutative diagram of schemes, with i and i' immersions. There is a canonical morphism of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X} \quad .$$

Proof. Locally we are in the situation:

$$\begin{tikzcd} \operatorname{Spec}(R/I) \arrow[r, hook, "i"] \arrow[d, "f"'] & \operatorname{Spec} R \arrow[d, "g"] \\ \operatorname{Spec}(R'/I') \arrow[r, hook, "i'"] & \operatorname{Spec} R' \end{tikzcd} \quad .$$

The required morphism $I'/(I')^2 \to I/I^2$ is deduced from $f^\sharp : R' \to R$ which maps I' to I. □

Lemma 6.3.4. *Let $Z \overset{j}{\hookrightarrow} Y \hookrightarrow X$ be two immersions. Then:*

$$j^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is an exact sequence of $\mathcal{O}_Z$-modules.

Proof. Locally one considers

$$\operatorname{Spec} A \to \operatorname{Spec} B \to \operatorname{Spec} C \quad ,$$

where $C \twoheadrightarrow B \twoheadrightarrow A$. Write $I := \ker(B \to A)$, $J := \ker(C \to A)$ and $K = \ker(C \to B)$. We want to show that the sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0$$

is exact. This follows immediately from $I = J/K$. □

Lemma 6.3.5. *Let $Z \hookrightarrow X$ be an immersion and $Z \overset{i'}{\hookrightarrow} Z' \longrightarrow X$ its first infinitesimal neighborhood. The commutative square*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, equal] & Z' \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

induces an isomorphism $\mathcal{C}_{Z/X} \xrightarrow{\sim} \mathcal{C}_{Z/Z'}$.

Proof. Follows immediately from the definition of Z', or from Lemma 6.3.4. □

6.4. **Cotangent sheaf: definition.** Let $f : X \to S$ be a morphism of schemes. Let $\Delta : X \to X \times_S X$ be the diagonal. The map Δ is an immersion, which is closed if and only if f is separated. The infinitesimal neighbourhoods of Δ parametrize couples of points of X "infinitesimally closed" one to another.

Definition 6.4.1. *Let $f : X \to S$ be a morphism of schemes. The sheaf $\Omega^1_{X/S}$ of Kähler differential forms of degree 1 is*

$$\Omega^1_{X/S} := \mathcal{C}_{X/X\times_S X} = \mathcal{I}/\mathcal{I}^2$$

where $\mathcal{I} \subset \mathcal{O}_{X\times_S X}$ denotes the ideal sheaf of $\Delta : X \to X \times_S X$.

Let

$$\begin{tikzcd} X \arrow[r, hook, "i'"] \arrow[rr, bend left, "\Delta"] & X' \arrow[r] \arrow[dr, shift left, "p_2"] \arrow[dr, shift right, "p_1"'] & X \times_S X \arrow[d, shift left] \arrow[d, shift right] \\ & & X \end{tikzcd}$$

be the first infinitesimal neighborhood of Δ. Consider the exact sequence of sheaves on X:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_{X/S} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "j_2"'] \arrow[l, bend left, "j_1"] & 0 \end{tikzcd}$$

where $j_i = p_i^\sharp : \mathcal{O}_X \to \mathcal{O}_{X'}$, $i = 1, 2$, is a ring morphism. Define

$$d_{X/S} := j_2 - j_1 : \mathcal{O}_X \to \Omega^1_{X/S} \quad .$$

Definition 6.4.2. *Recall that for $f : X \to S$ and $\mathcal{M}$ an $\mathcal{O}_X$-module one defines the abelian group of S-derivations from $\mathcal{O}_X$ to $\mathcal{M}$ by*

$$\mathrm{Der}_S(\mathcal{O}_X, \mathcal{M}) = \left\{ \begin{array}{r} D : \mathcal{O}_X \to \mathcal{M} \text{ morphism of } f^{-1}\mathcal{O}_S - \text{module } / \\ D(a \cdot b) = a \cdot Db + b \cdot Da \quad \forall\, a, b \in \mathcal{O}_X \end{array} \right\}.$$

Lemma 6.4.3. *$d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ is an S-derivation.*

The proof of Lemma 6.4.3 is immediate from the definition of $d_{X/S}$. In fact one shows that this construction provides the *universal derivation*:

Lemma 6.4.4. *Let $f : X \to S$ be a morphism of schemes. The functor* $\mathrm{Mod}(\mathcal{O}_X) \to$ **Sets** *which to $\mathcal{M}$ associates* $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{M})$ *is corepresented by $\Omega^1_{X/S}$:*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{M}) &\to \mathrm{Der}_S(\mathcal{O}_X, \mathcal{M}) \\ \alpha &\mapsto \alpha \circ d_{X/S} \quad . \end{aligned}$$

Recall the local description of $\Omega^1_{X/S}$. If $f : \operatorname{Spec} B \to \operatorname{Spec} A$ then $\Omega^1_{B/A}$ is the quotient of the free B-module generated by symbols db, $b \in B$, modulo the relations

$$d(b+b') - db - db', \quad b, b' \in B$$
$$d(b \cdot b') - b \cdot db' - b' \cdot db$$
$$da, \quad a \in A$$

Moreover the differential $d_{B/A} : B \to \Omega^1_{B/A}$ is just the map associating db to $b \in B$.

Definition 6.4.5. *The tangent sheaf $T_{X/S}$ is the dual $\mathcal{H}om(\Omega^1_{X/S}, \mathcal{O}_X)$ of the cotangent sheaf $\Omega^1_{X/S}$.*

Thus for any open subset U of X the sections of $T_{X/S}$ over U are $\Gamma(U, T_{X/S}) = \operatorname{Der}_S(\mathcal{O}_U, \mathcal{O}_U)$. For $S = \operatorname{Spec} \mathbb{C}$ we recover the classical definition of vector fields as derivations of functions.

6.5. **Cotangent sheaf: basic properties.** Let

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram of schemes. The morphism

$$\mathcal{O}_X \xrightarrow{f^\sharp} f_*\mathcal{O}_{X'} \xrightarrow{f_* d_{X'/S'}} f_*\Omega^1_{X'/S'}$$

is obviously an S-derivation, hence defines an $\mathcal{O}_X$-morphism

$$\Omega^1_{X/S} \to f_*\Omega^1_{X'/S'} \quad ,$$

or equivalently by adjunction a canonical map

$$f^*\Omega^1_{X/S} \to \Omega^1_{X'/S'} \quad .$$

The following three lemmas describe the basic properties of the cotangent sheaf:

Lemma 6.5.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$. Then the sequence of $\mathcal{O}_X$-modules*

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact.

Proof. This is the sheafified version of [Mat80, Th.57 p.186]. □

Lemma 6.5.2. *Let*

$$\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

be an immersion over S. The the sequence of $\mathcal{O}_Z$-modules

$$\mathcal{C}_{Z/X} \xrightarrow{\overline{d_{X/S}}} i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$$

is exact.

Remark 6.5.3. The canonical map $\overline{d_{X/S}}$ is defined as follows. As $\mathcal{I}$ is contained in $\mathcal{O}_X$ one can consider the restriction $d_{X/S} : \mathcal{I} \to \Omega^1_{X/S}$. As $d_{X/S}$ is a derivation it maps $\mathcal{I}^2$ to $\mathcal{I} \cdot \Omega^1_{X/S}$ hence induces a map

$$\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/S}/\mathcal{I} \cdot \Omega^1_{X/S}$$

which is $\mathcal{O}_X/\mathcal{I}$-linear. This defines $\overline{d_{X/S}} : \mathcal{C}_{Z/X} = \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{X/S}$ by the Lemma 6.3.1.

Proof. Locally $X = \operatorname{Spec} A$, $Z = \operatorname{Spec}(B = A/I)$, $S = \operatorname{Spec} C$ and one has a commutative diagram of rings:

$$\begin{tikzcd} A \arrow[r, two heads, "\varphi"] & B = A/I \\ C \arrow[u, "\alpha"] \arrow[r, equal] & C. \arrow[u, "\beta"'] \end{tikzcd}$$

We want to prove that the sequence

$$I/I^2 \to \Omega^1_{A/C} \otimes_A B \to \Omega^1_{B/C} \to 0$$

is exact.

Surjectivity on the right: $A \twoheadrightarrow B$ hence $\Omega^1_{A/C} \twoheadrightarrow \Omega^1_{B/C}$ by the description of Ω^1 by generators and relations. *A fortiori:* $\Omega^1_{A/C} \otimes_A B \twoheadrightarrow \Omega^1_{B/C}$.

The composite of the two arrows is zero: indeed let $f \in I$. Then the image of $df \in \Omega^1_{A/C}$ in $\Omega^1_{B/C}$ is $d\overline{f}$, where $\overline{f}$ is the class of f in $B = A/I$, hence 0.

Exactness in the middle: this is equivalent to showing that the kernel of the natural map $\Omega^1_{A/C} \to \Omega^1_{B/C}$ is generated as A-module by $I \cdot \Omega^1_{A/B}$ and df, $f \in I$. The explicit description of Ω^1 by generators and relations implies that this kernel is $< da >$, where $a \in A$ satisfy $\varphi(a) = \beta(c)$ for some $c \in C$. Write $a = \alpha(c) + (a - \alpha(c))$. Then $da = d(a - \alpha(c))$ as $d(\alpha(c)) = 0 \in \Omega^1_{A/C}$. But $a - \alpha(c) \in I$ as $\varphi(a - \alpha(c)) = \varphi(a) - \varphi(\alpha(c)) = \beta(c) - \beta(c) = 0$. This shows that the kernel of the map $\Omega^1_{A/C} \to \Omega^1_{B/C}$ is in fact generated as A-module by the df's, $f \in I$. $\square$

Lemma 6.5.4. *Let Y be a scheme and consider $\mathbb{A}^n_Y = Y[T_1, \cdots, T_n]$. Then $\Omega^1_{\mathbb{A}^n_Y/Y}$ is a free $\mathcal{O}_{\mathbb{A}^n_Y}$-module with basis $(dT_i)_{1 \le i \le n}$.*

6.6. **Digression: the De Rham complex.** Let $f : X \to S$ be a morphism of schemes. Define $\Omega^i_{X/S} := \bigwedge^i \Omega^1_{X/S}$. One easily shows that there exists a unique family of morphisms of $f^{-1}(\mathcal{O}_S)$-modules $d : \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ satisfying the following properties:

(i) d is an S-derivation of $\bigoplus_i \Omega^i_{X/S}$: $d(a \wedge b) = da \wedge b + (-1)^{\deg a} a \wedge db$ (a, b homogeneous).
(ii) $d^2 = 0$.
(iii) $da = d_{X/S} a$ if a is of degree 0.

The complex $(\Omega^\bullet_{X/S}, d)$ is called the De Rham complex of $f : X \to S$.

7. Smooth, net and étale morphisms

7.1. **Definitions.** Recall that $f : X \to Y$ is *locally of finite type* if for any $x \in X$ there exist $U = \operatorname{Spec} B$ an open affine neighborhood of x in X, $V = \operatorname{Spec} A$ an open affine neighborhood of $y = f(x)$ in Y such that $f(U) \subset V$ and $A \to B$ is of finite type (i.e. $B = A[T_1, \cdots, T_n]/I$). It is *locally of finite presentation* if moreover I can be chosen of finite type over A.

If Y is locally noetherian (i.e. covered by spectra of noetherian rings) then f is locally of finite presentation if and only if it is locally of finite type.

Definition 7.1.1. *Let $f : X \to S$ be a morphism of schemes. One says that f is smooth (resp. net or unramified, resp. étale) if:*

(i) *f is locally of finite presentation.*

(ii) *for any solid diagram*

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T_0 \arrow[rru, "g_0"] \arrow[r, hook, "i"'] & T \arrow[r] \arrow[ru, dotted, "g"] & S \end{tikzcd}$$

where i is a thickening of order 1, there exists, locally for the Zariski topology on T, one (resp. at most one, resp. a unique) S-morphism g making the diagram commute (one says that f is formally smooth, resp. net, resp. étale).

Remarks 7.1.2. (i) We could have defined smooth, net and étale morphisms right after defining thickenings. However the cotangent sheaf is a basic tool which enables nice characterisation for smoothness or netness, see below.

(ii) In this definition one can obviously replace order 1 by any finite order thickening.

Corollary 7.1.3. (a) *the composite of two smooth morphisms (resp. net, resp. étale) is smooth (resp. net, resp. étale).*

(b) *these notions are stable under base change $S' \to S$.*

(c) *from (a) and (b) it follows that if $f_i : X_i \to S$, $i = 1, 2$ is smooth (resp. net, resp. étale) then $X_1 \times_Y X_2 \to S$ is smooth (resp. net, resp. étale).*

(d) *$\mathbb{A}^n_S \to S$ is smooth.*

7.2. **Main properties.**

Proposition 7.2.1. (a) *The morphism $f : X \to S$ is net if and only if $\Omega^1_{X/S} = 0$.*

If $f : X \to S$ is smooth, the $\mathcal{O}_X$-module Ω^1_X is locally free of finite type and

$$\forall x \in X, \quad \operatorname{rk}_x \Omega^1_{X/S} = \dim_x X_{f(x)} \quad .$$

(b) *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ (situation of Lemma 6.5.1).*

If f is smooth then

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \tag{11}$$

is exact and locally split. In particular if f is étale then $f^*\Omega^1_{Y/S} \simeq \Omega^1_{X/S}$.

Conversely suppose that gf is smooth. If the sequence eq. (11) is exact and locally split then f is smooth. If $f^*\Omega^1_{Y/S} \simeq \Omega^1_{X/S}$ then f is étale.

(c) *Let*

$$\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

be an immersion over S (situation of Lemma 6.5.2).

If f is smooth then the sequence of $\mathcal{O}_Z$-modules

$$0 \to \mathcal{C}_{Z/X} \overset{\overline{d_{X/S}}}{\to} i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0 \tag{12}$$

is exact and locally split. In particular if f is étale then $\mathcal{C}_{Z/X} \overset{\sim}{\to} i^*\Omega^1_{X/S}$.

Conversely assume that g is smooth. If the sequence eq. (12) is exact locally split then f is smooth. If $\mathcal{C}_{Z/X} \overset{\sim}{\to} i^*\Omega^1_{X/S}$ then f is étale.

7.3. **Local coordinates.** Let $f : X \to S$ be a smooth morphism. Let $x \in X$ and let $s_1, \cdots, s_n$ be sections of $\mathcal{O}_X$ in a neighborhood of x such that $((ds_i)_x)_{1\le i\le n}$ is an $\mathcal{O}_{X,x}$-basis of $\left(\Omega^1_{X/S}\right)_x$. As $\Omega^1_{X/S}$ is $\mathcal{O}_X$-locally free of finite type the $(ds_i)_{1\le i\le n}$ are an $\mathcal{O}_X$-basis of $\Omega^1_{X/S}$ over some open neighborhood U of x in X. This defines a morphism

$$s = (s_1, \cdots, s_n) : U \to \mathbb{A}^n_S = S[T_1, \cdots, T_n] \ .$$

It follows from the converse part of Proposition 7.2.1(b) that the map s is étale.

Definition 7.3.1. *One says that the $(s_i)_{1\le i\le n}$ form a system of local coordinates of X over S in a neighborhood of $x \in X$.*

Corollary 7.3.2. *Any smooth morphism is locally the composite of the projection of a standard affine space with an étale morphism.*

7.4. **Jacobian criterion.** Let

$$\begin{tikzcd} Z \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

be an immersion over S (situation of Lemma 6.5.2). Suppose that g is smooth. Let $z \in Z$. In order for f to be smooth at z it is enough by Proposition 7.2.1(c) to exhibit sections $s_1, \cdots, s_r$ of the ideal I_Z in a neighborhood of z, generating $I_{Z,z}$ and such that the vectors $\{(ds_i)(z)\}_{1\le i\le r}$ are linearly independent in $\Omega^1_{X/S}(z) := \Omega^1_{X/S} \otimes k(z)$. This is the classical Jacobian criterion.

7.5. **Implicit functions theorem.** In the situation of Lemma 6.5.2 again, assume that f is smooth in a neighborhood of $z \in Z$. Sections $(s_i)_{1\le i\le r}$ of I_Z generating I_Z around z form a *minimal* system of generators of $I_{Z,z}$ (i.e. define a base of $I_Z \otimes k(z)$ or equivalently define a basis of $\mathcal{I}_Z/\mathcal{I}_Z^2 = \mathcal{C}_{Z/X}$ in a neighborhood of z) if and only if the $(ds_i(z))_{1\le i\le r}$ are linearly independent in $\Omega^1_{X/S}(z)$. In this case one can complete the $(s_i)_{1\le i\le r}$ by sections $(s_j)_{r+1\le j\le r+n}$ of $\mathcal{O}_X$ in a neighborhood of z so that the family

$(ds_i(z))_{1\leq i\leq r+n}$ is a basis of $\Omega^1_{X/S}(z)$. Hence the $(s_i)_{1\leq i\leq n+r}$ define an étale S-morphism s on a neighborhood U of z in X making the following diagram commutative:

$$\begin{tikzcd} U\cap Z \arrow[r] \arrow[d, "s"'] & U \arrow[d, "s"] \\ \mathbb{A}^n_S \arrow[r] & \mathbb{A}^{n+r}_S \end{tikzcd} .$$

This is the classical "implicit functions theorem".

7.6. **Proof of proposition 7.2.1.** We give part of the proof and refer to [EGAIV, 17.2] for more details.

Sub-lemma 7.6.1. *Given a commutative diagram of schemes*

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T_0 \arrow[urr, "g_0"] \arrow[r, hook, "/", "i"'] & T \arrow[ur, shift left, "g_2"] \arrow[ur, shift right, "g_1"'] \arrow[r] & S, \end{tikzcd}$$

where $T_0 \overset{/}{\underset{i}{\hookrightarrow}} T$ is a thickening of order 1 and ideal $\mathcal{I}$, the map

$$g_2^\sharp - g_1^\sharp : \mathcal{O}_X \to g_{0*}\mathcal{O}_T$$

factorizes trough $g_{0*}\mathcal{I}$. Moreover:

$$g_2^\sharp - g_1^\sharp \in \mathrm{Der}_S(\mathcal{O}_X, g_{0*}\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, g_{0*}\mathcal{I}) \quad .$$

Remark 7.6.2. Notice that T_0 and T have the same underlying topological space. In particular $g_{0*}\mathcal{O}_T$ makes sense and coincide with $g_{i*}\mathcal{O}_T$, $i=1,2$.

Proof. The proof is elementary. Locally one has a commutative diagram of rings

$$\begin{tikzcd} & & B \arrow[dll, "g_0"'] \arrow[dl, shift right, "g_1"'] \arrow[dl, shift left, "g_2"] \\ C_0 & C \arrow[l, two heads] & A. \arrow[l] \arrow[u, "f"'] \end{tikzcd}$$

Clearly the map $\varphi : B \to C$ defined by $\varphi(b) = (g_2 - g_1)(b)$ takes values in $I := \ker(C \to C_0)$.

We need to check that φ belongs to $\mathrm{Der}_A(B, I)$. As g_1 and g_2 are ring homomorphisms one immediately obtains $\varphi(ab) = a\varphi(b)$ for all $a \in A$ and $b \in B$. Moreover for any $b, b' \in B$:

$$\begin{aligned} \varphi(b\cdot b') &= g_2(b)g_2(b') - g_1(b)g_1(b') \\ &= g_2(b)(g_2(b') - g_1(b')) + g_1(b')(g_2(b) - g_1(b)) \\ &= b\varphi(b') + b'\varphi(b). \end{aligned}$$

□

7.6.1. *Proof that $f : X \to S$ is net if and only if $\Omega^1_{X/S} = 0$.* Let us suppose that $\Omega^1_{X/S} = 0$. We have to show that $g_1 = g_2$. But:

$$g_2^\sharp - g_1^\sharp \in \mathrm{Hom}(\Omega^1_{X/S}, g_{0*}I) = 0 \ ,$$

hence $g_2^\sharp - g_1^\sharp = 0$ and $g_2 = g_1$.

Conversely suppose that $f : X \to S$ is net. Consider the diagram

$$\begin{tikzcd}
 & & X \arrow[d, "f"] \\
X \arrow[r, hook, "/" marking, "i"'] \arrow[urr, "\mathrm{Id}"] \arrow[dr, "\Delta"'] & (X \times_S X)_1 \arrow[r] \arrow[ur, shift left, "p_2"] \arrow[ur, shift right, "p_1"'] \arrow[d] & S \\
 & X \times_S X &
\end{tikzcd}$$

where $(X \times_S X)_1$ denotes the first infinitesimal neighborhood of Δ. As f is net one obtains $p_2 = p_1$ hence

$$0 = p_2^\sharp - p_1^\sharp =: d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S} \ .$$

Notice that $d_{X/S}$ corresponds to $\mathrm{Id}_{\Omega^1_{X/S}}$ under the canonical isomorphism

$$\mathrm{Der}_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1_{X/S}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \Omega^1_{X/S}) \ .$$

Hence $\mathrm{Id}_{\Omega^1_{X/S}} = 0$ and $\Omega^1_{X/S} = 0$.

□

7.6.2. *Proof of Proposition 7.2.1(c).* Let

$$\begin{tikzcd}
Z \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\
 & S
\end{tikzcd}$$

be an immersion over S. We want to show that if f is smooth then the sequence of $\mathcal{O}_Z$-modules

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0$$

is exact and locally split. Consider the commutative diagram

$$\begin{tikzcd}
 & & Z \arrow[d, "f"] \\
Z \arrow[r, hook, "/" marking, "i_1"'] \arrow[urr, equal] & Z_1 \arrow[r] \arrow[ur, dotted, "r"'] & S
\end{tikzcd}$$

where $Z \overset{/}{\underset{i_1}{\hookrightarrow}} Z_1 \hookrightarrow X$ is the first infinitesimal neighborhood of Z in X and the (local) retraction r of i_1 is provided by the smoothness of f.

Define $\varphi : i^*\Omega^1_{X/S} \to \mathcal{C}_{Z/S}$ by $\varphi(da \mod \mathcal{I}) = (\mathrm{Id}_{Z_1} - i_1 \circ r)^* a \mod \mathcal{I}^2$ for $a \in \mathcal{O}_X$. One easily checks that φ is an inverse of the natural morphism $\overline{d_{X/S}} : \mathcal{C}_{Z/X} \to i^*\Omega^1_{X/S}$ hence the result.

□

7.6.3. *Extensions of schemes by quasicoherent modules.* The rest of the proof require some preliminaries.

Definition 7.6.3. *Let $f : X \to S$ be a morphism of schemes and $\mathcal{I} \in \mathrm{QCoh}(\mathcal{O}_X)$. A S-extension of X by $\mathcal{I}$ is an S-thickening X' of X of order 1, of ideal $\mathcal{I}$:*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & X' \arrow[d, "g"] \\ & S \end{tikzcd}$$

An isomorphism of S-extensions

$$(X \overset{i'}{\hookrightarrow} X') \overset{a}{\simeq} (X \overset{i''}{\hookrightarrow} X'')$$

is an S-morphism $a : X' \to X''$ such that $ai' = i''$ and a induces the identity map on $\mathcal{I}$. In particular the map a^{-1} is an isomorphism:

$$\begin{tikzcd} & & \mathcal{O}_{X'} \arrow[dr] & & \\ 0 \arrow[r] & \mathcal{I} \arrow[ur] \arrow[dr] & & \mathcal{O}_X \arrow[r] & 0 \\ & & \mathcal{O}_{X''} \arrow[uu, "a^{-1}"] \arrow[ur] & & \end{tikzcd}$$

Remarks 7.6.4.
(i) Notice that *a priori* there is no multiplicative structure on $\mathcal{I}$.
(ii) As a 1-thickening of X has the same space as X, the datum of an S-extension X' of X by $\mathcal{I}$ is equivalent to the datum of an extension

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r, "p"] & \mathcal{O}_X \arrow[r] & 0 \\ & & f^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & & \end{tikzcd}$$

where $\mathcal{O}_{X'}$ is an $f^{-1}(\mathcal{O}_S)$-algebra and p is a homomorphism of $f^{-1}(\mathcal{O}_S)$-algebras. Hence the problem of constructing extensions is similar to the problem of constructing extensions of modules over a ring.
(iii) This notion of extension plays a crucial role in deformation theory but we won't go there.

Definition 7.6.5. *We denote by $\mathrm{Ext}_S(X, \mathcal{I})$ the set of isomorphism classes of S-extensions of X by $\mathcal{I}$.*

Lemma 7.6.6. $\mathrm{Ext}_S(X, \mathcal{I})$ *is naturally an abelian group, with neutral element the trivial extension $D(I) := \mathcal{O}_X \oplus \mathcal{I}$ (dual numbers over $\mathcal{I}$)*

Proof. Let us define the addition. Given two isomorphism classes $c_i := [0 \to \mathcal{I} \to \mathcal{O}_{X_i} \to \mathcal{O}_X] \in \mathrm{Ext}_S(X, \mathcal{I})$, $i = 1, 2$ we first consider the pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \oplus \mathcal{I} \arrow[r] \arrow[d, equal] & \mathcal{O}_{X_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_2} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "\Delta"'] & 0 \\
0 \arrow[r] & \mathcal{I} \oplus \mathcal{I} \arrow[r] & \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \arrow[r] & \mathcal{O}_X \oplus \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

then the pushout:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \oplus \mathcal{I} \arrow[r] \arrow[d, "+"'] & \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X_3} \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

and define $c_1 + c_2 := [0 \to \mathcal{I} \to \mathcal{O}_{X_3} \to \mathcal{O}_X \to 0]$. One easily shows this class does not depend on the choices of representatives for c_1 and c_2. $\square$

Lemma 7.6.7. *Let $f : X \to S$ and $\mathcal{I} \in \mathrm{QCoh}(\mathcal{O}_X)$. Assume that f is smooth. Then the morphism*

$$\begin{array}{rccc}
\varphi : & \mathrm{Ext}_S(X, \mathcal{I}) & \to & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}) \\
& \left[\begin{tikzcd}[row sep=small, column sep=small] X \arrow[r, hook, "i"] \arrow[dr] & X' \arrow[d] \\ & S \end{tikzcd}\right] & \mapsto & [0 \to \mathcal{I} \to i^*\Omega^1_{X'/S} \to \Omega^1_{X/S} \to 0]
\end{array}$$

is an isomorphism.

Proof. One easily checks that φ is a homomorphism of abelian groups. Using that f is smooth one defines an inverse

$$\psi : \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}) \to \mathrm{Ext}_S(X, \mathcal{I})$$

to φ as follows. Given $[0 \to \mathcal{I} \xrightarrow{u} E \xrightarrow{v} \Omega^1_{X/S} \to 0] \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$ consider the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r, "{(0,u)}"] \arrow[dr] & \mathcal{O}_X \oplus E \arrow[r, "\mathrm{Id}\oplus v"] & \mathcal{O}_X \oplus \Omega^1_{X/S} \arrow[r] & 0 . \\
& & \mathcal{O}_{X'} \arrow[r, "q"'] \arrow[u, "p"'] & \mathcal{O}_X \arrow[r] \arrow[u, "\mathrm{Id}+d_{X/S}"'] & 0
\end{tikzcd}$$

Define $\psi(E) = \left[\begin{tikzcd}[row sep=small, column sep=small] X \arrow[r, hook] \arrow[dr] & X' \arrow[d] \\ & S \end{tikzcd}\right]$ The composition $\psi\varphi$ is obviously the identity. To prove that $\varphi \circ \psi = \mathrm{Id}$, note that $p - q : \mathcal{O}_{X'} \to E$ is naturally an S-derivation hence defines a morphism $\gamma : i^*\Omega^1_{X'/S} \to E$. The following commutative diagram whose second line is

$\varphi\psi(E)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & E \arrow[r] & \Omega^1_{X/S} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[u, equal] & i^*\Omega^1_{X'/S} \arrow[r] \arrow[u, "\gamma"] & \Omega^1_{X/S} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

shows that γ is an isomorphism and the result. ☐

7.6.4. *Proof that if $f : X \to S$ is smooth then $\Omega^1_{X/S}$ is locally free.* We just proved that

$$\forall \mathcal{I} \in \mathrm{QCoh}(\mathcal{O}_X), \quad \mathrm{Ext}_S(X, I) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}) \ .$$

Denoting by $\mathcal{E}xt_S(X, \mathcal{I})$ the Zariski sheaf on X associated to the presheaf $U \mapsto \mathrm{Ext}_S(U, \mathcal{I}_{|U})$ one concludes that

$$\mathcal{E}xt_S(X, \mathcal{I}) \simeq \mathcal{E}xt^1(\Omega^1_{X/S}, \mathcal{I}) \ .$$

As f is smooth any S-extension of X by $\mathcal{I}$ is locally trivial (as there exists a local retraction) hence $\mathcal{E}xt_S(X, \mathcal{I}) = 0$ thus $\mathcal{E}xt^1(\Omega^1_{X/S}, \mathcal{I}) = 0$. As this is true for any $\mathcal{I} \in \mathrm{QCoh}(X)$ and $\Omega^1_{X/S}$ is of finite type over $\mathcal{O}_X$,we conclude that $\Omega^1_{X/S}$ is locally free by the sublemma below.

☐

Sub-lemma 7.6.8. *Let X be a scheme, $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X)$ of finite type. Suppose that for any $\mathcal{I} \in \mathrm{QCoh}(X)$ the group $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$ vanishes. Then $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module.*

Proof. As $\mathcal{F}$ is of finite type there exists an exact sequence of the form

$$0 \to \mathcal{I} \to \mathcal{O}_X^n \to \mathcal{F} \to 0 \ . \tag{13}$$

In particular $\mathcal{I} \in \mathrm{QCoh}(\mathcal{O}_X)$. By hypothesis $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) = 0$ hence the exact sequence eq. (13) locally splits, which implies that $\mathcal{F}$ is locally free. ☐

7.6.5. *Proof that if $X \xrightarrow{f} Y \xrightarrow{g} S$ and f is smooth then the sequence $0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$ is exact locally split.* We start with the

Lemma 7.6.9. *Consider $X \xrightarrow{f} Y \xrightarrow{g} S$ with f affine. Let $\mathcal{I} \in \mathrm{QCoh}(\mathcal{O}_X)$. Then one has a canonical exact sequence of abelian groups*

$$\begin{aligned} 0 \to \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{I}) \to \mathrm{Der}_S(\mathcal{O}_X, \mathcal{I}) \to \mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I}) \\ \xrightarrow{\partial} \mathrm{Ext}_Y(X, \mathcal{I}) \to \mathrm{Ext}_S(X, \mathcal{I}) \to \mathrm{Ext}_S(Y, f_*\mathcal{I}), \end{aligned} \tag{14}$$

where all the maps except ∂ are defined via the obvious functorialities and if $D \in \mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$ one defines

$$\partial(D) : \quad \begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r, "p"] & \mathcal{O}_X \arrow[r] & 0 \\
 & & f^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

where the map $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \oplus I$ corresponds to $(f^\sharp, D) : \mathcal{O}_Y \to f_*\mathcal{O}_X \oplus f_*\mathcal{I}$.

Proof. The proof is long but easy, see [EGAIV, 0_{IV}20.2.3]. □

Suppose now that f is smooth. The assertion that the sequence

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact locally split is local. Hence we can assume that $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} C$. In particular f is affine. Showing that

$$0 \to C \otimes_B \otimes \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

is exact locally split is equivalent to showing that for any C-module I, the sequence of Abelian groups obtained by applying the functor $\operatorname{Hom}_C(\cdot, I)$ is exact, equivalently that the sequence

$$0 \to \operatorname{Der}_B(C, I) \to \operatorname{Der}_A(C, I) \to \operatorname{Der}_A(B, IB) \to 0$$

is exact. As f is smooth $\Omega^1_{X/Y}$ is locally free by the previous section, hence $\Omega^1_{C/B}$ is projective of finite type over C. Hence $\operatorname{Ext}^1_C(\Omega^1_{C/B}, I) = \operatorname{Ext}_Y(X, I) = 0$ and the result follows from the Lemma 7.6.9.

□

7.6.6. We leave the two converse statements of 7.2.1 to the reader. He will prove them using the techniques already developed.

7.7. **A remark on smoothness.** Differential calculus provides a simple characterisation for a morphism $f : X \to S$ to be net: $\Omega^1_{X/S} = 0$. If $f : X \to S$ is smooth, we showed that $\Omega^1_{X/S}$ is $\mathcal{O}_X$-locally free. This is not a characterization of smoothness.

Let us indeed consider the following example. Let A be a ring and $B = A[X, Y]/(g)$. Consider the diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, hook, "i"] \arrow[d, "f"'] & \mathbb{A}^2_A \arrow[dl] \\ \operatorname{Spec} A. & \end{tikzcd}$$

The associated exact sequence of B-modules

$$\mathcal{C}_{B/A[X,Y]} \simeq (g)/(g^2) \to \Omega^1_{A[X,Y]/A} \otimes_{A[X,Y]} B \to \Omega^1_{B/A} \to 0 \tag{15}$$

can be rewritten

$$B \to BdX \oplus BdY \to \Omega^1_{B/A} \to 0 \ ,$$

where one maps $1 \in B$ to the differential $\partial g/\partial X dX + \partial g/\partial Y dY$. The Jacobian criterion shows that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is smooth if and only if

$$< \partial g/\partial X, \partial g/\partial Y >= B \ .$$

In this case $\Omega^1_{B/A}$ is locally free of rank one over B.

However there are other cases where $\Omega^1_{B/A}$ is locally free. Suppose that A has characteristic p and $f = X^p + Y^p$. In this case $\Omega^1_{B/A}$ is free of rank 2. Clearly $\operatorname{Spec} B$ is still of relative dimension 1 over A and we don't want to call such a map smooth!

Remark 7.7.1. Still, there is a purely differential criterion for smoothness involving the cotangent complex and not only $\Omega^1_{X/S}$.

7.8. Smoothness, flatness and regularity.

7.8.1. *Smoothness and flatness.* In this section we relate the smoothness of a morphism $f : X \to S$ to the smoothness of its fibers:

Theorem 7.8.1. *Let $f : X \to S$ be locally of finite presentation. The following conditions are equivalent:*

(i) *f is smooth.*
(ii) *f is flat and for any $s \in S$ the fiber X_s/s is smooth.*

Proof. Let us show that (2) $\Rightarrow$ (1). Let $x \in X$, we want to show that $f : X \to S$ is smooth at x. Let $s = f(x)$. The problem is local on X and we may assume that X is embedded in some $Z := \mathbb{A}_S^{n+r}$ with ideal $\mathcal{I}$. We have the diagram

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[r, hook, "i"] \arrow[d, "f"] & Z \arrow[dl] \\ s \arrow[r] & S. & \end{tikzcd}$$

Consider the exact sequence

$$0 \to \mathcal{I}_x \to \mathcal{O}_{Z,x} \to \mathcal{O}_{X,x} \to 0 \ .$$

Since f is flat one obtains an exact sequence after tensoring with $k(s)$:

$$0 \to \mathcal{I}_x \otimes_{\mathcal{O}_{S,s}} k(s) \to \mathcal{O}_{Z_s,x} \to \mathcal{O}_{X_s,x} \to 0 \ .$$

As f_s is smooth at x one may choose $(g_1, \cdots, g_r)$ generating $\mathcal{I}_x \otimes_{\mathcal{O}_{S,s}} k(s)$ such that $dg_1(x), \cdots, dg_r(x)$ are linearly independent in $\Omega^1_{Z_s/s} \otimes_{\mathcal{O}_{Z_s,x}} k(x) = \Omega^1_{Z/S} \otimes_{\mathcal{O}_{Z,x}} k(x)$. Lift $(g_1, \cdots, g_r)$ to $(f_1, \cdots, f_r) \in \mathcal{I}_x$. Then $df_1(x), \cdots, df_r(x)$ are linearly independent in $\Omega^1_{Z/S} \otimes_{\mathcal{O}_{Z,x}} k(x)$. By Nakayama's lemma $\mathcal{I}_x$ is generated by $f_1, \cdots, f_r$. By the Jacobian criterion f is smooth at x.

Conversely let us prove that (1) $\Rightarrow$ (2). Assume that $f : X \to S$ is smooth. By Corollary 7.1.3 smoothness is stable under base change of the target thus X_s/s is smooth for any $s \in S$. It remains to show that $f : X \to S$ is flat. Let $s \in S$ and $x \in X_s$. Locally around x we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"] & Z = \mathbb{A}_S^{n+r} \arrow[dl] \\ S. & \end{tikzcd}$$

Notice that Z is obviously flat over S at x. To prove that $X \to S$ is flat, we introduce the notion of *regular immersion*:

Definition 7.8.2. *A closed immersion $i : X \hookrightarrow Z$ of locally Noetherian schemes is said* regular *at a point $x \in X$ if the ideal $\mathcal{I}$ of i can be locally defined by $(f_1, \cdots, f_r)$ at x, such that $(f_i)_x$ is a regular sequence in $\mathcal{O}_{Z,x}$.*

Proposition 7.8.3. *Let*

$$
\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d] & Z \arrow[dl] \\
S &
\end{tikzcd}
$$

be a closed immersion locally of finite type over a locally Noetherian scheme S. Let $s \in S$ and $x \in Z_s$. The following conditions are equivalent:

(1) *The closed immersion $i_s : X_s \hookrightarrow Z_s$ is regular at x and Z is flat over S at x (i.e. $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module).*
(2) *X is flat over S at x and $i : X \hookrightarrow Z$ is regular at x.*

In particular the closed immersion $i : X \hookrightarrow Z$ is regular and Z is flat over S if and only if X is flat over S and $i_s : Y_s \hookrightarrow X_s$ is regular for any $s \in S$.

Admitting Proposition 7.8.3 for a moment, we are reduced to prove that the closed immersion $X_s \hookrightarrow Z_s$ is regular.

Let $\mathcal{I}$ be the ideal of i and $f_1, \cdots, f_r$ local sections of $\mathcal{I}$ at x such that $(f_i)_x$ is a minimal system of generators of $\mathcal{I}_x$, i.e. $(f_i \otimes k(x))_{1\leq i\leq r}$ is a basis of $\mathcal{C}_{X/Z}(x)$. As f is smooth the sequence of $k(x)$-vector spaces

$$
0 \longrightarrow \mathcal{C}_{X/Z}(x) \longrightarrow \Omega^1_{Z/S} \otimes k(x) \longrightarrow \Omega^1_{X/S} \otimes k(x) \longrightarrow 0
$$

is exact. As the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_{X/Z}(x) \arrow[r] \arrow[dr] & \Omega^1_{Z/S} \otimes k(x) \arrow[r] & \Omega^1_{X/S} \otimes k(x) \arrow[r] & 0 \\
& & \mathfrak{m}_{X_s,x}/\mathfrak{m}^2_{X_s,x} \arrow[u, "d_{Z/S}"'] & &
\end{tikzcd}
$$

is commutative, it follows that the $k(x)$-linear map $\mathcal{C}_{X/Z}(x) \to \mathfrak{m}_{X_s,x}/\mathfrak{m}^2_{X_s,x}$ is injective. Hence the $(f_i)_x$'s, $1 \leq i \leq r$ form a regular sequence in $\mathcal{O}_{X_s,x}$ and the closed immersion $X_s \hookrightarrow Z_s$ is regular. This finishes the proof that $X \to S$ is flat, hence the proof of Theorem 7.8.1, assuming Proposition 7.8.3. $\square$

Proposition 7.8.3 is the special case of the following algebraic statement for $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{Z,x}$ and $M = \mathcal{O}_{Z,x}$:

Proposition 7.8.4. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local morphism of Noetherian local rings and $k = A/\mathfrak{m}_A$. Let M be a finitely generated B-module and $(f_1, \cdots, f_r) \in \mathfrak{m}_B$. The following conditions are equivalent:*

(1) *M is A-flat and $(f_1 \otimes k, \cdots, f_r \otimes k)$ is $(M \otimes k)$-regular.*
(2) *$(f_1, \cdots, f_r)$ is M-regular and $M/\sum_{i=1}^r f_i M$ is flat over A.*

Proof. We start with a few lemmas.

Lemma 7.8.5. *Let R be an Artinian local ring, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let M be an R-module. Then $M \otimes_R k = 0$ implies $M = 0$.*

Proof. Since R is local Artinian there exists an integer m such that $\mathfrak{m}^m = 0$. Then $M \otimes_R k = 0$ implies

$$M = \mathfrak{m}M = \mathfrak{m}^2 M = \cdots = \mathfrak{m}^n M = 0 \ .$$

□

Lemma 7.8.6. *Let R be an Artinian local ring and M an R-module. Then M is free if and only if M is flat.*

Proof. If M is free it is clearly flat. Conversely let $\mathfrak{m}$ be the maximal ideal of R and k its residue field. Choose $(x_\alpha)_{\alpha \in I}$ a family of elements of M lifting a basis of $M/\mathfrak{m}M$. Denote by F the free R-module with basis $(e_\alpha)_{\alpha \in I}$ and $g : F \to M$ the homomorphism of R-modules mapping e_α to x_α. Applying Lemma 7.8.5 to the $\mathrm{Coker}\, u$ shows that g is surjective, hence provides an exact sequence of R-modules

$$0 \to K \to F \xrightarrow{g} M \to 0 \ .$$

Writing the beginning of the long exact sequence associated to the functor $\cdot \otimes_R k$ one obtains

$$\mathrm{Tor}_1^R(M, k) = 0 \to K/\mathfrak{m}K \to F/\mathfrak{m}F \xrightarrow{\overline{g}} M/\mathfrak{m}M \to 0 \ .$$

Hence $K/\mathfrak{m}K = 0$, thus $K = 0$ by Lemma 7.8.5. □

Lemma 7.8.7. *$(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local morphism of Noetherian local rings and $k = A/\mathfrak{m}_A$. Let E, F be finitely generated B-modules and $u : E \to F$ a morphism of B-modules.*

Suppose that F is A-flat and $u \otimes k : E \otimes k \to F \otimes k$ is injective. Then u is injective and $\mathrm{Coker}\, u$ *is flat over A.*

Proof. (Raynaud) For n a non-negative integer let $A_n := A/\mathfrak{m}^{n+1}$, $E_n := E \otimes A_n$ and $F_n := F \otimes A_n$. We first show that $u_n : E_n \to F_n$ is injective and split. Since F_n is flat over A_n and A_n is Artinian, F_n is free over A_n by Lemma 7.8.6. Take a basis of $E_n \otimes k$ and lift its image in $F_n \otimes k$ into a part of basis of F_n, which forms a free A_n-submodule L of F_n. The diagram

$$\begin{tikzcd} L \arrow[d, "\varphi"'] \arrow[dr, hook] & \\ E_n \arrow[r] & F_n \end{tikzcd}$$

commutes (where φ is defined in the obvious way). In particular φ is injective. By Nakayama's lemma φ is also surjective. Hence φ is an isomorphism, and the sequence

$$0 \to E_n \xrightarrow{u_n} F_n \to \mathrm{Coker}\,(u_n) \to 0$$

is exact and split.

The fact that F_n is A_n-flat thus implies that $\mathrm{Coker}\,(u_n)$ is also A_n-flat. Consider the commutative diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d] & F \arrow[d] \\ \hat{E} := \mathrm{colim} E_n \arrow[r, hook] & \hat{F} := \mathrm{colim} F_n, \end{tikzcd}$$

where $E \hookrightarrow \hat{E}$ (and similarly $F \hookrightarrow \hat{F}$) by [B, III, 5, prop.2]. So $E \to F$ is injective and Coker (u) is A-flat by [B, III, 5, theor.1]. □

Lemma 7.8.8. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local morphism of Noetherian local rings. Let M be a finitely generated B-module and $f \in \mathfrak{m}_B$. If $M/f^{n+1}M$ is flat over A for any $m \geq 0$ then M is flat over A.*

Proof. It is enough to show that for any $N \hookrightarrow N'$ finitely generated A-module the induced morphism $u : M \otimes_A N' \to M \otimes_A N$ is injective.

Let $x \in \mathrm{Ker}(u)$. Fix $n \geq 0$. As $M/f^{n+1}M$ is A-flat, the morphism

$$M/f^{n+1}M \otimes_A N' \to M/f^{n+1}M \otimes_A N$$

is injective. Hence $x \in f^{n+1}(M \otimes_A N')$. Finally $x \in \cap_n f^{n+1}(M \otimes_A N')$.

As $M \otimes_A N'$ is a finitely generated B-module it is separated for the f-adic topology, hence $x = 0$. So u is injective. □

We now finish the proof of Proposition 7.8.4.

Let us show (1) $\Rightarrow$ (2). By induction on r we are reduced to the case $r = 1$. By assumption $f \otimes k : M \otimes k \to M \otimes k$ is injective and M is flat over A. Thus f is injective and M/fM is A flat by Lemma 7.8.7.

Conversely let us show (2) $\Rightarrow$ (1). Once more by induction on r we are reduced to the case $r = 1$. Consider the exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0 \quad . \tag{16}$$

Applying the functor $\cdot \otimes_A k$ to this exact sequence, one obtains that $f \otimes k : M \otimes \to M \otimes k$ is injective as M/fM is A-flat. It remains to show that M is flat over A. Consider the exact sequence

$$0 \to M/fM \xrightarrow{f^n} M/f^{n+1}M \to M/f^nM \to 0 \quad .$$

By induction on n we obtain that $M/f^{n+1}M$ is A-flat for any n. Hence M is A-flat by Lemma 7.8.8.

□

7.8.2. *Smoothness and regularity.* Via Theorem 7.8.1 we now relate the geometric Definition 7.1.1 of smoothness and étaleness to a more algebraic one (used for example in [SGA1]):

Theorem 7.8.9. *Let $f : X \to S$ a morphism of schemes locally of finite presentation. The following conditions are equivalent:*

(i) *f is smooth.*
(ii) *f is flat and the geometric fibers of f are regular schemes.*

Corollary 7.8.10. *Let $f : X \to S$ a morphism of schemes. The morphism f is étale if and only if f is locally of finite presentation, flat and net.*

7.8.3. *Regularity.* We start with classical facts on regularity.

Let A be a Noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field $k := A/\mathfrak{m}$. In general $d := \dim A \leq \mathrm{rk}_k \mathfrak{m}/\mathfrak{m}^2$ (see [Stacks Project, Commutative Algebra, 57]).

Definition 7.8.11. *A Noetherian local ring A of dimension d is said to be regular if the following equivalent conditions are satisfied:*

(i) $d = \mathrm{rk}_k \mathfrak{m}/\mathfrak{m}^2$.
(ii) *there exist $x_1, \dots, x_d \in \mathfrak{m}$ generating $\mathfrak{m}$.*

A sequence $(x_i)_{1 \leq i \leq d}$ as in (ii) is called a regular system of parameters *for the regular local ring A.*

The regularity of a Noetherian local ring is a homological property. Recall that the homological dimension $\mathrm{hdim}(A)$ of a ring A is the smallest integer n such that any A-module M has a projective resolution of length at most n (if such an integer does not exist one defines $\mathrm{hdim}(A) = +\infty$). One easily shows that $\mathrm{hdim}(A) \leq n$ if and only if for any ideal I of A and any A-module M the groups $\mathrm{Ext}^i_A(A/I, M)$, $i > n$, do vanish.

Theorem 7.8.12 (Serre)**.** *A local ring A is regular if and only if is has finite homological dimension. In this case* $\mathrm{hdim}(A) = \dim A$.

As a corollary regularity is stable under localization:

Corollary 7.8.13. *If A is a regular local ring and $\mathfrak{p} \in \mathrm{Spec}\, A$ then $A_\mathfrak{p}$ is regular.*

Proof. Let J be an ideal of $A_\mathfrak{p}$. Hence $J = I_\mathfrak{p}$, where I is an ideal of A. Similarly any $A_\mathfrak{p}$-module is of the form $M_\mathfrak{p}$, $M \in A-\mathrm{Mod}$. By localization:

$$\mathrm{Ext}^i_{A_\mathfrak{p}}(A_\mathfrak{p}/I_\mathfrak{p}, M_\mathfrak{p}) \simeq \mathrm{Ext}^i_A(A/I, M)_\mathfrak{p} = 0 \quad \text{for } i > d$$

as A is regular of dimension d. Hence $A_\mathfrak{p}$ has finite homological dimension, hence is regular by Serre's theorem. □

We also need to understand when a quotient of a regular ring is regular.

Lemma 7.8.14. *Le A be a regular local ring with maximal ideal $\mathfrak{m}$ and $\dim A = d$. Let $I \subset \mathfrak{m}$, $B = A/I$. The following properties are equivalent:*

(1) *B is regular.*
(2) *there exists a regular system of parameters $(x_1, \dots, x_d)$ of A such that $I = \sum_{i=1}^r x_i A$.*

Proof. Let us show that (2) implies (1). Assume that $(x_1, \cdots, x_r)$ is part of a regular system of parameters of A. Then $\dim B = d - r$ (see [EGA0, IV 16.3.7]). Let $\mathfrak{n} = \mathfrak{m}/I$ be the maximal ideal of B, then we have an exact sequence

$$0 \to (\mathfrak{m}^2 + I)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0 \quad . \tag{17}$$

Since the x_i, $1 \leq i \leq r$, generate I and have linearly independent images in $\mathfrak{m}/\mathfrak{m}^2$, $\dim_k(\mathfrak{m}^2 + I)/\mathfrak{m}^2 = r$, hence $\dim_k \mathfrak{n}/\mathfrak{n}^2 = d - r = \dim B$ hence B is regular.

Conversely let us show that (1) implies (2). Let $d-r$ be the dimension of B. Assuming that B is regular, one has the equality $d - r = \dim_k \mathfrak{n}/\mathfrak{n}^2$. The sequence eq. (17) implies that $\dim_k(\mathfrak{m}^2 + I)/\mathfrak{m}^2 = r$. Choose x_i, $1 \leq i \leq r$, having linearly independent images in $\mathfrak{m}/\mathfrak{m}^2$, and choose $x_{r+1}, \cdots, x_d \in \mathfrak{m}$ such that $(x_1, \cdots, x_d)$ is a regular system of parameters of A. Denote by $I' := \sum_{i=1}^r x_i A \subset A$ and consider the exact sequence

$$0 \to I/I' \to A/I' \to A/I \to 0 \ .$$

As A/I is regular, A/I is a domain. Hence I/I' is prime in A/I'.

On the other hand it follows from (2) $\Rightarrow$ (1) that A/I' is regular and $\dim A/I' = d - r = \dim A/I$. The fact that I/I' is prime then implies $I = I'$. ☐

Let us globalize the notion of regularity.

Definition 7.8.15. *A scheme X is called* regular *if it is locally Noetherian and for any point x in X the Noetherian local ring $\mathcal{O}_{X,x}$ is regular.*

Corollary 7.8.16. *Let X be a Noetherian scheme. If $\mathcal{O}_{X,x}$ is regular for all closed points x of X then it is regular for all points x of X.*

Proof. As X is Noetherian it is quasi-compact. Hence any point has a closed point in its closure (see [Stacks Project, Schemes, 27.5.8]) and we can assume that $X = \operatorname{Spec} A$ is affine. Let $\mathfrak{p} \in \operatorname{Spec} A$. There exists a maximal ideal $\mathfrak{m} \supset \mathfrak{p}$. Hence $A_{\mathfrak{p}} = (A_{\mathfrak{m}})_{\mathfrak{p}}$ is regular by Corollary 7.8.13. ☐

7.8.4. *Schemes of finite type over a field.* Let k be a field and X/k a scheme of finite type. Recall that $x \in X$ is a closed point if and only if $[k(x) : k] < +\infty$ by the Hilbert Nullstellensatz. Moreover $\dim X = \dim \mathcal{O}_{X,x}$ in this case.

Proposition 7.8.17. *Let k be a field and X/k a scheme of finite type. The following conditions are equivalent:*

(i) *X/k is étale.*
(ii) *$\Omega^1_{X/k} = 0$, i.e. X/k is net.*
(iii) *$X = \operatorname{Spec} \prod_{i=1}^n K_i$, where K_i/k is a finite separable extension.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious.

For $(ii) \Rightarrow (iii)$: We can assume that $X = \operatorname{Spec} A$ is affine. We want to show that if $\overline{k}$ is an algebraic closure of k then $A \otimes_k \overline{k} \simeq \overline{k}^N$ (this characterises separable extensions). Let $Z := \operatorname{Spec}(A \otimes_k \overline{k})$ and $x \in Z$ a closed point (hence $k(x) = \overline{k}$). Thus $\Omega^1_{Z/\overline{k}} = \Omega^1_{X/k} \otimes_k \overline{k} = 0$ by assumption. The diagram

$$\begin{tikzcd} x \arrow[r, hook] \arrow[dr, "f"'] & Z \arrow[d] \\ & \operatorname{Spec} \overline{k} \end{tikzcd}$$

gives, as $f = \mathrm{Id}$ is obviously smooth:

$$0 \to \mathcal{C}_{x/Z} = \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega^1_{Z/\overline{k}} \otimes_{\overline{k}} k(x) \to \Omega^1_{x/\overline{k}} = 0 \to 0 \ .$$

Hence $\mathfrak{m}_x/\mathfrak{m}_x^2 \simeq \Omega^1_{Z/\overline{k}} \otimes_{\overline{k}} k(x) = 0$ thus $\mathfrak{m}_x = 0$ and $\mathcal{O}_{Z,x} = k(x) = \overline{k}$ as required.

For $(iii) \Rightarrow (i)$: without loss of generality one can assume that $X = \operatorname{Spec} K$, K/k finite separable. Write $K = k[T]/(f)$ with $f'(T) \neq 0$ in K. Consider the diagram:

$$\begin{tikzcd}
X = \operatorname{Spec} K \arrow[r, hook, "i"] \arrow[d] & \operatorname{Spec} k[T] \arrow[dl] \\
\operatorname{Spec} k. &
\end{tikzcd}$$

The Jacobian criterion implies that X/k is smooth, obviously of relative dimension zero, hence étale. ∎

Theorem 7.8.18. *Let k be a field and X/k be a scheme of finite type.*

(1) *if X/k is smooth then X is regular. If moreover X is integral then $\operatorname{rk}_k \Omega^1_{X/k} = \dim X$.*
(2) *If k is perfect and X is regular then X/k is smooth.*

Proof. For (1): by Corollary 7.8.16 it is enough to show that for any closed point x of X the local ring $\mathcal{O}_{X,x}$ is regular. Let $x \in X$ be a closed point. In particular $[k(x) : k] < +\infty$. Locally the following diagram holds:

$$\begin{tikzcd}
X \arrow[r, hook, "i", "/" marking] \arrow[d] & Z := \mathbb{A}^{n+r}_k \arrow[dl] \\
\operatorname{Spec} k &
\end{tikzcd}$$

with X of ideal $\mathcal{I}$ in Z. As X/k is smooth one can choose $(f_i)_{1 \leq i \leq r}$ in $\mathcal{O}_Z$ with $\mathcal{I}_x = \sum_{i=1}^r (f_i)_x \mathcal{O}_{Z,x}$ and $(d_{Z/k} f_i \otimes k(x))_{1 \leq i \leq r}$ linearly independent. Denote $\mathfrak{m} := \mathfrak{m}_{Z,x}$. Then:

$$\begin{tikzcd}
\mathcal{I}/\mathcal{I}^2 \otimes k(x) \arrow[r, hook, "\overline{d_{Z/k}}"] \arrow[dr] & \Omega^1_{Z/k} \otimes k(x) \\
 & \mathfrak{m}/\mathfrak{m}^2. \arrow[u, "d_{Z/k}"']
\end{tikzcd}$$

Hence the $[(f_i)_x]_{1 \leq i \leq r} \mod \mathfrak{m}^2$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. I.e. they are part of a regular system of parameters for $\mathcal{O}_{Z,x}$.

Hence $\mathcal{O}_{X,x} = \mathcal{O}_{Z,x}/\mathcal{I}_x$ is regular by the Lemma 7.8.14.

Suppose moreover X integral. As X is smooth over k the following sequence is exact:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Z/k} \otimes \mathcal{O}_X \to \Omega^1_{X/k} \to 0 \quad .$$

But $\operatorname{rk}(\Omega^1_{Z/k} \otimes \mathcal{O}_X) = n + r$ and $\operatorname{rk}(\mathcal{I}/\mathcal{I}^2) = r$ hence $\operatorname{rk} \Omega^1_{X/k} = n = \dim X$.

For (2): consider once more

$$\begin{tikzcd}
X \arrow[r, hook, "i", "/" marking] \arrow[d] & Z := \mathbb{A}^{n+r}_k \arrow[dl] \\
\operatorname{Spec} k &
\end{tikzcd}$$

with X of ideal $\mathcal{I}$ in Z. For $x \in X$ a closed point we want to show that

$$d_{Z/k} \otimes k(x) : \mathcal{I}/\mathcal{I}^2 \otimes k(x) \hookrightarrow \Omega^1_{Z/k} \otimes k(x) \quad ,$$

hence X/k is smooth thanks to the Jacobian criterion.

As k is perfect the extension $k(x)/k$ is separable hence $\Omega^1_{k(x)/k} = 0$. Consider the two exact sequences:

$$\mathcal{I}/\mathcal{I}^2 \otimes k(x) \to \Omega^1_{Z/k} \otimes k(x) \to \Omega^1_{X/K} \otimes k(x) \to 0 \tag{18}$$

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega^1_{X/k} \otimes k(x) \to \Omega^1_{k(x)/k} \to 0 \tag{19}$$

The first one implies that

$$\dim \Omega^1_{X/k} \otimes k(x) \geq \dim \Omega^1_{Z/k} \otimes k(x) - r = n \quad .$$

On the other hand the second one implies:

$$\dim \Omega^1_{X/k} \otimes k(x) \leq n \quad .$$

Hence $\dim \Omega^1_{X/k} \otimes k(x) = n$ and $d_{Z/k} \otimes k(x)$ is injective.

□

Corollary 7.8.19. *Let k be a field and X/k be a scheme of finite type. The following assertions are equivalent:*

(i) *X/k is smooth.*
(ii) *For any extension k'/k the scheme $X \otimes k'$ is regular.*
(iii) *There exist a perfect extension k' of k such that $X \otimes k'$ is regular.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are obvious. Let us show (3) ⇒ (1). As $X \otimes k'$ is regular and k' is perfect, it follows from Theorem 7.8.18(c) that $X \otimes k'$ is smooth over k'. As X/k is of finite type there exists a closed immersion $i : X \hookrightarrow \mathbb{A}^n_k$, we denote by $\mathcal{C}$ its conormal sheaf. By base change it induces a closed immersion $i' : X \otimes k' \hookrightarrow \mathbb{A}^n_{k'}$, with conormal sheaf $\mathcal{C}'$. Let x be a point of X and x' a point of X' over x. As X'/k is smooth the linear map

$$d_{\overline{\mathbb{A}^n_{k'}/k'}} \otimes k(x') : \mathcal{C}' \otimes k(x') \to \Omega^1_{\mathbb{A}^n_{k'}} \otimes k(x')$$

is injective by Proposition 7.2.1(c). Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C} \otimes k(x) \arrow[r, "\overline{d_{\mathbb{A}^n_k}} \otimes k(x)"] \arrow[d] & i^*\Omega^1_{\mathbb{A}^n_k/X} \otimes k(x) \arrow[d] \\
\mathcal{C}' \otimes k(x') \arrow[r, "\overline{d_{\mathbb{A}^n_{k'}}} \otimes k'(x)"] & i'^*\Omega^1_{\mathbb{A}^n_{k'}/X'} \otimes k(x')
\end{tikzcd} \quad .$$

As $k \to k'$ is flat one shows that the vertical maps of this diagram are injective. Hence $\overline{d_{\mathbb{A}^n_k}} \otimes k(x)$ is injective. It follows from Proposition 7.2.1(c, converse) that X/k is smooth.

□

7.8.5. *Proof of Theorem 7.8.9.* As any algebraically closed field is perfect, it follows from Theorem 7.8.18 that Theorem 7.8.9 is equivalent to Theorem 7.8.1.

□

7.9. **Examples of étale morphisms.**

Example 7.9.1. We relate the notion of étale morphisms to classical facts of algebraic number theory. Let L/K be an extension of number fields. Consider the morphism $f : \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ between their rings of integers. The ramification locus of this morphism is an ideal of $\mathcal{O}_L$, called the different $\mathcal{D}_{L/K}$, which is nothing else than the annihilator of $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$. The discriminant of this morphism, an ideal of $\mathcal{O}_K$, is the norm of the different, i.e. $f_*\mathcal{D}_{L/K}$. If one defines $X := \operatorname{Spec} \mathcal{O}_L \setminus \mathcal{D}_{L/K}$, the morphism $f : X \to \operatorname{Spec} \mathcal{O}_K$ is unramified. As any local homomorphism of DVR is flat, $f : X \to \operatorname{Spec} \mathcal{O}_K$ is in fact étale. Denote by Y the complement of the discriminant in $\operatorname{Spec} \mathcal{O}_K$, the morphism $f : X \to Y$ is finite étale.

Example 7.9.2. [Ray70, p.66]

Lemma 7.9.3. *Let A be a ring and $B = A[T]/(T^n - a)$. Then B is étale over A if and only if $n = 1$ or na is invertible in A.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} A$ and let $k := k(\mathfrak{p})$. Let $\overline{B} := B \otimes_A k = k[T]/(T^n - \alpha)$ where α denotes the image of a in k. By the Jacobian criterion $\overline{B}$ is étale over k if and only if nT^{n-1} and $T^n - \alpha$ are relatively prime in $k[T]$. This holds true if $n = 1$ or if $n\alpha \neq 0$ in k and is not true if n is a multiple of $\operatorname{char} k$ or if $n \neq 1$ and $\alpha = 1$. □

Remark 7.9.4. For $a = 1$ the spectrum of B is nothing else than the finite group scheme μ_n over A of n-roots of unity.

Example 7.9.5. [Ray70, p.70] Let k be a field and $B = k[X, Y]$ with the action of $G := \mathbb{Z}/2\mathbb{Z}$ by central symmetry mapping (X, Y) to $(-X, -Y)$. Then $A := B^G$ is generated over k by $u = X^2$, $v = Y^2$, $w = XY$. Hence $A = k[u, v, w]/(uv - w^2)$. The algebra B is finite over A and $B = A[X, Y]/(X^2 - u, Y^2 - v, XY - w)$. The Jacobian matrix has 2×2-minors equal to $4XY, -2X^2, -2Y^2$. By the Jacobian criterion B is étale over A outside the origin.

8. Etale fundamental group

We give a light introduction to the étale fundamental group, following [Mi80], and refer to [SGA1] for much more material.

8.1. **Reminder on the topological fundamental group.** Let X be a connected topological space. We assume that X is arcwise connected and locally simply connected. Let x be a point in X. The fundamental group $\pi_1(X, x)$ is the group of loops in X through x, up to homotopy. This definition can hardly generalize to schemes and we will use a more categorical one.

Recall that $\pi : Y \to X$ is a covering of X if any point x in X admits a neighbourhood U such that $\pi^{-1}(U) \simeq \coprod_i U_i$ with $\pi_{|U_i} : U_i \to U$ a homeomorphism. Denote by $\mathbf{Cov}(X)$ the category whose objects are coverings of X with a finite number of connected components (and the obvious morphisms). The functor

$$F_x : \begin{array}{ccc} \mathbf{Cov}(X) & \to & \mathbf{Sets} \\ [\pi : Y \to X] & \mapsto & \pi^{-1}(x) \end{array}$$

associating to any covering its fiber over x is representable by the universal cover $\tilde{X} \to X$:

$$\forall \pi : Y \to X, \quad F_x(Y) \simeq \mathrm{Hom}_X(\tilde{X}, Y) \ .$$

The group $\pi_1(X, x) := \mathrm{Aut}_X(\tilde{X})$ acts on $\tilde{X}$ on the right, hence on $\mathrm{Hom}(\tilde{X}, Y)$ on the left. This enriches the functor F_x as:

$$F_x : \mathbf{Cov}(X) \to \pi_1(X, x) - \mathbf{Sets}$$

and defines an equivalence of categories between $\mathbf{Cov}(X)$ and the category of $\pi_1(X, x)$-sets with a finite number of orbits.

We will generalize this picture to schemes.

8.2. **The étale fundamental group.** Let X be a scheme. Let FEt/X be the category of finite étale morphisms $\pi : Y \to X$ (with X-morphisms). Let us fix $\overline{x} \to X$ a geometric point of X (hence $\overline{x} = \mathrm{Spec}\, k$ with k separably closed) and consider the functor

$$\begin{array}{rlcl} F_{\overline{x}}: & \mathrm{FEt}/X & \to & F\mathbf{Sets} \\ & [\pi : Y \to X] & \mapsto & Hom_X(\overline{x}, Y) \end{array}$$

which associates to any finite étale cover of X its fiber over $\overline{x}$ (where $F\mathbf{Sets}$ denotes the category of finite sets).

The functor $F_{\overline{x}}$ is usually not representable. Consider for example $X = \mathbb{A}^1_k \setminus \{0\}$ over an algebraically closed field k of characteristic 0. One easily checks that the only schemes in FEt/X are the $X_n = X \stackrel{t \mapsto t^n}{\to} X$, $n \in \mathbb{N}^*$. There is no "biggest" such scheme, hence no universal cover. Notice that if $k = \mathbb{C}$ the topological universal cover which dominates all the X_n is given by $\exp : \mathbb{C} \to \mathbb{C}^*$ which is not an algebraic morphism.

However $F_{\overline{x}}$ is pro-representable: there exists a projective system $\tilde{X} = (X_i)_{i \in I}$ of objects $X_i \to X \in \mathrm{FEt}/X$ indexed by a directed set I such that

$$F_{\overline{X}}(Y) = \mathrm{Hom}_X(\tilde{X}, Y) := \mathrm{colim}_I \mathrm{Hom}_X(X_i, Y) \ .$$

One can always choose the X_i/X Galois, i.e. of degree equal to $|\mathrm{Aut}_X(X_i)|$. Let us define

$$\pi_1^{\text{ét}}(X, \overline{x}) = \mathrm{Aut}_X(\tilde{X}) := \lim_I \mathrm{Aut}_X(X_i) \ .$$

As $\mathrm{Aut}_X(X_i)$ is a finite group the group $\pi_1^{\text{ét}}(X, \overline{x})$ is naturally a profinite group.

Example 8.2.1. Consider again the case $X = \mathbb{A}^1_k \setminus \{0\}$, $k = \overline{k}$ of characteristic zero and X_n as above. Then $\mathrm{Aut}_X(X_n) = \mu_n(k)$ (where $\xi \in \mu_n(k)$ acts on X_n by $\xi(x) = \xi \cdot x$). Hence

$$\pi_1^{\text{ét}}(\mathbb{A}^1_k \setminus \{0\}) = \lim_n \mu_n(k) \simeq \hat{\mathbb{Z}} \ .$$

Example 8.2.2. Let $X/\mathbb{C}$ be a smooth quasi projective variety. The Riemann's existence theorem (due in this generality to Grauert and Remmert) states that the natural functor

$$\begin{array}{lcl} \mathrm{FEt}/X & \to & F\mathbf{Cov}(X^{\mathrm{an}}) \\ [\pi : Y \to X] & \mapsto & [\pi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}] \end{array}$$

is an equivalence of categories (where $F\mathbf{Cov}(X^{\mathrm{an}})$ denotes the category of finite coverings). Hence $\pi_1^{\text{ét}}(X)$ and $\pi_1(X^{\mathrm{an}})$ have the same finite quotients. As $\pi_1^{\text{ét}}(X)$ is profinite this implies that $\pi_1^{\text{ét}}(X) \simeq \pi_1(X^{\mathrm{an}})^\wedge$, the profinite completion of $\pi_1(X^{\mathrm{an}})$.

Exercice 8.2.3. Show that $\pi_1^{\text{ét}}(\mathbb{P}^1_k) = \{1\}$ for any separably closed field k.

Example 8.2.4. Let $X = \operatorname{Spec} k$, k a field. Choose $X_i = \operatorname{Spec} K_i$ where K_i ranges through the finite extensions of k in k^s. Thus $\pi_1^{\text{ét}}(X) = \operatorname{Gal}(k^s/k)$.

Example 8.2.5. Let X be a normal irreducible scheme with generic point x. Write $\overline{x} := \operatorname{Spec} k(x)^s$ and define X_i as the normalisation of X in K_i where K_i ranges through the finite Galois extensions of $k(x)$ in $k(x)^s$ such that X_i/X is unramified. Thus $\pi_1^{\text{ét}}(X) = \operatorname{Gal}(k(x)^{\text{ur}}/k(x))$.

Theorem 8.2.6. *Let X be a connected scheme and $\overline{x} \to X$ a geometric point. Then*

$$F_{\overline{x}} : \mathrm{FEt}/X \to \pi_1(X, \overline{x}) - F\mathbf{Sets}$$

is an equivalence of categories, where $\pi_1(X, \overline{x}) - F\mathbf{Sets}$ denotes the category of finite sets with a continuous *$\pi_1(X, \overline{x})$-action).*

9. Sites and sheaves

9.1. **Presheaves.** Recall that a category is *small* if its objects and its morphisms form sets.

Definition 9.1.1. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be any category. A presheaf on $\mathcal{C}$ with value in $\mathcal{D}$ is a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$. We denote by $\mathbf{PSh}(\mathcal{C}, \mathcal{D})$ the category of presheaves on $\mathcal{C}$ with value in $\mathcal{D}$.*

Definition 9.1.2. *We write $\mathbf{PSh}(\mathcal{C}) := \mathbf{PSh}(\mathcal{C}, \mathbf{Sets})$ and $\mathbf{PAb}(\mathcal{C}) := \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$. If Λ is a ring, we denote by $\Lambda - \mathrm{Mod}$ the category of Λ-modules and by $\mathbf{P\Lambda - Mod}(\mathcal{C}) := \mathbf{PSh}(\mathcal{C}, \Lambda - \mathrm{Mod})$.*

Example 9.1.3. Let $X \in \mathcal{C}$. Then

$$\begin{array}{rccl} h_X : & \mathcal{C}^{\text{op}} & \to & \mathbf{Sets} \\ & U & \mapsto & h_X(U) := \operatorname{Hom}_{\mathcal{C}}(U, X) \end{array}$$

is the presheaf represented by X.

Lemma 9.1.4. *(Yoneda) Let $\mathcal{C}$ be a category. Then for any $F \in \mathbf{PSh}(\mathcal{C})$ there is a functorial isomorphism*

$$F(X) \simeq \operatorname{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_X, F) \ .$$

In particular the functor $\mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ mapping X to h_X is fully faithful.

9.2. **Sheaves on topological spaces.** We recall some classical facts concerning sheaves on topological spaces.

Let (X, τ) be a topological space i.e. X is a set and τ is the set of open subsets of X. Hence τ is a subset of $\mathcal{P}(X)$ such that:

(a) $\emptyset \in \tau$, $X \in \tau$.
(b) If I is a set and $(U_i)_{i \in I} \in \tau^I$ then $\bigcup_{i \in I} U_i \in \tau$.
(c) $\forall\, U, V \in \tau,\ \ U \cap V \in \tau$.

One associates canonically a category X_τ to (X, τ). Its objects are the elements of τ and $\operatorname{Hom}_{X_\tau}(U, V)$ is empty if $U \not\subset V$, the set with one element otherwise. By definition a presheaf on (X, τ) is a presheaf on X_τ.

Definition 9.2.1. *Let $\mathcal{F}$ be a presheaf of sets on (X, τ). It is a sheaf if the following conditions are satisfied:*

(1) *For any $U = \bigcup_{i \in I} U_i \in \tau$, for any $s, t \in \mathcal{F}(U)$ such that $s_{|U_i} = t_{|U_i} \in \mathcal{F}(U_i)$ for all $i \in I$ then $s = t \in \mathcal{F}(U)$.*

(2) *For any $U = \bigcup_{i \in I} U_i \in \tau$ and any $(s_i \in \mathcal{F}(U_i))_{i \in I}$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all $i, j \in I$, there exists $s \in \mathcal{F}(U)$ such that $s_i = s_{|U_i}$.*

In other words: $F \in \mathbf{Sh}(X_\tau)$ if and only if for any $U \in \tau$, for any decomposition $U = \bigcup_{i \in I} U_i$, the natural sequence of sets

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact.

Remark 9.2.2. In this theorem and in the rest of the text: a sequence of sets is said to be exact if this is an equalizer.

Definition 9.2.3. *One defines $\mathbf{Sh}(X_\tau)$, resp. $\mathbf{Ab}(X_\tau)$, resp. $\mathbf{\Lambda} - \mathbf{Mod}(X_\tau)$, as the full subcategory of sheaves in $\mathbf{PSh}(X_\tau)$, resp. in $\mathbf{PAb}(X_\tau)$, resp. in $\mathbf{P\Lambda} - \mathbf{Mod}(X_\tau)$.*

9.3. **Sites.**

9.3.1. *Sieves.*

Definition 9.3.1. *Let $\mathcal{C}$ be a small category and $S \in \mathcal{C}$. A sieve of S is a subfunctor $\mathcal{U} \subset h_S = \mathrm{Hom}(\cdot, S)$. In other words this is a collection of morphisms $T \to S$ stable by precomposition.*

Definition 9.3.2. *Let $\mathcal{U} \subset h_S$ be a sieve and $f : T \to S$. The pull-back of $\mathcal{U}$ is $f^*\mathcal{U} = \mathcal{U} \times_{h_S} h_T \subset h_T$.*

9.3.2. *Topology.*

Definition 9.3.3. *A Grothendieck topology τ on a small category $\mathcal{C}$ is the datum, for every object $S \in \mathcal{C}$, of a family $\mathrm{Cov}_\tau(S)$ of sieves of S, called* covering sieves *of S, satisfying the following axioms:*

(GT1) $\forall\, S \in \mathcal{C}, \quad h_S \in \mathrm{Cov}_\tau(S)$.

(GT2) $\forall\, f : T \to S \in \mathcal{C}, \quad \forall\, \mathcal{U} \in \mathrm{Cov}_\tau(S), \ f^*\mathcal{U} \in \mathrm{Cov}_\tau(T)$.

(GT3) *If $\mathcal{V} \in \mathrm{Cov}_\tau(S)$ and $\mathcal{U} \subset h_S$ are such that for any $g : T \to S \in \mathcal{V}$, $g^*(\mathcal{U}) \in \mathrm{Cov}_\tau(T)$ then $\mathcal{U} \in \mathrm{Cov}_\tau(S)$.*

Definition 9.3.4. *A site $\mathcal{C}_\tau$ is a small category $\mathcal{C}$ equipped with a Grothendieck topology τ.*

Lemma 9.3.5. *Let $\mathcal{C}_\tau$ be a site.*

(i) *If $\mathcal{U} \subset \mathcal{V} \subset h_S$ and $\mathcal{U} \in \mathrm{Cov}_\tau(S)$ then $\mathcal{V} \in \mathrm{Cov}_\tau(S)$.*

(ii) *If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}_\tau(S)$ then $\mathcal{U} \cap \mathcal{V} \in \mathrm{Cov}_\tau(S)$.*

Proof. For (i): it is enough to notice that if $f : T \to S \in \mathcal{U}$ the pull-back $f^*\mathcal{V}$ is the sieve of T of arrows $X \to T$ whose composite with f is in $\mathcal{V}$. As $f \in \mathcal{U}$ and $\mathcal{U}$ is a sieve, $f^*(\mathcal{V}) = h_T \in \mathrm{Cov}_\tau(T)$ by $(GT1)$. It follows from $(GT3)$ that $\mathcal{V} \in \mathrm{Cov}_\tau(S)$.

For (ii): obviously $\mathcal{U} \cap \mathcal{V}$ is a sieve. Let $g : T \to S \in \mathcal{V}$. The sieve $g^*(\mathcal{U} \cap \mathcal{V})$ of T coincide with $g^*\mathcal{U}$, which belongs to $\mathrm{Cov}_\tau(T)$ by $(GT2)$. The result follows then from $(GT3)$. ∎

9.3.3. *Pre-topologies.*

Definition 9.3.6. *Let $\mathcal{C}$ be a small category with fiber products. A Grothendieck pre-topology on $\mathcal{C}$ is the datum of* covering families $(S_i \to S)_{i\in I}$ *for all objects $S \in \mathcal{C}$ such that:*

(PT1) *For any $S \in \mathcal{C}$, any isomorphism $S' \simeq S$ in $\mathcal{C}$ is a covering family of S.*
(PT2) *If $(S_i \to S)_{i\in I}$ is a covering family and if $T \to S \in \mathcal{C}$ is any morphism then the family $S_i \times_S T \to T$ is a covering family of T.*
(PT3) *If $(S_i \to S)_{i\in I}$ and $(S_{i,j} \to S_i)_{j\in J_i}$ are covering families for S and S_i respectively then $(S_{i,j} \to S)_{i,j}$ is a covering family for S.*

Lemma 9.3.7. *Let $\mathcal{C}$ be a small category with a Grothendieck pre-topology. Define a covering sieve $\mathcal{U} \in h_S$ as any sieve containing a covering family of S. Then this family of covering sieves define a Grothendieck topology on $\mathcal{C}$.*

Proof. Exercice. □

Example 9.3.8. Let $\mathcal{C}$ be any small category. Define the collection of covering sieves for $S \in \mathcal{C}$ as being reduced to h_S. The associated topology is called the chaotic topology. Sheaves for this topology are just presheaves.

Example 9.3.9. Let (X, τ) be any topological space. One defines a covering family of $U \in X_\tau$ as any family $(U_i \to U)_{i\in I}$ in τ such that $U = \bigcup_{i\in I} U_i$. This makes X_τ a site.

Example 9.3.10. Let G be a group. Let T_G be the category of G-sets (with G-equivariant morphisms). The covering families are the $(f_i : U_i \to U)_{i\in I}$ such that $U = \bigcup_{i\in I} f_i(U_i)$. This makes T_G a site.

9.3.4. *Topologies on categories of schemes.*

Definition 9.3.11. *Let S be a scheme. One denotes by $\mathbf{Sch}/S$ the category of schemes over S.*

Lemma 9.3.12. *Let $\mathcal{C}$ be a subcategory of $\mathbf{Sch}/S$ with fiber products. Let (P) be a property of morphisms of $\mathcal{C}$ satisfying:*

(i) *(P) is true for isomorphisms of $\mathcal{C}$.*
(ii) *(P) is stable by base-change.*
(iii) *(P) is stable by composition.*

Define a family $(f_i : T_i \to T)_{i\in I}$ in $\mathcal{C}$ to be a covering family if for any $i \in I$ the arrow $f_i : T_i \to T$ satisfies (P), and $|T| = \bigcup_{i\in I} f_i(|T_i|)$. This defines a (pre)-topology on $\mathcal{C}$.

Proof. It is enough to check $(PT2)$. This follows from the fact that the underlying set of a fiber product of schemes surjects onto the fiber product of the underlying sets. □

Definition 9.3.13. *Being an open immersion, an étale morphism, a smooth morphism or a faithfully flat morphism of finite presentation are properties (P) satisfying the conditions of Lemma 9.3.12. These properties define respectively the sites $(\mathbf{Sch}/S)_{\mathrm{Zar}}$, $(\mathbf{Sch}/S)_{\text{ét}}$, $(\mathbf{Sch}/S)_{\mathrm{smooth}}$, $(\mathbf{Sch}/S)_{\mathrm{fppf}}$.*

Lemma 9.3.14. *Let $\tau \in \{\mathrm{Zar}, \text{ét}, \mathrm{smooth}, \mathrm{fppf}\}$. Let $T \in \mathbf{Sch}/S$ be an affine scheme and let $(T_i \to T)_{i\in I}$ be a τ-covering family. Then there exists a τ-covering $(U_j \to T)_{1\le j\le m}$ which is a refinement of $(T_i \to T)_{i\in I}$ such that each U_j is open affine in some T_i.*

This last property, which is crucial for reducing oneself to finite coverings, is not automatically satisfied for more general flat families. Hence we define:

Definition 9.3.15. *Let $T \to S$ be a scheme over S. An fpqc covering of T is a family $(f_i : T_i \to T)_{i \in I}$ such that:*

(i) *each $T_i \to T$ is a flat morphism and $|T| = \bigcup_{i \in I} f_i(|T_i|)$.*
(2) *For each affine open $U \subset T$ there exists a finite set $J \subset I$ and affine opens $U_j \subset T_j$ such that $U = \bigcup_{j \in J} f_j(U_j)$.*

This defines a site $(\mathbf{Sch}/S)_{\mathrm{fpqc}}$.

Example 9.3.16. (i) If $f : T' \to T$ is flat surjective and quasi-compact then this is an fpqc-covering.
(ii) For k an infinite field, the morphism $\varphi : \coprod_{x \in \mathbb{A}^n_k} \mathrm{Spec}\,(\mathcal{O}_{\mathbb{A}^n_k, x}) \to \mathbb{A}^n_k$ is flat and surjective but it is not quasicompact hence it is not an fpqc-covering.
(iii) Write $\mathbb{A}^2_k = \mathrm{Spec}\, k[x,y]$. The family $(D(x) \hookrightarrow \mathbb{A}^2_k, D(y) \hookrightarrow \mathbb{A}^2_k, \mathrm{Spec}\, k[[x,y]] \to \mathbb{A}^2_k)$ is an fpqc-covering (where $D(x)$ and $D(y)$ are the standard Zariski open subsets).

9.4. **Sheaves on a site.**

9.4.1. *Sections of a presheaf on a sieve.* Let $F \in \mathbf{PSh}(\mathcal{C})$ and $S \in \mathcal{C}$. By Yoneda's lemma:

$$F(S) \simeq \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(h_S, F) \ .$$

Hence it is natural to make the following:

Definition 9.4.1. *Let $F \in \mathbf{PSh}(\mathcal{C})$ and $\mathcal{U} \subset h_S$ a sieve of $S \in \mathcal{C}$. One defines $F(\mathcal{U}) :=$ $\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(\mathcal{U}, F)$.*

In down-to-earth terms: if $\mathcal{U} = \{f : U_f \to S\}$, a section $s \in F(\mathcal{U})$ is a collection

$$(s_f) \in \prod_{f \in \mathcal{U}} F(U_f) \ \text{ such that } \ F(g)s_f = s_{fg} \ ,$$

for any $f : U_f \to S \in \mathcal{U}$ and any $g : X \to U_f$.

9.4.2. *Sheaves: definition.*

Definition 9.4.2. *Let $\mathcal{C}_\tau$ be a site. A presheaf $F \in \mathbf{PSh}(\mathcal{C})$ is a τ-sheaf (resp. is τ-separated) if for any $S \in \mathcal{C}$ and any $\mathcal{U} \in \mathrm{Cov}_\tau(S)$ the restriction map*

$$F(S) \to F(\mathcal{U})$$

is bijective (resp. injective).

Definition 9.4.3. *One defines $\mathbf{Sh}(\mathcal{C}_\tau)$ as the full subcategory of $\mathbf{PSh}(\mathcal{C})$ whose objects are τ-sheaves.*

9.4.3. *Sheafification.* Let $\mathcal{C}_\tau$ be a site.

Definition 9.4.4. *Let $F \in \mathbf{PSh}(\mathcal{C})$ and $S \in \mathcal{C}$. Define*

$$F^+(S) := \mathrm{colim}_{\mathcal{U} \in \mathrm{Cov}_\tau(S)} F(\mathcal{U}) \quad .$$

Lemma 9.4.5. *For any $F \in \mathbf{PSh}(\mathcal{C})$, $F^+ \in \mathbf{PSh}(\mathcal{C})$.*

Proof. Let $f : T \to S \in \mathcal{C}$ and $\mathcal{U} \in \mathrm{Cov}_\tau(S)$. Taking the colimit on $\mathrm{Cov}_\tau(S)$ of the arrows

$$F(\mathcal{U}) \to F(f^*(\mathcal{U})) \to \mathrm{colim}_{\mathcal{V} \in \mathrm{Cov}_\tau(T)} F(\mathcal{V}) = F^+(T)$$

defines a map $F^+(S) \stackrel{f^*}{\to} F^+(T)$. □

Lemma 9.4.6. *For any $F \in \mathbf{PSh}(\mathcal{C})$ the presheaf F^+ is τ-separated.*

One has a natural morphism of functors $F \to F^+$ in $\mathbf{PSh}(\mathcal{C})$ hence a morphism of functors

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F^+, \cdot) \to \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, \cdot) \quad .$$

Lemma 9.4.7. *If $G \in \mathbf{Sh}_\tau(\mathcal{C})$ then $\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F^+, G) \simeq \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, G)$. In particular if F is a sheaf one has a canonical isomorphism $F \simeq F^+$.*

Lemma 9.4.8. *If F is separated then F^+ is a sheaf.*

Definition 9.4.9. *One defines the τ-sheafification $F^\sharp \in \mathbf{Sh}(\mathcal{C}_\tau)$ of $F \in \mathbf{PSh}(\mathcal{C})$ as*

$$F \to F^+ \to F^{++} =: F^\sharp \quad .$$

Lemma 9.4.10. *One has a natural adjunction*

$$\cdot^\sharp : \mathbf{PSh}(\mathcal{C}) \rightleftarrows \mathbf{Sh}(\mathcal{C}_\tau) : \mathrm{i} \quad .$$

9.4.4. *Properties of sheafification.* As the sheafification functor $\cdot^\sharp$ has a right adjoint it commutes with all colimits. In particular: for any family $(F_i)_{i \in I}$ of $\mathbf{Sh}(\mathcal{C}_\tau)$,

$$\mathrm{colim}_{\mathbf{Sh}(\mathcal{C}_\tau)} F_i = (\mathrm{colim}_{\mathbf{PSh}(\mathcal{C})} F_i)^\sharp \quad .$$

In the category **Sets** the filtered colimits commute with finite limits. As the functor $\cdot^+$ is defined using filtered colimits it preserves small limits. In particular it preserves algebraic structures: the sheafification of an abelian presheaf is an abelian sheaf, etc...

9.4.5. *Sheaves and pre-topologies.* Suppose the site $\mathcal{C}_\tau$ is defined by a pre-topology given by covering families $(U_i \to X)_{i \in I}$. Let $\mathcal{U}$ be a covering sieve of $X \in \mathcal{C}$ generated by a covering family $(U_i \to X)_{i \in I}$. Then for all $F \in \mathbf{PSh}(\mathcal{C})$ the following sequence of sets is exact:

$$F(\mathcal{U}) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)) \quad .$$

The presheaf F is a sheaf if and only if the map $F \to F^+$ is an isomorphism. Hence it is enough that for any object X there exists a cofinal set of covering sieves $\mathcal{U}$ of $\mathrm{Cov}_\tau(X)$ such that the natural map $F(X) \to F(\mathcal{U})$ is an isomorphism. Hence F is a τ-sheaf if and only if for any object $X \in \mathcal{C}$ and any covering family $(U_i \to X)_{i \in I}$ the following sequence of sets is exact:

$$F(X) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)) \quad .$$

Exercice 9.4.11. Let T_G be the site defined in Example 9.3.10. Show that $\mathbf{Sh}(T_G)$ is naturally equivalent to the category of G-sets.

9.5. **The abelian category of abelian sheaves; cohomology.** In this section, for simplicity of notations we denote by the same symbol a site and its underlying category.

Theorem 9.5.1. *Let $\mathcal{C}$ be a site. The category $\mathbf{Ab}(\mathcal{C})$ is Abelian. Moreover:*

(1) *if $\varphi : F \to G$ is a morphism in $\mathbf{Ab}(\mathcal{C})$ then $\ker \varphi = \ker i(\varphi)$ and $\mathrm{Coker}\ \varphi = (\mathrm{Coker}\ i(\varphi))^\sharp$.*
(2) *A sequence $F \to G \to H$ in $\mathbf{Ab}(\mathcal{C})$ is exact in G if and only if for any $U \in \mathcal{C}$ and any section $s \in G(U)$ whose image in $H(U)$ is zero, there exists $(U_i \to U)_{i\in I} \in \mathrm{Cov}(U)$ such that $s_{|U_i}$ lies in the image of $F(U_i) \to G(U_i)$.*

Proof. We first state the following lemma in categorical algebra, whose proof is left to the reader:

Lemma 9.5.2. *Let $b : \mathcal{B} \rightleftarrows \mathcal{A} : a$ be an adjoint pair of categories. Suppose that:*

(i) *$\mathcal{A}$, $\mathcal{B}$ are additive and a, b are additive functors.*
(ii) *$\mathcal{B}$ is abelian and b is left exact (i.e. commutes with finite limits).*
(iii) *$ba = \mathrm{Id}_{\mathcal{A}}$.*

Then $\mathcal{A}$ is abelian and if $\psi : A_1 \to A_2 \in \mathcal{A}$ then $\ker \psi = b(\ker(a\psi))$ and $\mathrm{Coker}\ \psi = b(\mathrm{Coker}\,(a\psi))$.

Applying this lemma to $\cdot^\sharp : \mathbf{PAb}(\mathcal{C}) \rightleftarrows \mathbf{Ab}(\mathcal{C}) : i$ we obtain that $\mathbf{Sh}(\mathcal{C})$ is abelian and the description of $\mathrm{Coker}\ \varphi$. For $\ker \varphi$: notice that the kernel is a finite limite and $\cdot^\sharp$ commutes with finite limits hence the result. This finishes the proof of (1) in Theorem 9.5.1. The assertion (2) follows immediately as $\mathrm{Im} = \ker \circ \mathrm{Coker}$. □

We state the following general result without proof:

Theorem 9.5.3. *Let $\mathcal{C}$ be a site. The Abelian category $\mathbf{Ab}(\mathcal{C}_\tau)$ has enough injectives.*

Definition 9.5.4. *Let $\mathcal{C}_\tau$ be a site. Let $X \in \mathcal{C}$ and $F \in \mathbf{Ab}(\mathcal{C}_\tau)$. One defines the cohomology groups of F on X as the right-derived functors of the functor of global sections $H^0(X, \cdot) : \mathbf{Ab}(\mathcal{C}_\tau) \to \mathbf{Ab}$:*

$$\forall\ X \in \mathcal{C}, \quad \forall F \in \mathbf{Ab}(\mathcal{C}_\tau), \quad H^p(X, F) := R^p H^0(X, \cdot)(F) = H^p(H^0(X, I^\bullet))\ ,$$

where $F \to I^\bullet$ is an injective resolution in $\mathbf{Ab}(\mathcal{C}_\tau)$.

9.6. **Functoriality.** Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. It induces canonically a functor:

$$\begin{array}{cccc} u^p : & \mathbf{PSh}(\mathcal{D}) & \to & \mathbf{PSh}(\mathcal{C}) \\ & F & \mapsto & F \circ u\ . \end{array}$$

Suppose now that $\mathcal{C}$ and $\mathcal{D}$ are sites. We would like u^p to map sheaves to sheaves.

Definition 9.6.1. *A functor $u : \mathcal{C} \to \mathcal{D}$ between two sites is continuous if it preserves coverings and fiber products: for all $(V_i \to V)_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ then*

(1) *$(u(V_i) \to u(V))_{i\in I} \in \mathrm{Cov}(\mathcal{D})$.*
(2) *$\forall\ T \to V \in \mathcal{C}$ then $u(T \times_V V_i) \simeq u(T) \times_{u(V)} u(V_i)$.*

This definition is tailored so that one obtains the:

Lemma 9.6.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites. If $F \in \mathbf{Sh}(\mathcal{D})$ then $u^p F \in \mathbf{Sh}(\mathcal{C})$.*

Definition 9.6.3. *We denote by $u^s : \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(C)$ the functor deduced from u^p.*

On the other hand the functor u^p always has a left adjoint

$$u_p : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$$

defined as

$$(u_p F)(V) = \mathrm{colim}_{\mathcal{I}_V^{\mathrm{op}}} F_V \quad ,$$

where:

- $\mathcal{I}_V$ is the category whose objects are pairs (U, φ), $U \in \mathcal{C}$, $\varphi : V \to u(U)$ and the morphisms between such pairs are the obvious ones.
- $F_V : \mathcal{I}_V^{\mathrm{op}} \to \mathbf{Sets}$ is the functor associating $F(U)$ to an object (U, φ) of $\mathcal{I}_V^{\mathrm{op}}$.

Lemma 9.6.4. *The functor*

$$\begin{array}{cccc} u_s : & \mathbf{Sh}(\mathcal{C}) & \to & \mathbf{Sh}(\mathcal{D}) \\ & G & \mapsto & (u_p G)^\sharp \end{array}$$

is left adjoint to u^s.

Definition 9.6.5. *A morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is a continuous functor $u : \mathcal{C} \to \mathcal{D}$ (notice the inverse direction!) such that $u_s : \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ is left exact (hence exact as it has a right adjoint). We write*

$$f^{-1} := u_s : \ \mathbf{Sh}(\mathcal{C}) \rightleftarrows \mathbf{Sh}(D) \ : u^s =: f_* \quad .$$

9.6.1. *Digression on Topoi.*

Definition 9.6.6. *A topos is a category $\mathbf{Sh}(\mathcal{C})$ for some site $\mathcal{C}$. A morphism of topoi from $\mathbf{Sh}(\mathcal{D})$ to $\mathbf{Sh}(\mathcal{C})$ is an adjoint pair*

$$f^{-1} : \ \mathbf{Sh}(\mathcal{C}) \rightleftarrows \mathbf{Sh}(\mathcal{D}) \ : f_* \quad .$$

such that f^{-1} is left exact (hence exact).

Example 9.6.7. $\mathcal{C} = \{pt\}$ with one object, one morphism, one covering. Then $\mathbf{Sh}(\{pt\}) = \mathbf{Sets}$.

Remark 9.6.8. If $f : \mathcal{D} \to \mathcal{C}$ is a morphism of sites then

$$f^{-1} := u_s : \ \mathbf{Sh}(\mathcal{C}) \rightleftarrows \mathbf{Sh}(\mathcal{D}) \ : u^s =: f_*$$

is a morphism of topoi.

10. Sheaves on schemes; fpqc sheaves

10.1. **Cohomology of sheaves on schemes.** Let $\tau \in \{\mathrm{Zar}, \text{ét}, \mathrm{smooth}, \mathrm{fppf}\}$ and $(\mathbf{Sch}/S)_\tau$ the corresponding site. If $X \in \mathbf{Sch}/S$ and $F \in \mathbf{Ab}((\mathbf{Sch}/S)_\tau)$ Definition 9.5.4 particularizes to define $H^\bullet(X_\tau, F)$ as the right derived functors of the functor of global sections $H^0(X, \cdot) : \mathbf{Ab}((\mathbf{Sch}/S)_\tau) \to \mathbf{Ab}$.

Let $\tau \in \{$étale, Zariski$\}$. It is obvious that if $f : X \to S$ and $g : Y \to S$ are open immersions then any S-morphism from X to Y is an open immersion. It follows from their very definition that étale morphisms satisfy a similar property:

Lemma 10.1.1. *If $f : X \to S$ and $g : Y \to S$ are étale morphisms then any S-morphism from X to Y is étale.*

Hence for $\tau \in \{$étale, Zariski$\}$ we can consider the restriction of τ to the subcategory of $\mathbf{Sch}/S$ whose objects are the étale maps $f : X \to S$, resp. the open immersions: this still defines a site, denoted S_τ and called the *small τ-site* of S.

If $X \in S_\tau$, then any $F \in \mathbf{Ab}((\mathbf{Sch}/S)_\tau$ is in particular an element of $\mathbf{Ab}(S_\tau)$. In particular, while we defined the cohomology groups $H^\bullet(X, F)$ in terms of the big site of S, an alternative definition would be to consider the derived functors of $H^0(X, \cdot) : \mathbf{Ab}(S_\tau) \to \mathbf{Ab}$. However one can show that these two definitions give canonically isomorphic groups.

10.2. **A criterion to be a sheaf on $(\mathbf{Sch}/S)_\tau$.** We have the following continuous functors of sites:

$$(\mathbf{Sch}/S)_{\mathrm{Zar}} \stackrel{\mathrm{id}}{\to} (\mathbf{Sch}/S)_{\text{ét}} \stackrel{\mathrm{id}}{\to} (\mathbf{Sch}/S)_{\mathrm{lisse}} \stackrel{\mathrm{id}}{\to} (\mathbf{Sch}/S)_{\mathrm{fppf}} \stackrel{\mathrm{id}}{\to} (\mathbf{Sch}/S)_{\mathrm{fpqc}} \ . \tag{20}$$

Hence any τ-sheaf, $\tau \in \{$ fpqc, fppf, lisse, étale, Zariski$\}$ is a Zariski sheaf. The following lemma characterizes τ-sheaves among Zariski sheaves.

Lemma 10.2.1. *Let $\tau \in \{$ fpqc, fppf, lisse, étale, Zariski$\}$ and let $\mathcal{C} = (\mathbf{Sch}/S)_\tau$, or S_τ. A presheaf F on $\mathcal{C}$ is a sheaf if and only if:*

(i) *it is a Zariski-sheaf.*
(ii) *For any $V \to U \in \mathrm{Cov}_{\mathcal{C}}(U)$, with U and V affine in $\mathcal{C}$, the sequence*

$$F(U) \longrightarrow F(V) \rightrightarrows F(V \times_U V)$$

is exact.

Proof. The fact that F is a Zariski sheaf implies that $F(\coprod U_i) = \prod F(U_i)$. Hence the sheaf condition for the covering $(U_i \to U)_{i \in I} \in \mathrm{Cov}_{\mathcal{C}}(U)$ is equivalent to the sheaf condition for the covering $\coprod_{i \in I} U_i \to U$ as

$$(\coprod U_i) \times_U (\coprod U_j) = \coprod_{i,j} U_i \times_U U_j \ .$$

This implies in particular that the sheaf condition is satisfied for coverings $(U_i \to U)_{i \in I}$ such that $|I|$ is finite and each U_i is affine for then $\coprod_{i \in I} U_i$ is affine.

Let $f : U' \to U \in \mathrm{Cov}_{\mathcal{C}}(U)$. Choose an open affine covering $U = \cup_i U_i$ and write $f^{-1}(U_i) = \cup_k U'_{ik}$ a finite open affine covering (this is possible as f is quasi-compact).

Hence $U' = \cup_{i,k} U'_{ik}$ is an open affine covering. Consider the commutative diagram:

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & F(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(U' \times_U U') \arrow[d] \\
\prod_i F(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i F(U'_{ik}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i,k,l} F(U'_{ik} \times_U U'_{il}) \\
\prod_{i,j} F(U_i \times_U U_j) \arrow[r] & \prod_{i,j,k,l} F(U'_{ik} \times_U U'_{jl}) &
\end{tikzcd}$$

The two columns on the left are exact as F is a Zariski-sheaf while the second row is exact as all the schemes considered are affine and for each i the sets of corresponding indices j and k are finite. It follows first that $F(U) \hookrightarrow F(U')$ (i.e. F is separated), hence the row on the bottom is injective and F is a sheaf by diagram chasing. □

10.3. fpqc sheaves and faithfully flat descent. Although our main object of interest are étale sheaves, we start by studying a few fpqc sheaves as any such sheaf is in particular an étale sheaf by eq. (20).

Lemma 10.3.1. *Let $S \in \mathbf{Sch}$ and $F \in \mathrm{QCoh}(S)$. Then the presheaf*

$$\begin{array}{rccc} F: & \mathbf{Sch}/S & \to & \mathbf{Ab} \\ & [f : T \to S] & \mapsto & \Gamma(T, f^*F) \end{array}$$

is an fpqc sheaf, in particular an étale sheaf.

Proof. That F is a Zariski sheaf is a classical fact. Thanks to Lemma 10.2.1 we are reduced to showing that for any $A \to B$ a faithfully flat ring morphism and writing the coherent sheaf F as $\tilde{M}$ on $\operatorname{Spec} A$, the sequence of A-module

$$0 \to M \to B \otimes_A M \to B \otimes_A \otimes B \otimes_A M \tag{21}$$

is exact. This follows from the results below on faithfully flat descent. □

Grothendieck's topologies appeared originally as a residue of his theory of descent, whose goal is to define locally global objects via a glueing procedure. Let us develop a bit the problem of descent for quasi-coherent sheaves. Let $X \in \mathbf{Sch}/S$ and let $\mathcal{U} \subset h_X$ be a covering sieve for a topology τ on $(\mathbf{Sch}/S)$. A quasi-coherent module "given $\mathcal{U}$-locally" $E_{\mathcal{U}}$ is the following set of data:

(a) for all $U \in \mathcal{U}$, a module $E_U \in \mathrm{QCoh}(U)$.
(b) for all $U, V \in \mathcal{U}$ and any X-morphism $\varphi : V \to U$, an isomorphism $\rho_\varphi : E_V \overset{\sim}{\to} \varphi^* E_U$, such that
(c) for all $W \overset{\psi}{\to} V \overset{\varphi}{\to} U$ the diagram

$$\begin{tikzcd}
E_W \arrow[rr, "\rho_{\psi\circ\varphi}"] \arrow[dr, "\rho_\psi"'] & & \psi^*\varphi^* E_U \\
& \psi^* E_V \arrow[ur, "\psi^*\rho_\varphi"'] &
\end{tikzcd}$$

commutes.

Of course any $E \in \mathrm{QCoh}(X)$ defines by pull-back a quasi-coherent module $E_{\mathcal{U}}$ given $\mathcal{U}$-locally. Descent theory deals with the converse problem: does every $E_{\mathcal{U}}$ comes from some $E \in \mathrm{QCoh}(X)$? This is true by the very definition of an $\mathcal{O}_X$-module if $\tau = Zar$, but not very useful as many modules are naturally given locally for coverings defined only in finer topologies. Notice that the local $E_{\mathcal{U}}$'s naturally form a category $\mathrm{QCoh}(\mathcal{U})$. The first main result of descent theory is the following:

Theorem 10.3.2. *Let $(U_i \to X)_{i\in I}$ be an fpqc-covering family and let $\mathcal{U}$ be the sieve generated by $(U_i \to X)_{i\in I}$. Then the functor*

$$\begin{array}{rccc} \psi: & \mathrm{QCoh}(X) & \to & \mathrm{QCoh}(\mathcal{U}) \\ & E & \mapsto & E_{\mathcal{U}} \end{array}$$

is an equivalence of categories.

Proof. As in Lemma 10.2.1 one easily reduces to the case of a covering defined by a faithfully flat morphism $U \to X$, U and X both affine.

Notice that the statement is obvious if $U \to X$ admits a section. In this case $X \in \mathcal{U}$ hence for any $E_{\mathcal{U}} \in \mathrm{QCoh}(\mathcal{U})$ the module $E := E_X \in \mathrm{QCoh}(X)$ is well-defined and one easily checks that $E_{\mathcal{U}} \simeq \psi(E)$. We will reduce ourselves to this case.

Let $E_{\mathcal{U}} \in \mathrm{QCoh}(\mathcal{U})$. One easily checks that the datum of $E_{\mathcal{U}}$ is equivalent to the datum of a diagram

$$E' \rightrightarrows E'' \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} E'''$$

cartesian over

$$U \leftleftarrows U \times_X U \mathrel{\substack{\textstyle\leftarrow\\[-0.6ex]\textstyle\leftarrow\\[-0.6ex]\textstyle\leftarrow}} U \times_X U \times_X U$$

i.e. in terms of modules:

$$E' \rightrightarrows E'' \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} E'''$$

cartesian over

$$B \rightrightarrows B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} B \otimes_A B \otimes_A B$$

(by cartesian we mean that each natural map $\partial_i : E' \otimes_{B,\partial_i} (B \otimes_A B) \to E''$ is an isomorphism and similarly for the other maps). In this language the functor ψ can be described as

$$\begin{array}{rccl} \psi: & \mathrm{Mod}(A) & \to & \mathrm{QCoh}(\mathcal{U}) \\ & E & \mapsto & \Big(M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} M \otimes_A B \otimes_A B \otimes_A B \Big). \end{array}$$

It admits a natural right-adjoint functor, which associates to $\left(E' \rightrightarrows E'' \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} E'''\right)$ the A-module $\ker(E' \rightrightarrows E'')$.

We are thus reduced to prove that the two adjunction arrows

$$E \to \ker(\, E \otimes_A B \rightrightarrows E \otimes_A B \otimes_A B \,) \tag{22}$$

and

$$\ker(\, E' \rightrightarrows E'' \,) \otimes_A B \to E' \tag{23}$$

are isomorphisms.

Remark 10.3.3. Notice that eq. (22) being an isomorphism is equivalent to our original claim that the sequence eq. (21) is exact.

It is enough to prove the result after a faithfully flat base change $A \to A'$, as faithfully flat maps preserves exact sequences. Taking $A' = B$ the structure map $A \to B$ becomes $B \to B \otimes_A B$ mapping b to $b \otimes 1$, which admits a section $b \otimes b' \mapsto bb'$. Hence we are done thanks to the previous case.

$\square$

Remark 10.3.4. More generally given $A \to B$ a ring morphism one can consider the complex

$$(B/A)^\bullet : \; B \rightrightarrows B \otimes_A B \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; B \otimes_A B \otimes_A B \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots$$

Lemma 10.3.5. *If $A \to B$ is faithfully flat then for any A-module M the complex $(B/A)^\bullet \otimes_A M$ is acyclic and $H^0((B/A)^\bullet \otimes_A M) = M$.*

10.4. **The fpqc sheaf defined by a scheme.** Another kind of fpqc sheaf is provided by the following:

Lemma 10.4.1. *Let $X \in \mathbf{Sch}/S$. Then $h_X \in \mathbf{Sh}((\mathbf{Sch}/S)_{\mathrm{fpqc}})$ (hence also $h_X \in \mathbf{Sh}((\mathbf{Sch}/S)_{\text{ét}})$.*

Proof. Clearly h_X is a Zariski sheaf. We have to show that if $A \to B$ is faithfully flat then

$$X(A) \longrightarrow X(B) \rightrightarrows X(B \otimes_A B)$$

is exact. One easily reduces to the case $X = \operatorname{Spec} C$ is affine, in which case we have to show that

$$\mathrm{Hom}_{A-alg}(C, A) \longrightarrow \mathrm{Hom}_{A-alg}(C, B) \rightrightarrows \mathrm{Hom}_A(C, B \otimes_A B)$$

is exact. This follows immediately from eq. (21). $\square$

Remark 10.4.2. One can show that on any category there exists a finest topology such that all representable presheaves are sheaves: the canonical topology. Hence the fpqc topology (and *a fortiori* the étale topology) is coarser than the canonical topology: one says it is subcanonical.

Remark 10.4.3. Let $S = \operatorname{Spec} k$ and $F \in \mathbf{Sh}(S_{\text{ét}})$. Let $E := \operatorname{colim} F(K_i)$, where K_i/k is a finite separable extension. The set E has a continuous $G := \mathrm{Gal}(k^s/k)$ action, hence can be written $E = \coprod E_i$ where E_i is finite, $E_i = G/H_i$ with H_i an open subgroup of G. Then F is represented by $\coprod U_i$ where $U_i = \operatorname{Spec} K_i$, $K_i := (k^s)^{H_i}$. Hence F is an ind-object in $(\operatorname{Spec} k)_{\text{ét}}$.

Remark 10.4.4. On $\mathcal{C} = (\mathbf{Sch}/S)_\tau$ or S_τ, the sheaf $(\mathcal{O}_S)_\tau$ associated to the quasi-coherent sheaf $\mathcal{O}_S$ coincide with the sheaf $\mathbb{G}_{a,S}$ hence is representable. The presheaf $\mathcal{O}_S^*$ of $\mathcal{O}_S$ is easily seen to be an fpqc subsheaf which coincide with $\mathbb{G}_{m,S}$.

10.4.1. *Roots of unity.* Let n be a positive integer. Define the fpqc sheaf

$$\mu_{n,S} := \ker(\mathbb{G}_{m,S} \stackrel{(\cdot)^n}{\to} \mathbb{G}_{m,S}) \;.$$

Proposition 10.4.5. *If n is invertible on S then the sequence of $\mathbf{Ab}(S_{\text{ét}})$*

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \stackrel{(\cdot)^n}{\to} \mathbb{G}_{m,S} \to 0$$

is exact.

Remark 10.4.6. This is not true for the Zariski topology! Usually an element of $\Gamma(U, \mathcal{O}_U^*)$ is not Zariski-locally a n-th power.

Proof. Let $U \in S_{\text{ét}}$ and $a \in \mathbb{G}(U) = \Gamma(U, \mathcal{O}_U^*)$. The integer n is invertible on S hence on U, thus $T^n - a$ is separable over $\mathcal{O}_U$. By the Jacobian criterion this implies that $U' := \operatorname{Spec} \mathcal{O}_U[T]/(T^n - a)$ is étale over U. As $U' \to U$ is surjective this is an étale covering family. As a admits an n-th root on U' we conclude. $\square$

10.5. **Constant sheaf.** Let C be an abelian group. The Zariski sheafification of the constant presheaf C on S_{Zar} is the sheaf

$$C_S : U \mapsto C^{\pi_0(U)} \quad .$$

As it is representable by the group scheme $S \times C$ this is also an fpqc-sheaf (hence an étale sheaf).

We will be especially interested in $(\mathbb{Z}/n\mathbb{Z})_S$.

11. Etale sheaves

We now turn to a more detailed study of étale sheaves.

11.1. Neighborhoods and stalks.

Definition 11.1.1. *Let X be a scheme and x a point of X.*

(i) *An étale neighborhood of (X, x) is an étale morphism $(U, u) \to (X, x)$.*

(ii) *If $\overline{x} : \operatorname{Spec} k^s \to X$ is a geometric point of X of image x, an étale neighborhood of $\overline{x}$ is a commutative diagram*

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \operatorname{Spec} k^s \arrow[ur, "\overline{u}"] \arrow[r, "\overline{x}"'] & X, \end{tikzcd}$$

where $\varphi : (U, u) \to (X, x)$ is an étale neighborhood of (X, x). One writes $(U, \overline{u}) \to (X, \overline{x})$.

(iii) *Morphisms of étale neigborhoods are defined in an obvious way.*

Definition 11.1.2. *Let $F \in \mathbf{Sh}(X_{\text{ét}})$. The fiber of F at $\overline{x}$ is the set*

$$F_{\overline{x}} := \operatorname{colim}_{(U, \overline{u})} F(U) \quad ,$$

where the colimit is taken over the cofiltered category of étale neighborhoods of $(X, \overline{x})$.

Proposition 11.1.3. *Let X be a scheme.*

(i) *A morphism $f : F \to G \in \mathbf{Sh}(X_{\text{ét}})$ is a monomorphism (resp. an epimorphism) if and only if for any geometric point $\overline{x} \to X$ the morphism $f_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is a monomorphism (resp. an epimorphism).*

(ii) *A sequence*

$$0 \to F \to G \to H \to 0 \in \mathbf{Ab}(X_{\text{ét}})$$

is exact if and only if for any geometric point $\overline{x}$ of X the sequence of abelian groups

$$0 \to F_{\overline{x}} \to G_{\overline{x}} \to H_{\overline{x}} \to 0$$

is exact.

Proof. Let us prove the abelian case. First the surjectivity.

Suppose that $F \twoheadrightarrow G$. Consider the exact sequence defining the cokernel Λ:

$$F_{\overline{x}} \to G_{\overline{x}} \to \Lambda \to 0 \ .$$

Let us define $\Lambda^{\overline{x}} \in \mathbf{Sh}(X_{\text{ét}})$ by

$$\Lambda^{\overline{x}}(U) := \bigoplus_{\mathrm{Hom}_X(\overline{x}, U)} \Lambda \ ,$$

it obviously satisfies the adjunction

$$\mathrm{Hom}_{\mathbf{Sh}(X_{\text{ét}})}(F, \Lambda^{\overline{x}}) = \mathrm{Hom}_{\mathbf{Ab}}(F_{\overline{x}}, \Lambda) \ .$$

If x is closed in X this is the skyscraper sheaf at $\overline{x}$ with value Λ. The morphism $G_{\overline{x}} \to \Lambda$ defines a morphism of sheaves $G \to \Lambda^{\overline{x}}$. The composite

$$F \to G \to \Lambda^{\overline{x}}$$

is zero as it corresponds to the composite $F_{\overline{x}} \to \Lambda$. If $\Lambda \neq 0$ this contradicts the assumption $F \twoheadrightarrow G$.

Conversely, suppose that $F_{\overline{x}} \to G_{\overline{x}}$ is surjective for all $\overline{x}$. Let $U \to X \in X_{\text{ét}}$ and $\overline{u} \to U$ a geometric point with image $\overline{x} \to X$. Clearly $F_{\overline{u}} \simeq F_{\overline{x}}$ hence we can assume that $U = X$. Let $s \in G(X)$. Fix $\overline{x} \to X$ a geometric point. As $F_{\overline{x}} \twoheadrightarrow G_{\overline{x}}$ there exists an étale neighborhood $(V, \overline{v}) \to (X, \overline{x})$ such that $s_{|V} \in \mathrm{Im}\ (F(V) \to G(V)$. Arguing this way for sufficiently many $\overline{x}$ one can cover X by the union of the V's. Hence the result by Theorem 9.5.1(2).

For the injectivity: a colimit of exact sequences is exact hence

$$0 \to F(U) \to G(U)$$

implies

$$0 \to F_{\overline{x}} \to G_{\overline{x}} \ .$$

□

11.2. Strict localisation.

Definition 11.2.1. *The strict localization of X at $\overline{x}$ is the ring $\mathcal{O}_{X,\overline{x}}$.*

As any Zariski neigborhood of x is also an étale neighborhood of x one obtains a morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\overline{x}}$. Similarly for any étale neighborhood $(U, \overline{u}) \to (X, \overline{x})$ one obtains a ring morphism $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ and clearly $\mathcal{O}_{X,\overline{x}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{O}_{U,u}$.

Lemma 11.2.2. *The ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselianisation $\mathcal{O}^{sh}_{X,x}$ of $\mathcal{O}_{X,x}$.*

Let us recall a few facts on henselian rings.

Definition 11.2.3. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring.*

(i) *The ring R is said to be henselian if for any monic $f \in R[t]$ and $a_o \in \kappa$ such that $\overline{f}(a_o) = 0$ and $\overline{f'}(a_0) \neq 0$ then there exists a unique $a \in R$ with image a_0 such that $f(a_0) = 0$.*

(ii) *The ring R is said to be strictly henselian if moreover κ is separably closed.*

Lemma 11.2.4. *The following statement are equivalent:*

(1) *the ring R is henselian.*
(2) *if $f \in R[t]$ is monic and $\overline{f} = \overline{g} \cdot \overline{h}$ with $\overline{g}, \overline{h} \in \kappa[T]$ monic satisfying $\overline{g} \wedge \overline{h} = 1$, there exists unique relatively prime monic $g, h \in R[t]$ with image $\overline{g}$ and $\overline{h}$ such that $f = gh$.*
(3) *any finite extension of R is a product of local rings.*
(4) *for any étale morphism $R \to S$ and $\mathfrak{q} \in \operatorname{Spec} S$ over $\mathfrak{m}$ with $\kappa(\mathfrak{q}) = \kappa$ there exists a section $\tau : S \to R$ of $R \to S$.*

Example 11.2.5. Any complete local ring is henselian.

Lemma 11.2.6. *Let $(R, \mathfrak{m}, \kappa)$ be an henselian local ring. Then reduction mod $\mathfrak{m}$ establishes an equivalence of categories between the category of finite étale extensions $R \to S$ and the category of finite étale extensions $\kappa \to \overline{S}$.*

Definition 11.2.7. *Let R be a local ring. A local homomorphism $R \to R^h$ is called the henselianization of R if it is universal among henselian extensions:*

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S \\ & R^h \arrow[u, dotted] \end{tikzcd} .$$

Similarly for the strict henselianization.

11.3. **Direct image and inverse image, the étale case.** Let $f : Y \to X$ be a morphism of schemes. Let $u : X_{\text{ét}} \to Y_{\text{ét}}$ be the corresponding functor, in fact one easily checks this is a morphism of sites. Hence:

$$f^{-1}(:= u_s) : \ \mathbf{Sh}(X_{\text{ét}}) \rightleftarrows \mathbf{Sh}(Y_{\text{ét}}) \ : (u^s =:) f_*$$

One has a canonical morphism $(f_*F)_{\overline{x}} \to F_{\overline{y}}$ which is neither injective nor surjective in general.

11.3.1. *Direct image.*

Lemma 11.3.1. (a) *If $j : U \hookrightarrow X$ then* $(j_*F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \textit{if} \quad x \in U, \\ ? & \text{otherwise.} \end{cases}$

(b) *If $i : Z \hookrightarrow X$ then* $(i_*F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \text{if} \quad x \in Z, \\ 0 & \text{otherwise.} \end{cases}$

(c) *Let $f : Y \to X$ be a finite morphism. Then $(f_*F)_{\overline{x}} = \bigoplus_{y \mapsto x} F_{\overline{y}}^{d(y)}$ where $d(y)$ is the separable degree of the extension of residues fields $\kappa(y)/\kappa(x)$.*

Proof. For (a): by definition $(j_*F)_{\overline{x}} = \operatorname{colim}_{(V,\overline{v})}(j_*F)(V)$ where $(V, \overline{v})$ ranges through the étale neighborhoods of $(X, \overline{x})$. If $x \in U$ the image of such sufficiently small étale neighborhoods is contained in U. Thus the étale neighborhoods of $(U, \overline{x})$ are cofinal in the étale neighborhoods of $(X, \overline{x})$ hence $(j_*F)_{\overline{x}} = F_{\overline{x}}$ in this case.

For (b): If $x \notin Z$ the image in X of a sufficiently small étale neighborhood of $(X, \overline{x})$ does not meet Z hence $(i_*F)_{\overline{x}} = 0$. If $x \in Z$ it is enough to show that an étale neigborhood of $(Z, \overline{x})$ extends to an étale neighborhood of $(X, \overline{x})$. Locally $X = \operatorname{Spec} A$

and $Z = \mathrm{Spec}\,(A/a)$ with a an ideal in A. Let us write $\overline{A} = A/a$ and let $\overline{A} \to \overline{B}$ be an étale ring homomorphism. Hence one can write $\overline{B} = (\overline{A}[T]/\overline{f}(T))_{(\overline{b})}$ for some $\overline{b} \in \overline{A}[T]/\overline{f}(T)$, where $\overline{f}[T] \in \overline{A}[T]$ with $\overline{f'}[T]$ invertible in $\overline{B}$. Choose $f(T) \in A[T]$ lifting $\overline{f}$ and set $B := (A[T]/f(t))_{(b)}$. For an appropriate b lifting $\overline{b}$, the extension $A \to B$ is étale and extends $\overline{A} \to \overline{B}$.

For (c): left as an exercice. □

Corollary 11.3.2. *If $f : Y \to X$ is finite then $f_* : \mathbf{Ab}(Y_{\text{ét}}) \to \mathbf{Ab}(X_{\text{ét}})$ is exact.*

Proof. Check on stalks using Lemma 11.3.1(c). □

11.3.2. *Inverse image.* if $f : X \to Y$ then $f^p : \mathbf{PAb}(Y_{\text{ét}}) \to \mathbf{PAb}(X_{\text{ét}})$ is defined by

$$(f^p F)(U) = \mathrm{colim}_V F(V)$$

where V ranges through the commutative diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d, "\text{ét}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and $f^* = (f^p)^\sharp$.

Remark 11.3.3. If f is étale then f^* is just the usual restriction functor.

Remark 11.3.4. If $i_{\overline{x}} : \overline{x} \to X$ is a geometric point then $i_{\overline{x}}^* F = F_{\overline{x}}$ by definition.

Lemma 11.3.5. *Let $f : X \to Y$. Then $(f^*F)_{\overline{x}} = F_{\overline{f(x)}}$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd} \overline{x} \arrow[r, "i_{\overline{x}}"] \arrow[dr, "i_{\overline{f(x)}}"'] & X \arrow[d, "f"] \\ & Y . \end{tikzcd}$$

Notice that $(g \circ f) = f^* g^*$ by unicity of the left adjoint to $(g \circ f)_* = g_* f_*$. Hence:

$$(f^*F)_{\overline{x}} = i_{\overline{x}}^*(f^*F) = i_{\overline{f(x)}} F = F_{\overline{f(x)}} \quad .$$

□

Corollary 11.3.6. *The functor f^* is exact.*

Corollary 11.3.7. *$f_* : \mathbf{Ab}(Y_{\text{ét}}) \to \mathbf{Ab}(X_{\text{ét}})$ send injectives to injectives.*

Proof. This is a formal consequence of the fact that f_* admits a left adjoint functor f^* which is left exact. Indeed let I be an injective in $\mathbf{Ab}(Y_{\text{ét}})$. Completing the solid diagram

$$\begin{tikzcd} F \arrow[r, hook] \arrow[dr] & G \arrow[d, dotted] \\ & f_* I \end{tikzcd}$$

is equivalent by adjunction to completing the solid diagram (the upper row remains injective as f^* is left exact)

$$\begin{tikzcd} f^*F \arrow[r, hook] \arrow[dr] & f^*G \arrow[d, dotted] \\ & I. \end{tikzcd}$$

This follows from the injectivity of I. □

Remark 11.3.8. At this point one easily showa that $\mathbf{Ab}(X_{\text{ét}})$ has sufficiently many injectives (hence we prove in this particular case the Theorem 9.5.3 stated without proof for the category of abelian sheaves on any site). Indeed consider the monomorphism

$$F \hookrightarrow \prod_{\overline{x}\to X} i_{\overline{x}*}i_{\overline{x}}^*F \quad .$$

Choose a monomorphism $i_{\overline{x}}^*F \hookrightarrow I_{\overline{x}}$ in $\mathbf{Ab}$ with $I_{\overline{x}}$ injective in $\mathbf{Ab}$. This exists as $\mathbf{Ab}$ has sufficiently many injectives. Thus $F \hookrightarrow \prod_{\overline{x}\to X} i_{\overline{x}*}I_{\overline{x}}$ and the term on the right is an injective sheaf by the corollary above.

11.4. Extension by zero.

Lemma 11.4.1. *Let $j : U \to X$ be an étale morphism (for example an open immersion). Then $j^* : \mathbf{Ab}(X_{\text{ét}}) \to \mathbf{Ab}(U_{\text{ét}})$ has a left adjoint $j_! : \mathbf{Ab}(U_{\text{ét}}) \to \mathbf{Ab}(X_{\text{ét}})$ which is exact (in particular j^* maps injectives to injectives).*

Proof. Let $F \in \mathbf{Ab}(U_{\text{ét}})$. For $V \overset{\varphi}{\to} X$ define

$$F_!(V) = \bigoplus_{\begin{tikzcd}[row sep=small, column sep=small] V \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & U \arrow[d, "j"] \\ & X \end{tikzcd}} F(V) \quad .$$

Notice that if $j : U \overset{o}{\hookrightarrow} X$ is an open immersion then

$$F_!(V) = \begin{cases} F(V) & \text{if } \varphi(V) \subset U, \\ 0 & \text{otherwise .} \end{cases}$$

Hence $F_! \in \mathbf{PAb}(X_{\text{ét}})$ and clearly $F_!$ is left adjoint to j^p. Set $j_!F := (F_!)^\sharp$. If $G \in \mathbf{Ab}(X_{\text{ét}})$ then

$$\mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(j_!F, G) = \mathrm{Hom}_{\mathbf{PAb}(X_{\text{ét}})}(F_!, G) = \mathrm{Hom}_{\mathbf{PAb}(U_{\text{ét}})}(F, j^pG) = \mathrm{Hom}_{\mathbf{Ab}(U_{\text{ét}})}(F, j^*G) \quad .$$

This proves the existence of the left adjoint functor $j_!F$.

One easily shows from the definition that

$$(j_!F)_{\overline{x}} = \begin{cases} \bigoplus_{\substack{\overline{u}\to U \\ j(\overline{u})=\overline{x}}} F_{\overline{u}} & \text{if } x \in j(U) \ , \\ 0 & \text{otherwise .} \end{cases}$$

This implies immediately that $j_!$ is exact. □

11.5. **The fundamental triangle.** Consider the following geometric situation:

$$U := X \setminus Z \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z \quad .$$

Lemma 11.5.1. *Let $F \in \mathbf{Ab}(X_{\text{ét}})$. The following sequence of $\mathbf{Ab}(X)_{\text{ét}}$*

$$0 \to j_! j^! F \to F \to i_* i^* F \to 0$$

is exact (where we set $j^! := j^*$).

Proof. Consider the corresponding stalks. If $x \in U$ one obtains

$$0 \to F_{\overline{x}} \overset{\text{id}}{\to} F_{\overline{x}} \to 0 \to 0$$

which is obviously exact. If $x \in Z$ then

$$0 \to 0 \to F_{\overline{x}} \overset{\text{id}}{\to} F_{\overline{x}} \to 0$$

which is also exact. □

Given $F \in \mathbf{Ab}(X_{\text{ét}})$ we set

$$\begin{array}{rcl} F_U & := & j^* F \in \mathbf{Ab}(U_{\text{ét}}), \\ F_Z & := & i^* F \in \mathbf{Ab}(Z_{\text{ét}}) \ . \end{array}$$

By adjunction one obtains a canonical map $F \to j_* j^* F = j_* F_U$. Applying i^* gives:

$$F_Z \to i^* j_* F_U \quad .$$

Proposition 11.5.2. *Let us denote by $\mathcal{T}$ the category of triplets*

$$(F_Z \in \mathbf{Ab}(Z_{\text{ét}}), F_U \in \mathbf{Ab}(U_{\text{ét}}), F_Z \overset{\phi}{\to} i^* j_* F_U)$$

with the obvious morphisms. The functor

$$\begin{array}{rcl} \mathbf{Ab}(X_{\text{ét}}) & \to & \mathcal{T} \\ F & \mapsto & (F_Z, F_U, F_Z \to i^* j * F_U) \end{array}$$

is an equivalence of categories.

Proof. We construct an inverse functor as follows. Starting from $(F_Z, F_U, F_Z \overset{\phi}{\to} i^* j_* F_U)$ let us define $\tilde{F} \in \mathbf{Ab}(X_{\text{ét}})$ as the cartesian product

$$\begin{tikzcd} \tilde{F} \arrow[r, dotted] \arrow[d, dotted] & j_* F_U \arrow[d] \\ i_* F_Z \arrow[r, "i_* \varphi"'] & i_* i^* j_* F_U \ . \end{tikzcd}$$

If now $F \in \mathbf{Ab}(X_{\text{ét}})$ the natural maps $F \to j_* F_U$ and $F \to i_* F_Z$ defines a morphism $F \to \tilde{F}$. We have to check this is an isomorphism. Hence we have to show that the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & j_* F_U \arrow[d] \\ i_* F_Z \arrow[r, "i_* \varphi"'] & i_* i^* j_* F_U \end{tikzcd}$$

is cartesian. As stalks at geometric points commute with fiber product and form a conservative family we have to check that the corresponding diagrams of stalks are Cartesian. For $x \in U$ we obtain

$$\begin{tikzcd} F_{\overline{x}} \arrow[r] \arrow[d] & F_{\overline{x}} \arrow[d] \\ 0 \arrow[r] & 0. \end{tikzcd}$$

For $x \in Z$:

$$\begin{tikzcd} F_{\overline{x}} \arrow[r] \arrow[d] & (j_*j^*F)_{\overline{x}} \arrow[d] \\ F_{\overline{x}} \arrow[r] & (j_*j^*F)_{\overline{x}}. \end{tikzcd}$$

Both squares are Cartesian hence we are done. □

Definition 11.5.3. *Let $F \in \mathbf{Ab}(X_{\text{ét}})$. If Y is any subscheme of X we say that F has support contained in Y if $F_{\overline{x}} = 0$ for any $x \notin Y$.*

Corollary 11.5.4. *Let $Z \xrightarrow[i]{} X$. Then*

$$i_* : \mathbf{Ab}(X_{\text{ét}}) \to \mathbf{Ab}(X_{\text{ét}})$$

induces an equivalence of categories between $\mathbf{Ab}(Z_{\text{ét}})$ and the full subcategory of $\mathbf{Ab}(Z_{\text{ét}})$ of sheaves with support contained in Z.

Proof. Notice that $F \in \mathbf{Ab}(X_{\text{ét}})$ has support contained in Z if and only if it is of the form $(F_Z, 0, 0)$ in the description of Proposition 11.5.2. □

We summarize our results through the following diagram of adjunctions:

$$\begin{tikzcd} \mathbf{Ab}(U_{\text{ét}}) \arrow[d, shift right=8, "j_!"'] \arrow[d, shift left=8, "j_*"] \\ \mathbf{Ab}(X_{\text{ét}}) \arrow[u, "j^!=j^*"] \arrow[d, shift right=8, "i^*"'] \arrow[d, shift left=8, "i^!"] \\ \mathbf{Ab}(Z_{\text{ét}}) \arrow[u, "i_*=i_!"] \end{tikzcd}$$

satisfying the following identities

$$\mathrm{id} \xrightarrow{\sim} j^!j_! \quad ,$$
$$j^*j_* \xrightarrow{\sim} \mathrm{id} \quad ,$$
$$i^*i_* \xrightarrow{\sim} \mathrm{id} \quad ,$$
$$\mathrm{id} \xrightarrow{\sim} i^!i_! \quad ,$$
$$j^*i_* = 0 \quad \text{hence} \quad i^!j_* = i^*j_! = 0 \quad ,$$

where we defined the functor of sections with support in Z:

$$i^! : (F_Z, F_U, \varphi : F_Z \to i^*j_*F_U) \mapsto \ker\varphi \quad .$$

12. Etale cohomology

Let X be a scheme. Consider the left exact functor

$$\Gamma(X,\cdot) := \mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(\mathbb{Z}_X,\cdot) : \begin{array}{ccc} \mathbf{Ab}(X_{\text{ét}}) & \to & \mathbf{Ab} \\ F & \mapsto & \Gamma(X,F) = \mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(\mathbb{Z}_X,F) \end{array} .$$

One considers its right derived functors

$$R\Gamma(X,\cdot) = R\,\mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(\mathbb{Z}_X,\cdot) : D^+\mathbf{Ab}(X_{\text{ét}}) \to D^+\mathbf{Ab} .$$

and define

$$H^i(X_{\text{ét}},F) := R^i\,\mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(\mathbb{Z}_X,\cdot) .$$

12.1. **Cohomology with support.** Let $Z \overset{i}{\hookrightarrow} X$ and $F \in \mathbf{Ab}(X_{\text{ét}})$. Define $U := X \setminus Z$ and

$$\Gamma_Z(X,F) := \ker(\Gamma(X,F) \to \Gamma(U,F_{|U}))$$

the group of sections of F with support in Z. The functor $\Gamma_X(X,\cdot)$ is clearly left exact, hence we can define its right derived functors.

Definition 12.1.1. *We define the cohomology groups of F with support in Z as*

$$H^r_Z(X,F) := R^r\Gamma_Z(X,F) .$$

Theorem 12.1.2. *The following long sequence of abelian groups is exact:*

$$\cdots \to H^r_Z(X,F) \to H^r(X,F) \to H^r(Z,F) \to H^{r+1}_Z(X,F) \to \cdots$$

Proof. Consider the *Ext*-long exact sequence obtained by applying $\mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(\cdot,F)$ to the exact sequence of sheaves provided by Lemma 11.5.1:

$$0 \to j_!j^!\mathbb{Z}_X \to \mathbb{Z}_X \to i_*i^*\mathbb{Z}_X \to 0 .$$

Notice that

$$\mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(j_!j^!\mathbb{Z}_X,G) = \mathrm{Hom}_{\mathbf{Ab}(U_{\text{ét}})}(j^*\mathbb{Z}_X,j^*G) = G(U) ,$$

hence by considering an injective resolution $F \simeq I^\bullet$:

$$\mathrm{Ext}^r_{\mathbf{Ab}(X_{\text{ét}})}(j_!j^!\mathbb{Z}_X,F) = H^r(U_{\text{ét}},F_{|U}) .$$

Looking at the beginning of the *Ext* long exact sequence:

$$0 \to \mathrm{Hom}_{\mathbf{Ab}(X_{\text{ét}})}(i_*i^*\mathbb{Z}_X,F) \to F(X) \to F(U)$$

is exact hence the left hand term is necessarily $\Gamma_Z(X,F)$. Applying to an injective resolution of F we deduce:

$$\mathrm{Ext}^r_{\mathbf{Ab}(X_{\text{ét}})}(i_*i^*\mathbb{Z}_X,F) \simeq H^r_Z(X_{\text{ét}},F) .$$

The result follows. ∎

12.2. **Nisnevich excision.** The excision theorem for usual cohomology says that cohomology with support in Z depends only on a neighborhood of Z in X. Similarly:

Theorem 12.2.1. *Let*

$$\begin{tikzcd} & X' \arrow[d, "f"] & \\ Z \arrow[r, hook, "i"'] & X & U := X \setminus Z \arrow[l, hook', "j"] \end{tikzcd}$$

with f an étale map such that $f_{|f^{-1}(Z)^{\mathrm{red}}} : f^{-1}(Z)^{\mathrm{red}} \simeq Z$ (such an étale covering of X is called an elementary Nisnevich covering). Then:

$$H^r_Z(X_{\text{ét}}, F) \simeq H^r_Z(X'_{\text{ét}}, f^*F) \ .$$

Proof. We proved that f^* is exact. Moreover as f is étale f^* preserve injectives by Lemma 11.4.1. Hence it is enough to prove the result for $r = 0$.

Consider the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_{Z'}(X', f^*F) \arrow[r] & \Gamma(X', f^*F) \arrow[r] & \Gamma(U', f^*F) \\ 0 \arrow[r] & \Gamma_Z(X, F) \arrow[r] \arrow[u, "\varphi"] & \Gamma(X, F) \arrow[r] \arrow[u] & \Gamma(U, F) \ . \arrow[u] \end{tikzcd}$$

We have to show that φ is an isomorphism.

For the injectivity: suppose that $s \in \Gamma_X(X, F)$ is mapped to zero in $\Gamma_{Z'}(X', f^*F)$. Hence s, seen as an element of $\Gamma(X, F)$, maps to zero in $\Gamma(U, F)$ and $\Gamma(X', f^*F)$. But $(X' \xrightarrow{f} X, U \to X)$ is an étale covering of X and F is a sheaf hence $s = 0$.

For the surjectivity: let $s' \in \Gamma_{Z'}(X', f^*F)$. One easily checks that the pair $(s, o) \in \Gamma(X', f^*F) \times \Gamma(U, F)$ maps to zero on intersections hence comes from $s \in \Gamma(X, F)$ as F is an étale sheaf. ☐

13. Čech cohomology and étale cohomology of quasi-coherent sheaves

The goal of this section is to prove:

Theorem 13.0.1. *Let S be a scheme and $F \in \mathrm{QCoh}(S)$. Then*

$$H^p(S_{\mathrm{Zar}}, F) = H^p(S_{\text{ét}}, F) = H^p(S_{\mathrm{fpqc}}, F) \ .$$

The basic tool will be Čech cohomology, a cohomology theory for presheaves.

13.1. Čech cohomology for coverings.

Definition 13.1.1. *Let $\mathcal{C}$ be a category and $\mathcal{U} = (U_i \to U)_{i \in I}$ any family of morphism to $U \in \mathcal{C}$. Let $F \in \mathbf{PAb}(\mathcal{C})$.*

The Čech complex of F with respect to $\mathcal{U}$ is the complex $\check{C}^\bullet(\mathcal{U}, F) \in D^+\mathbf{Ab}$:

$$\check{C}^\bullet(\mathcal{U}, F) : \prod_{i_0} F(U_{i_0}) \to \prod_{i_0, i_1} F(U_{i_0, i_1}) \to \prod_{i_0, i_1, i_2} F(U_{i_0, i_1, i_2}) \to \cdots$$

where $U_{i_0, \dots, i_n} := U_{i_0} \times_U \cdots \times_U U_{i_n}$.

The Čech cohomology of F on U is $\check{H}^p(\mathcal{U}, F) := H^p(\check{C}^\bullet(\mathcal{U}, F))$.

Proposition 13.1.2. *$\check{H}^\bullet(\mathcal{U},\cdot) : \mathbf{PAb}(\mathcal{C}) \to \mathbf{Ab}$ is a universal δ-functor.*

Proof. We first show that $\check{H}^\bullet : (\mathcal{U},\cdot)$: is a δ-functor. Given an exact sequence of presheaves

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

one immediately obtains that the sequence of complexes

$$0 \to \check{C}^\bullet(\mathcal{U}, F_1) \to \check{C}^\bullet(\mathcal{U}, F_2) \to \check{C}^\bullet(\mathcal{U}, F_3) \to 0$$

is exact, hence we obtain the required long exact sequence

$$\cdots \to \check{H}^r(\mathcal{U}, F_1) \to \check{H}^r(\mathcal{U}, F_2) \to \check{H}^r(\mathcal{U}, F_3) \to \check{H}^{r+1}(\mathcal{U}, F_1) \to \cdots \quad .$$

Recall the universality means that given any other δ-functor $T^\bullet : \mathbf{PAb}(\mathcal{C}) \to \mathbf{Ab}$ and any morphism $\check{H}^0(\mathcal{U},\cdot) \to T^0$, there exist compatible morphisms $\check{H}^\bullet(\mathcal{U},\cdot) \to T^\bullet$. We have to show that for any $i > 0$ the functor $\check{H}^i(\mathcal{U},\cdot)$ is effaceable i.e. for any $F \in \mathbf{PAb}(\mathcal{C})$ there exists a monomorphism $F \hookrightarrow I$ with $\check{H}^i(\mathcal{U}, I) = 0$.

Given $V \in \mathcal{C}$ we denote by $\mathbb{Z}_V \in \mathbf{PAb}(\mathcal{C})$ the presheaf defined by $\mathbb{Z}_V(W) = \mathbb{Z}[\mathrm{Hom}_\mathcal{C}(W, V)]$. In other words $\mathbb{Z}_\bullet$ is the left adjoint functor to the inclusion $\mathbf{PAb}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{C})$ and $\mathbb{Z}_V := \mathbb{Z}_{h_V}$. Notice that:

$$\begin{aligned}
\check{C}^\bullet(\mathcal{U}, F) &= \left(\prod_{i_0} \mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}(\mathbb{Z}_{U_{i_0}}, F) \to \prod_{i_0,i_1} \mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}(\mathbb{Z}_{U_{i_0,i_1}}, F) \to \cdots\right) \\
&= \mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}\left(\left(\bigoplus_{i_0} \mathbb{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0,i_1} \mathbb{Z}_{U_{i_0,i_1}} \leftarrow \bigoplus_{i_0,i_1,i_2} \mathbb{Z}_{U_{i_0,i_1,i_2}} \leftarrow \cdots\right), F\right) \\
&= \mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}\left(\left(\mathbb{Z}_{\coprod_{i_0} U_{i_0}} \leftarrow \mathbb{Z}_{\coprod_{i_0,i_1} U_{i_0,i_1}} \leftarrow \mathbb{Z}_{\coprod_{i_0,i_1,i_2} U_{i_0,i_1,i_2}} \leftarrow \cdots\right), F\right) \quad .
\end{aligned}$$

Lemma 13.1.3. *The complex of $\mathbf{PAb}(\mathcal{C})$*

$$\mathbb{Z}^\bullet_\mathcal{U} : \left(\mathbb{Z}_{\coprod_{i_0} U_{i_0}} \leftarrow \mathbb{Z}_{\coprod_{i_0,i_1} U_{i_0,i_1}} \leftarrow \mathbb{Z}_{\coprod_{i_0,i_1,i_2} U_{i_0,i_1,i_2}} \leftarrow \cdots\right)$$

is exact in positive degrees.

Proof. Let $V \in \mathcal{C}$. Then

$$\begin{aligned}
\mathbb{Z}^\bullet_\mathcal{U}(V) &= \left(\mathbb{Z}\left[\coprod_{i_0} \mathrm{Hom}_\mathcal{C}(V, U_{i_0})\right] \leftarrow \mathbb{Z}\left[\coprod_{i_0,i_1} \mathrm{Hom}_\mathcal{C}(V, U_{i_0,i_1})\right] \leftarrow \cdots\right) \\
&= \bigoplus_{\varphi: V \to U} \left(\mathbb{Z}\left[\coprod_{i_0} \mathrm{Hom}_\varphi(V, U_{i_0})\right] \leftarrow \mathbb{Z}\left[\coprod_{i_0,i_1} \mathrm{Hom}_\varphi(V, U_{i_0}) \times \mathrm{Hom}_\varphi(V, U_{i_1})\right] \leftarrow \cdots\right)
\end{aligned}$$

where $\mathrm{Hom}_\varphi(V, U_i) = \left\{ \begin{tikzcd} V \arrow[r] \arrow[dr, "\varphi"'] & U_i \arrow[d] \\ & U \end{tikzcd} \right\}$. Set $S_\varphi := \coprod_i \mathrm{Hom}_\varphi(V, U_i)$. Thus

$$\mathbb{Z}^\bullet_\mathcal{U}(V) = \bigoplus_{\varphi: V \to U} (\mathbb{Z}[S_\varphi] \leftarrow \mathbb{Z}[S_\varphi \times S_\varphi] \leftarrow \mathbb{Z}[S_\varphi \times S_\varphi \times S_\varphi] \leftarrow \cdots) \quad .$$

Hence it is enough to show that for any set E the complex of abelian groups

$$\mathbb{Z}[E] \leftarrow \mathbb{Z}[E \times E] \leftarrow \mathbb{Z}[E \times E \times E] \leftarrow \cdots$$

is exact in positive degrees. This follows immediately from the contractibility of the simplicial set $\Delta^\bullet$. □

Lemma 13.1.4. *If $I \in \mathbf{PAb}(\mathcal{C})$ is injective then $\check{H}^p(\mathcal{U}, I) = 0$ for any $p > 0$.*

Proof. We showed that $\check{H}^p(\mathcal{U}, I) = H^p(\mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}(\mathbb{Z}^\bullet_{\mathcal{U}}, I))$. As $\mathbb{Z}^\bullet_{\mathcal{U}}$ is exact in positive degree by the previous lemma and $\mathrm{Hom}_{\mathbf{PAb}(\mathcal{C})}(\cdot, I)$ is exact as I is injective, the result follows. □

This finishes the proof of Proposition 13.1.2 □

Theorem 13.1.5. *$\check{H}^p(\mathcal{U}, \cdot) = R^p\check{H}^0(\mathcal{U}, \cdot)$ in $\mathbf{PAb}(\mathcal{C})$.*

Proof. Both functors are universal δ-functors and coincide in degree zero. □

Remark 13.1.6. Up to now we did not use the topology on $\mathcal{C}$.

13.2. **Čech to cohomology spectral sequence.**

Theorem 13.2.1. *Let $\mathcal{C}$ be a site. Let $U \in \mathcal{C}$, $\mathcal{U} \in \mathbf{Cov}(U)$ and $F \in \mathbf{Ab}(\mathcal{C})$. There is a natural spectral sequence, called the Čech to cohomology spectral sequence:*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U, F) \quad ,$$

where $\mathcal{H}^q(F) : U \mapsto H^q(U, F) \in \mathbf{PAb}(\mathcal{C})$.

Proof. Recall the following:

Theorem 13.2.2. *(Grothendieck's spectral sequence for composition of functors) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories. Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors and assume that FI is G-acyclic for any injective $I \in \mathcal{A}$. There is a canonical spectral sequence*

$$E_2^{p,q} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(G \circ F)(A) \quad .$$

We apply this result to

$$\begin{tikzcd} \mathbf{Ab}(\mathcal{C}) \arrow[r, "i"] \arrow[rr, bend right, "H^0"'] & \mathbf{PAb}(\mathcal{C}) \arrow[r, "\check{H}^0"] & \mathbf{Ab} \end{tikzcd} \quad ,$$

noticing that $i : \mathbf{Ab}(\mathcal{C}) \to \mathbf{PAb}(\mathcal{C})$ maps injectives to injectives (indeed the functor i admits as left adjoint functor the sheafification functor $\cdot^\sharp$ which is left exact) and that $(R^qiF)(V) = \mathcal{H}^q(F)(V)$ by definition. □

Lemma 13.2.3. *(locality of cohomology) Let $\mathcal{C}$ be a site and $F \in \mathbf{Ab}(\mathcal{C})$. Let $U \in \mathcal{C}$ and $\xi \in H^p(U, F)$ for some $p > 0$. There exists a covering family $(U_i \to U)_{i \in I}$ of U such that $\xi_{|U_i} = 0$ for any $i \in I$.*

Proof. Choose an injective resolution $F \simeq I^\bullet$ in $\mathbf{Ab}(\mathcal{C})$ and $\tilde{\xi} \in I^p(U)$ lifting ξ. In particular $d^p\tilde{\xi} = 0$. As the sequence $I^{p-1} \xrightarrow{d^{p-1}} I^p \xrightarrow{d^p} I^{p+1}$ is exact there exists a covering family $(U_i \to U)_{i \in I}$ of U and element $\xi_i \in I^{p-1}(U_i)$ such that $\tilde{\xi}_{|U_i} = d^{p-1}\xi_i$. Hence $\xi_{|U_i} = 0$. □

13.3. **Proof of Theorem 13.0.1.** We only sketch the proof.

The result for $p=0$ is equivalent to the fact that F is an fpqc sheaf.

For $p>0$, the main step consists in proving the result for S affine: we want to show in this case that $H^p(S_{\text{fpqc}}, F) = 0$ for any $p>0$. The proof is by induction on p.

For $p=1$: let $\xi \in H^1_{\text{fpqc}}(S,F)$. By Lemma 13.2.3 there exists an fpqc-covering family $(U_i \to S)_{i\in I}$ such that $\xi_{|U_i} = 0$ for any $i \in I$. Without loss of generality we can assume that each U_i is affine (in particular $H^r(U_i, F) = 0$ for any $r>0$) and I is finite. Let $\mathcal{U} = (V := \coprod_i U_i \to S)$. Hence ξ comes, via the Čech to Cohomology spectral sequence, of a class $\check{\xi} \in \check{H}^1(\mathcal{U}, F)$. Write $S = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, $F = \tilde{M}$. One easily checks that

$$\check{C}^\bullet(\mathcal{U}, F) = (B/A)^\bullet \otimes_A M \quad ,$$

hence $\check{H}^1(\mathcal{U}, F) = 0$ by faithfully flat descent. Hence $\check{\xi} = 0$ and $\xi = 0$.

For $p>1$: Notice that each $U_{i_0,\dots,i_n}$ is affine hence $E_2^{i,j} = \check{H}^i(\mathcal{U}, \mathcal{H}^j(F)) = 0$ for $0<j<p$ by induction hypothesis and the same argument provides the induction.

□

13.4. **Other applications of Čech cohomology.**

13.4.1. *Čech cohomology at the colimit.* We continue with the notations of Theorem 13.2.1. Let $\mathcal{V} = (V_j \to U)_{j\in J}$ be a refinement of $\mathcal{U} = (U_i \to U)_{i\in I}$, meaning that there exists a map $\tau : J \to I$ such that for every $j \in J$ one has a factorization

$$\begin{tikzcd} V_j \arrow[r, dashed] \arrow[dr] & U_{\tau(j)} \arrow[d] \\ & U. \end{tikzcd}$$

This gives rise to a canonical restriction map:

$$\rho_{\mathcal{V},\mathcal{U}} : \check{H}^\bullet(\mathcal{U}, F) \to \check{H}^\bullet(\mathcal{V}, F) \quad ,$$

and one defines:

$$\check{H}^\bullet(U, F) = \operatorname{colim}_{\mathcal{U}} \check{H}^\bullet(\mathcal{U}, F)$$

where $\mathcal{U}$ ranges through all coverings of U. Taking the colimit of the Čech to Cohomology spectral sequences for $\mathcal{U}$ leads to the spectral sequence:

$$E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, F) \quad . \tag{24}$$

In this language the locality of cohomology (Lemma 13.2.3) can be rewritten as:

$$\check{H}^0(U, \mathcal{H}^q(F)) = 0 \ \forall\, q>0 \quad .$$

In particular the spectral sequence eq. (24) looks like:

$$E_2^{p,q} = \begin{array}{cccc} 0 & \star & \star & \cdots \\ 0 & \star & \star & \cdots \\ 0 & \star & \star & \cdots \\ \star & \star & \star & \cdots \end{array}$$

hence

$$\check{H}^0(U,F) \simeq H^0(U,F) \ ,$$
$$\check{H}^1(U,F) \simeq H^1(U,F) \ ,$$

and the following sequence is exact:

$$0 \to \check{H}^2(U,F) \to H^2(U,F) \to \check{H}^1(U,\mathcal{H}^1(F)) \to \check{H}^3(U,F) \to H^3(U,F) \ .$$

13.4.2. *Mayer-Vietoris exact sequence in étale cohomology.*

Lemma 13.4.1. *(Mayer-Vietoris) Let $U = U_0 \cup U_1$ be a Zariski-open decomposition of U. Then for any $F \in \mathbf{Ab}(U)$ the following long exact sequence holds:*

$$\ldots \to H^s(U_{\text{ét}},F) \to H^s((U_0)_{\text{ét}},F) \oplus H^s((U_1)_{\text{ét}},F) \to H^s((U_0 \cap U_1)_{\text{ét}},F) \to H^{s+1}(U_{\text{ét}},F) \to \ldots$$

Proof. Consider the subcomplex $\check{C}^\bullet_{\text{alt}}(\mathcal{U},F) \subset \check{C}^\bullet(\mathcal{U},F)$ of alternate cochains:

$$c(i_0,\ldots,i_n) = 0 \text{ if } i_j = i_k \text{ for some } j<k.$$
$$c(i_{\sigma(0)},\ldots,i_{\sigma(n)}) = \varepsilon(\sigma)c(i_0,\ldots,i_n) \ .$$

If $\mathcal{U}$ is a Zariski covering one can show that $\check{C}^\bullet_{\text{alt}}(\mathcal{U},F) \subset \check{C}^\bullet(\mathcal{U},F)$ is a quasi-isomorphism (this is completely wrong in general!). For the covering $\mathcal{U} = (U_0 \to U, U_1 \to U)$ this implies that $\check{H}^s(\mathcal{U},\cdot) = 0$ for any $s \geq 2$. The Čech to Cohomology spectral sequence degenerates immediately and gives rise to the Mayer-Vietoris long exact sequence. ☐

13.5. **Flasque sheaves.**

Definition 13.5.1. *A sheaf $F \in \mathbf{Ab}(X_{\text{ét}})$ is said to be flasque if*

$$\mathcal{H}^q(F) = 0 \qquad \forall\, q > 0 \ .$$

Theorem 13.5.2 (Verdier)**.** *The following conditions are equivalent:*

(1) *the sheaf F is flasque.*
(2) *for any $U \in X_{\text{ét}}$, for any étale covering $\mathcal{U}$ of U, $\check{H}^q(\mathcal{U},F) = 0$ for all $q>0$.*
(3) *for any $U \in X_{\text{ét}}$, $\check{H}^q(U,F) = 0$ for all $q > 0$.*

Proof. (1) $\Rightarrow$ (2): consider the Čech to Cohomology spectral sequence:

$$E_2^{p,q} = \check{H}^p(\mathcal{U},\mathcal{H}^q(F)) \Rightarrow H^{p+q}(U_{\text{ét}},F) \ .$$

By assumption only the row $q=0$ is non-zero. Hence

$$\check{H}^p(\mathcal{U},F) = E_2^{p,0} \simeq E_\infty^{p,0} \simeq H^p(U_{\text{ét}},F) = 0 \quad \text{for } p>0 \ .$$

(2) $\Rightarrow$ (3): take the colimit over all $\mathcal{U}$'s.
(3) $\Rightarrow$ (1): consider the Čech to Cohomology spectral sequence:

$$E_2^{p,q} = \check{H}^p(U,\mathcal{H}^q(F)) \Rightarrow H^{p+q}(U_{\text{ét}},F) \ .$$

Hence

$$E_2^{p,q} = \begin{array}{cccc} 0 & \star & \star & \cdots \\ 0 & \star & \star & \cdots \\ 0 & \star & \star & \cdots \\ \star & 0 & 0 & \cdots \end{array}$$

Let us prove by induction on $n > 0$ that $H^n(U_{\text{ét}}, F) = 0$ for any $U \in X_{\text{ét}}$.

For $n = 1$: this is OK as $E_2^{0,1} = E_2^{1,0} = 0$.

Suppose by induction that $\mathcal{H}^i(F) = 0$ for $i \leq n$. Then $E_2^{p,q} = 0$ for any $p + q \leq n + 1$ and the result. □

Corollary 13.5.3. *Let $f : X \to Y$. If $F \in \mathbf{Ab}(X_{\text{ét}})$ is flasque then $f_*F \in \mathbf{Ab}(Y_{\text{ét}})$ is flasque.*

Proof. By the previous proposition f_*F is flasque if and only if for any covering $\mathcal{U}$ of $U \in Y_{\text{ét}}$ the Čech cohomology $\check{H}^q(\mathcal{U}, f_*F)$ vanishes for $q > 0$. But $\check{H}^q(\mathcal{U}, f_*F) = \check{H}^q(f^{-1}(\mathcal{U}), F) = 0$ as F is flasque. □

13.5.1. *Godement flasque resolution.* Let $F \in \mathbf{Ab}(X_{\text{ét}})$. We define

$$\mathrm{God}^0(F) = \prod_{\overline{x}} i_{\overline{x}*} i_{\overline{x}}^* F \quad .$$

Notice that any sheaf on $\overline{x}$ is obviously flasque, hence $i_{\overline{x}}^* F$ is flasque. It follows from Corollary 13.5.3 that $\mathrm{God}^0(F)$ is flasque. Define $\mathrm{God}^1(F) = \mathrm{God}^0(\mathrm{Coker}\,(F \hookrightarrow \mathrm{God}^0(F)))$ and by induction:

$$\mathrm{God}^{i+1}(F) = \mathrm{God}^0(\mathrm{Coker}\,(\mathrm{God}^{i-1}(F) \to \mathrm{God}^i(F))) \quad .$$

We thus obtain a canonical flasque resolution: $F \simeq \mathrm{God}^\bullet(F)$. For any $f : X \to Y$ it satisfies:

$$f^*(\mathrm{God}^\bullet(F)) = \mathrm{God}(f^*F) \quad .$$

13.5.2. *Flasque implies flabby.*

Corollary 13.5.4. *Let $V \subset U$ be an open immersion in $X_{\text{ét}}$ and let $F \in \mathbf{Ab}(X_{\text{ét}})$ be flasque. Then $F(U) \twoheadrightarrow F(V)$.*

Proof. Set $W = U \coprod_V U$. Denote by U_0, U_1 the two copies of U covering W. Considering the Mayer-Vietoris exact sequence (Lemma 13.4.1) we obtain:

$$0 \to F(W) \to F(U) \oplus F(U) \xrightarrow{-} F(V) \to H^1(W_{\text{ét}}, F) = 0$$

where the right hand term vanishes as F is flasque. The result follows. □

Remark 13.5.5. While the converse holds in the topological setting (any flabby sheaf is flasque) this does not hold in the étale setting. Indeed let k be a field and set $X = \mathrm{Spec}\, k$. An open inclusion $V \subset U$ in $X_{\text{ét}}$ is necessarily of the form $U = V \coprod V'$ hence any sheaf on $X_{\text{ét}}$ is necessarily flabby. However it is not flasque in general as the Galois cohomology of k is usually non-trivial.

13.6. **The Leray spectral sequence.** One computes the cohomology of topological spaces by using classical dévissages (Künneth formula, Leray spectral sequence, simplicial decompositions, excision...). One is reduced to compute the cohomology of the fundamental building block in topology: the interval $I = [0, 1]$.

In étale cohomology, the situation is similar (we use dévissage, like the Leray spectral sequence or proper base change) but the fundamental blocks are more complicated. We will be reduced to compute:

- the cohomology of points.

- the cohomology of curves over algebraically closed fields.

Let us start by giving one tool for dévissage: the Leray spectral sequence.

Proposition 13.6.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $F \in \mathbf{Ab}(X_{\text{ét}})$. There is a spectral sequence:*

$$E_2^{p,q} = R^p g_* \, R^q f_*(F) \Rightarrow R^{p+q}(gf)_* F \quad .$$

In particular:

$$E_2^{p,q} = H^p(Y_{\text{ét}}, R^q f_* F) \Rightarrow H^{p+q}(X_{\text{ét}}, F) \quad .$$

Proof. This is just the Grothendieck's spectral sequence for a composition of functors (noticing that f_* maps injectives to injectives, in particular to g_*-acyclic). □

Corollary 13.6.2. *If $R^q f_* F = 0$ for all $q > 0$ then $H^p(Y_{\text{ét}}, f_* F) = H^p(X_{\text{ét}}, F)$.*

To apply this corollary it will be necessary to compute the stalks of $R^q f_* F$.

Proposition 13.6.3. *Let $f : X \to Y$ be quasi-compact quasi-separated (recall this means that the diagonal $X \to X \times_Y X$ is quasi-compact). Let $F \in \mathbf{Ab}(X_{\text{ét}})$ and $\overline{y} \to Y$ a geometric point. Then $(R^q f_* F)_{\overline{y}} = H^q((X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\overline{y}})_{\text{ét}}, F)$ (we do not indicate the pull-back map from X to $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\overline{y}}$).*

Proof. By definition $(R^q f_* F)_{\overline{y}} = \operatorname{colim}_{(V,\overline{v})} H^q((X \times_Y V)_{\text{ét}}, F_{|X \times_Y V})$, where $(V, \overline{v})$ ranges through the étale neighborhoods of $(Y, \overline{y})$. By definition $\operatorname{Spec} \mathcal{O}_{Y,\overline{y}} = \lim_{(V,\overline{v})} V$. As the fiber product commutes with limits we are reduced to show that in our situation "cohomology commutes with limits". This follows from the following result (for details we refer to [Stacks Project, Etale Cohomology Th.52.1]):

Theorem 13.6.4. *Let $X = \lim_{i \in I} X_i$ be the limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X'_i \to X_i$. Assume that X_i is quasi-compact quasi-separated for any i and that the following data are given:*

1. $F_i \in \mathbf{Ab}((X_i)_{\text{ét}})$.
2. *for $i' \geq i$, $\varphi_{i',i} : f_{i'i}^{-1} F_i \to F'_i$ such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1} \varphi_{i'i}$ for $i'' \geq i' \geq i$.*

Set $f_i : X \to X_i$ and $F := \operatorname{colim}_i f_i^{-1} F_i$. Then

$$\operatorname{colim}_{i \in I} H^p((X_i)_{\text{ét}}, F_i) = H^p(X_{\text{ét}}, F) \text{ for all } p \geq 0 \quad .$$

□

Corollary 13.6.5. *Let $f : X \to Y$ be a finite morphism and $F \in \mathbf{Ab}(X_{\text{ét}})$. Then $R^q f_* F = 0$ for any $q > 0$.*

Proof. By Proposition 13.6.3 one has

$$(R^q f_* F)_{\overline{y}} = H^q((X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\overline{y}})_{\text{ét}}, F) \quad .$$

As $f : X \to Y$ is finite the scheme $X \times_Y \mathcal{O}_{Y,\overline{y}}$ is a finite extension of the strictly henselian ring $\mathcal{O}_{Y,\overline{y}}$, hence is a product of strictly henselian rings. The result follows from the following:

Lemma 13.6.6. *Let R be a local strictly henselian ring and $S := \operatorname{Spec} R$. Then $\Gamma(S, F) = F_{\overline{s}}$. In particular $\Gamma(S, \cdot)$ is an exact functor.*

Proof. Any étale surjective morphism onto S has a section as R is strictly henselian hence $\mathrm{id} : (S, \bar{s}) \to (S, \bar{s})$ is cofinal among étale neighborhoods of $(S, \bar{s})$. □

□

13.7. **Cohomology of points: Galois cohomology.** Let k be a field and $X = \operatorname{Spec} k$. Denote by G the Galois group $\mathrm{Gal}(k^s/k)$. We already proved the following:

Proposition 13.7.1. *There is an equivalence of categories*

$$\begin{array}{lcl} \{\text{k-finite étale algebras}\} & \simeq & \{\text{finite sets with continuous G-action}\} \\ A & \mapsto & \mathrm{Hom}_k(A, k^s) \ . \end{array}$$

Proposition 13.7.2. *There is an equivalence of categories*

$$\begin{array}{lcl} \mathbf{Sh}((\operatorname{Spec} k)_{\text{ét}}) & \simeq & \{\text{continuous G-sets}\} \\ F & \mapsto & F_{k^s} \ . \end{array}$$

In this proposition the inverse functor associates to a continuous G-set F_{k^s} the sheaf F defined by $F(U) = \mathrm{Hom}_{G-\text{sets}}(U(k^s), F_{k^s})$. In particular

$$F(\operatorname{Spec} k) = \mathrm{Hom}_{G-\text{sets}}(\star, F_{k^s}) = F_{k^s}^G \ .$$

By considering only abelian sheaves and taking the derived functors:

$$H^q(X_{\text{ét}}, F) = R^q\Gamma(X_{\text{ét}}, F) = (R^q(\cdot^G))(F_{k^s}) = H^q(G, F_{k^s})$$

hence the étale cohomology of points coincide with their Galois cohomology.

14. Cohomology of curves over an algebraically closed field

In this section we will prove the

Theorem 14.0.1. *Let k be an algebraically closed field and X a smooth curve over k. Then: $H^0(X_{\text{ét}}, \mathbb{G}_m) = H^0(X_{\text{Zar}}, \mathbb{G}_m)$, $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$ and $H^q(X_{\text{ét}}, \mathbb{G}_m) = 0$ for $q \geq 2$.*

Remark 14.0.2. If $\operatorname{char} k = p > 0$ and one only assumes that k is separably closed, the same proof will show that $H^0(X_{\text{ét}}, \mathbb{G}_m) = H^0(X_{\text{Zar}}, \mathbb{G}_m)$, $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$ and for $q \geq 2$ the group $H^q(X_{\text{ét}}, \mathbb{G}_m)$ is p-torsion.

Corollary 14.0.3. *Let k be an algebraically closed field and X a smooth projective curve over k. Let n be a positive integer invertible in k. Then $H^0(X_{\text{ét}}, \boldsymbol{\mu}_n) = \boldsymbol{\mu}_n(k)$, $H^1(X_{\text{ét}}, \boldsymbol{\mu}_n) = \operatorname{Pic}^0(X)_n$, $H^2(X_{\text{ét}}, \boldsymbol{\mu}_n) = \mathbb{Z}/n$ and $H^q(X_{\text{ét}}, \boldsymbol{\mu}_n) = 0$ for any $q > 2$.*

Proof. The Kummer exact sequence in $\mathbf{Ab}(X_{\text{ét}})$ is

$$1 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \overset{x \mapsto x^n}{\to} \mathbb{G}_m \to 1 \ . \tag{25}$$

Writing the corresponding long exact sequence, it follows from Theorem 14.0.1 that $H^q(X_{\text{ét}}, \boldsymbol{\mu}_n) = 0$ for any $q > 2$. In small degree the surjectivity of the elevation to the n-th power on k^* gives

$$0 \to H^0(X, \boldsymbol{\mu}_n) \to k^* \overset{x \mapsto x^n}{\to} k^* \to 0 \ .$$

Hence $H^0(X_{\text{ét}}, \boldsymbol{\mu}_n) = \boldsymbol{\mu}_n(k)$. The remaining part of the long exact sequence gives

$$0 \to H^1(X_{\text{ét}}, \boldsymbol{\mu}_n) \to \operatorname{Pic}(X) \overset{\times n}{\to} \operatorname{Pic}(X) \to H^2(X_{\text{ét}}, \boldsymbol{\mu}_n) \to 0 \ .$$

The exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \stackrel{\deg}{\to} \mathbb{Z} \to 0$$

gives $H^1(X_{\text{ét}}, \boldsymbol{\mu}_n) = \mathrm{Pic}^0(X)_n$. Moreover $\mathrm{Pic}^0(X) = (\mathrm{Pic}^0 X)(k)$ where $\mathrm{Pic}^0 X$ is the Jacobian of X. As n is invertible in k and k is algebraically closed, the multiplication by n is surjective on the k-points of the Abelian variety $\mathrm{Pic}^0 X$, hence the result. □

14.1. **The divisorial exact sequence.** Recall that a scheme X is said to be normal if for any point x of X the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain. In particular X is locally integral. If moreover it is Noetherian and connected then it is integral (hence irreducible in particular). The main tool in the proof of Theorem 14.0.1 is the following:

Proposition 14.1.1. *Let X be a connected Noetherian normal scheme with generic point η. The following sequence of $\mathbf{Ab}(X_{\text{ét}})$ is exact (surjective on the right if X is moreover regular):*

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_{m,\eta} \to \bigoplus_{x \in X^{(1)}} i_{x*}\mathbb{Z}_x \dashrightarrow 0 \quad . \tag{26}$$

Proof. We have to show that for any geometric point $\overline{y} \to X$, the corresponding sequence of stalks

$$0 \to (\mathbb{G}_m)_{\overline{y}} \to (j_*\mathbb{G}_{m,\eta})_{\overline{y}} \to \bigoplus_{x \in X^{(1)}} (i_{x*}\mathbb{Z}_x)_{\overline{y}} \dashrightarrow 0$$

is exact. These stalks are obtained by taking filtered colimits over the étale neighborhoods $(U, \overline{u})$ of $(X, \overline{y})$. As filtered colimits preserve exactness, it is enough to show that for any $U \to X$ in $X_{\text{ét}}$, the restriction of the sequence eq. (26) to U_{Zar} is exact.

As X is Noetherian normal (resp. regular) the scheme U is Noetherian normal too (resp. regular). Hence Proposition 14.1.1 follows from the analogous Zariski statement:

Lemma 14.1.2. *Let X be a connected Noetherian normal scheme. The following sequence of $\mathbf{Ab}(X_{\text{Zar}})$ is exact (surjective on the right if X is moreover regular):*

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_{m,\eta} \to \bigoplus_{x \in X^{(1)}} i_{x*}\mathbb{Z}_x \dashrightarrow 0 \quad . \tag{27}$$

Proof. We denote by K the function field of X, by $K_X^\times$ the constant Zariski sheaf defined by $K^\times$ on X and by Div the Zariski sheaf on X associated to the presheaf $U \mapsto \mathrm{Div}(U)$, with $\mathrm{Div}(U)$ the group of Weil divisors of U. The sequence eq. (27) can be rewritten as:

$$0 \to \mathcal{O}_X^* \to K_X^\times \to \mathrm{Div} \dashrightarrow 0.$$

Let $U = \mathrm{Spec}\, A$ be a Zariski open subset of X. Hence A is an integrally closed domain. Consider the sequence

$$0 \to A^* \to K^\times \to \bigoplus_{\mathrm{ht}\mathfrak{p}=1} \mathbb{Z} \dashrightarrow 0 \quad , \tag{28}$$

where the map on the right associates to $a \in K^*$ the collection $(v_{\mathfrak{p}}(a))$. Here v_p denotes the valuation of the discrete valuation ring $A_{\mathfrak{p}}$ (recall that a local ring of dimension one is a discrete valuation ring if and only if it is integrally closed if and only if it is regular).

We claim that the solid sequence eq. (28) is exact if A is integrally closed. Indeed in this case $A = \cap_{\mathrm{ht}\mathfrak{p}=1} A_{\mathfrak{p}}$ (see [Mat80, Th.38 p.124]). This finishes the proof of Lemma 14.1.2 in the case X normal.

The surjectivity of the dashed arrow is equivalent to saying that any prime ideal $\mathfrak{p}$ of height 1 in A is principal, or equivalently (see [Mat80, p.141]) that the Noetherian integral domain A is factorial. But any regular local ring is factorial. Thus the dashed sequence eq. (28) is exact for any regular local ring A, which finishes the proof of Lemma 14.1.2 in the case X regular. □

□

14.2. **Proof of Theorem 14.0.1.** From now on X is a smooth projective curve over an algebraically closed field k. We will compute $H^\bullet(X_{\text{ét}}, \mathbb{G}_m)$ from the exact sequence eq. (26).

Lemma 14.2.1. *$H^q(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) = 0$ for all $q > 0$.*

Proof. Apply the Leray spectral sequence to $j : \eta \to X$:

$$H^q(\eta_{\text{ét}}, \mathbb{G}_{m,\eta}) = H^q(X_{\text{ét}}, Rj_*\mathbb{G}_{m,\eta}) \quad .$$

Our claim then follows from the following two results:

Sub-lemma 14.2.2. *$R^p j_*\mathbb{G}_{m,\eta} = 0$ for all $p > 0$.*

Hence $H^q(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) = H^q(X_{\text{ét}}, Rj_*\mathbb{G}_{m,\eta}) = H^q(\eta_{\text{ét}}, \mathbb{G}_{m,\eta})$.

Sub-lemma 14.2.3. *$H^q(\eta_{\text{ét}}, \mathbb{G}_{m,\eta}) = 0$ for all $q > 0$.*

To prove Sub-lemma 14.2.2 one argues as follows.

As X is a scheme of finite type over the algebraically closed field k, it is enough to show that for any closed point x of X the stalk $(R^q j_*\mathbb{G}_{m,\eta})_{\overline{x}}$ vanishes.

It follows from Proposition 13.6.3 that for any closed point $x \in X$:

$$(R^q j_*\mathbb{G}_{m,\eta})_{\overline{x}} = H^q(\eta \times_X \operatorname{Spec} \mathcal{O}_{X,\overline{x}}, \mathbb{G}_m) \quad .$$

Let $\operatorname{Spec} A$ be some affine neighbourhood of x in X. Let K be the fraction field of A, hence $\eta = \operatorname{Spec} K$. Then $\eta \times_X \operatorname{Spec} \mathcal{O}_{X,\overline{x}} = \operatorname{Spec} (\mathcal{O}_{X,\overline{x}} \otimes_A K)$. The ring $\mathcal{O}_{X,\overline{x}} \otimes_A K$ is a localisation of the discrete valuation ring $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{\mathrm{sh}}_{X,x}$, hence it is either $\mathcal{O}_{X,\overline{x}}$ or its fraction field. As any local uniformizer of $\mathcal{O}_{X,\overline{x}}$ gets inverted in $\mathcal{O}_{X,\overline{x}} \otimes_A K$, we obtain that $\eta \times_X \operatorname{Spec} \mathcal{O}_{X,\overline{x}} = \operatorname{Spec} \operatorname{Frac} \mathcal{O}_{X,\overline{x}}$.

As every element of $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{\mathrm{sh}}_{X,x}$ is algebraic over $\mathcal{O}_{X,x}$, the extension $\operatorname{Frac} \mathcal{O}_{X,\overline{x}}$ of K is algebraic, hence an extension of k of transcendence degree 1. Thus both Sub-lemma 14.2.2 and Sub-lemma 14.2.3, hence the proof of Lemma 14.2.1, follow from the following:

Proposition 14.2.4. *Let k be an algebraically closed field and K/k an extension of transcendence degree 1. Then $H^q((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_m) = 0$ for all $q > 0$.*

□

Let us for the moment admit Proposition 14.2.4 and finish the proof of Theorem 14.0.1.

Lemma 14.2.5. *$H^q(X_{\text{ét}}, \bigoplus_{x \in X_{(0)}} i_{x*}\mathbb{Z}_x) = 0$ for all $q > 0$.*

Proof. The scheme X is quasi-compact quasi-separated hence étale cohomology on X commutes with colimits. Hence it is enough to show the vanishing of $H^q(X_{\text{ét}}, i_{x*}\mathbb{Z}_x)$ for all $x \in X_{(0)}$, $q > 0$. As $i_x : x \to X$ is a finite morphism $R^q i_{x*}\mathbb{Z}_x = 0$ for all $q > 0$ by Corollary 13.6.5. It follows from the Leray spectral sequence for i_x that $H^q(X_{\text{ét}}, i_{x*}\mathbb{Z}_x) = H^q(x_{\text{ét}}, \mathbb{Z}_x)$, which vanishes because x is separably closed. □

We deduce from Lemma 14.2.1 and Lemma 14.2.5 and from the exact sequence eq. (26) of étale sheaves that:
- $H^q(X_{\text{ét}}, \mathbb{G}_m) = 0$ for $q \geq 2$;
- the following sequence is exact:

$$0 \to H^0(X_{\text{ét}}, \mathbb{G}_m) \to H^0(X_{\text{ét}}, j_*\mathbb{G}_{m,\eta}) \to H^0(X_{\text{ét}}, \bigoplus_{x \in X_{(0)}} i_{x*}\mathbb{Z}_x) \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0 \ .$$

Comparing this sequence with the corresponding Zariski sequence

$$0 \to H^0(X_{\text{Zar}}, \mathbb{G}_m) \to H^0(X_{\text{Zar}}, j_*\mathbb{G}_{m,\eta}) \to H^0(X_{\text{Zar}}, \bigoplus_{x \in X_{(0)}} i_{x*}\mathbb{Z}_x) \to H^1(X_{\text{Zar}}, \mathbb{G}_m) = \operatorname{Pic} X \to 0$$

and as the H^0's coincide, we conclude that $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$.

□

14.3. Brauer groups and the proof of proposition 14.2.4.

The main tool for the proof of Proposition 14.2.4 is the Brauer group.

14.3.1. *Summary on Brauer groups.* Let k be a field with algebraic closure $\overline{k}$. In this section an algebra over k is an associative, possibly non-commutative, unital ring A equipped with a ring morphism from k to the center $Z(A)$ of A mapping 1 to 1. An A-module is a right A-module. The k-algebra A is said to be central, resp. simple, resp. finite, if $Z(A) = k$, resp. A has no non-trivial two-sided ideals, resp. A is a finite dimensional k-vector space. It is a division algebra if every element has a multiplicative inverse.

Theorem 14.3.1. *The following statements are equivalent:*

1. *A is a central finite simple k-algebra.*
2. *there exists a positive integer d such that $A \otimes_k \overline{k} \simeq \operatorname{Mat}(d \times d, \overline{k})$.*
3. *there exists a positive integer d and a finite extension k'/k such that $A \otimes_k k' \simeq \operatorname{Mat}(d \times d, k')$.*
4. *$A \simeq \operatorname{Mat}(n \times n, D)$ where D is a division algebra of center k.*

Remark 14.3.2. The integer d in (2) and (3) is called the *degree* of A.

Definition 14.3.3. *We define a relation on finite simple central k-algebras as follows: $A_1 \sim A_2$ if there exist $m, n > 0$ such that*

$$\operatorname{Mat}(n \times n, A_1) \simeq \operatorname{Mat}(m \times m, A_2)$$

Equivalently, the division algebras associated to A_1 and A_2 by the Theorem 14.3.1(4) coincide.

One checks (see [Stacks Project, Brauer Groups, Lemma 5.1]) that the relation $\sim$ on finite simple central k-algebras is an equivalence relation.

Definition 14.3.4. *Let k be a field. The Brauer group of k is the set* $\mathrm{Br}(k)$ *of equivalence classes of finite central simple algebras over k, endowed with the abelian group law* $[A_1]+[A_2] := [A_1 \otimes_k A_2]$.

In this definition the existence of inverses is given by

Lemma 14.3.5. *Let A be a central finite simple k-algebra. Then:*

$$\begin{array}{ccc} A \otimes_k A^{\mathrm{op}} & \simeq & \mathrm{End}_k(A) \\ a \otimes a' & \mapsto & (x \mapsto axa') \end{array}$$

Hence we can define $-[A] := [A^{\mathrm{op}}]$.

Notice that $\mathrm{Br}(k) = \cup_{n\in\mathbb{N}} \mathrm{Br}(n,k)$, where $\mathrm{Br}(n,k)$ denotes the torsion subgroup of classes $[A]$ such that there exists k'/k finite with $A_{k'} \simeq \mathrm{Mat}(n \times n, k')$. Now $\mathrm{Br}(n,k)$ is easy to describe: it is the group of k-forms of $\mathrm{Mat}(n \times n, \overline{k})$. Hence:

$$\mathrm{Br}(n,k) \simeq H^1(G, \mathrm{Aut}\,\mathrm{Mat}(n \times n, \overline{k})) = H^1(G, PGL(n, \overline{k})) \tag{29}$$

as all automorphisms of $\mathrm{Mat}(n \times n, \overline{k})$ are interior.

The short exact sequence of G-groups

$$1 \to \overline{k}^* \to GL(n, \overline{k}) \to PGL(n, \overline{k}) \to 1$$

give rise to boundary maps of cohomology groups

$$H^1(G, PGL(n, \overline{k})) \to H^2(G, \overline{k})$$

which are compatible. Composing with eq. (29) one obtains a canonical map:

$$\delta : \mathrm{Br}(k) \to H^2(G, \overline{k}) \quad .$$

Theorem 14.3.6. *The map* $\delta : \mathrm{Br}(k) \to H^2(G, \overline{k})$ *is an isomorphism.*

Proof. Exercice, see [Stacks Project, Etale Cohomology, Th.60.6]. ☐

14.3.2. *Brauer groups and Galois cohomology.* The link between Brauer groups and our problem lies in the following:

Proposition 14.3.7. *Let K be a field with algebraic closure $\overline{K}$ and* $G := \mathrm{Gal}(\overline{K}/K)$. *Suppose that for any finite extension K'/K the Brauer group* $\mathrm{Br}(K')$ *vanishes. Then:*

(i) $H^q(G, \overline{K}^*) = 0$ *for all* $q > 0$.
(ii) $H^q(G, F) = 0$ *for any torsion G-module F and any* $q \geq 2$.

Proof. See [Se97, Chapter II, Section 3, Proposition 5]. ☐

14.3.3. *Tsen's theorem.* As $H^q((\mathrm{Spec}\, K)_{\text{ét}}, \mathbb{G}_m) = H^q(G, \overline{K}^*)$, Proposition 14.2.4 will follow from Proposition 14.3.7 if we prove that $\mathrm{Br}(K) = 0$ for K/k an extension of transcendence degree 1, with k algebraically closed.

Definition 14.3.8. *A field K is said to be C_r if any polynomial* $f \in K[T_1, \dots, T_n]$ *homogeneous of degree d with* $1 < d^r < n$ *admits a non-trivial zero.*

Proposition 14.3.9. *If K is C_1 then* $\mathrm{Br}(K) = 0$.

Proof. Let D be a K-division algebra. Hence $D \otimes_K \overline{K} \simeq \mathrm{Mat}(d \times d, \overline{K})$, the isomorphism being uniquely defined up to interior automorphisms. In particular the determinant

$$\det : \mathrm{Mat}(d \times d, \overline{K}) \to \overline{K}$$

is G-invariant, hence descend to

$$N_{\mathrm{red}} : D \to K \quad .$$

This reduced norm is a homogeneous polynomial in d^2 variables of degree d over K. Hence if $d > 1$ there exists $x \neq 0 \in D$ satisfying $N_{\mathrm{red}}(X) = 0$: contradiction to the invertibility of x.

Thus $d = 1$ and $\mathrm{Br}(K) = 0$. □

Theorem 14.3.10. *(Tsen) The function field of a variety X of dimension r over an algebraically closed field k is a C_r-field.*

Proof. Without loss of generality we can assume that X is projective. Let $f \in K[T_1, \ldots, T_n]$ homogeneous of degree d, $1 < d^r < n$ (where $K = k(X)$). The coefficients of f can be assumed to lie in $\Gamma(X, \mathcal{O}_X(H))$ where H is some ample line bundle on X. Fix a positive integer e and consider $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\Gamma(X, \mathcal{O}_X(eH))$. Then $f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. We want to show that the equation $f(\alpha)$ has a non trivial zero.

The number of possible variables α is

$$n \cdot \dim_k \Gamma(X, \mathcal{O}_X(eH)) \sim n \cdot \frac{e^r}{r!}(H^r)$$

by the Riemann-Roch theorem.

The number of equations is

$$\dim_k \Gamma(X, \mathcal{O}_X((de+1)H)) \sim \frac{(de+1)^r}{r!}(H^r)$$

again by the Riemann-Roch theorem.

As $n > d^r$ there are more variables than equations hence $f(\alpha) = 0$ has a non-trivial solution. □

14.3.4. *Proof of Proposition 14.2.4.* Let K/k be of transcendence degree 1. We have to show that if K'/K is finite then $\mathrm{Br}(K') = 0$. Any such K' can be written as a colimit of extensions K'' *of finite type* of k, of transcendence degree 1. Any such extension K'' is the function field of a curve over k. Hence $\mathrm{Br}(K') = \mathrm{colim}_{K''} \mathrm{Br}(K'') = 0$ by Tsen's theorem.

□

15. Constructible sheaves

Classical topology study constant sheaves and their natural generalisation: locally constant sheaves. These locally constant sheaves have a bad functorial behaviour: the direct image of a locally constant sheaf is hardly ever locally constant. This leads to the notion of constructible sheaf. We follow the same path for étale topology, with a significant difference: one only considers torsion sheaves.

15.1. **Pathology of the étale constant sheaf $\mathbb{Z}$.** The étale constant sheaf $\mathbb{Z}$ is cohomologically uninteresting, as the following lemma shows:

Lemma 15.1.1. *Let X be a regular scheme. Then $H^1(X_{\text{ét}}, \mathbb{Z}_X) = 0$.*

Proof.

Sub-lemma 15.1.2. *Let X be a scheme and $x \xrightarrow{i_x} X$ a (non-necessarily closed) point. Then $H^1(X_{\text{ét}}, i_{x*}\mathbb{Z}) = 0$.*

Proof. The Leray spectral sequence for i_x

$$E_2^{p,q} = H^p(X_{\text{ét}}, R^q i_{x*}\mathbb{Z})) \Rightarrow H^{p+q}(x_{\text{ét}}, \mathbb{Z})$$

implies readily $H^1(X_{\text{ét}}, i_{x*}\mathbb{Z})) \subset H^1(x_{\text{ét}}, \mathbb{Z})$. But

$$\begin{aligned} H^1(x_{\text{ét}}, \mathbb{Z}) &= H^1(\mathrm{Gal}(\overline{k(x)}/k(x)), \mathbb{Z}) \\ &= \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{k(x)}/k(x)), \mathbb{Z}) \\ &= 0 \quad , \end{aligned}$$

where the first equality comes from our identification of the étale cohomology of points with Galois cohomology of their residue fields and the vanishing of Galois cohomology follows from the fact that $\mathrm{Gal}(\overline{k(x)}/k(x))$ is a profinite groupe while $\mathbb{Z}$ has no torsion. □

Let us finish the proof of Lemma 15.1.1. As X is regular one can assume that X is connected, hence irreducible. Let $j : \eta \to X$ be the generic point of X. Lemma 15.1.1 follows immediately from Sub-lemma 15.1.2 applied to j and the following:

Sub-lemma 15.1.3. *The adjunction map $\mathbb{Z}_X \to j_*\mathbb{Z}_\eta$ is an isomorphism.*

Proof. We have to show that for any geometric point $\overline{x} \to X$ the map of stalks $\mathbb{Z}_{X,\overline{x}} \to (j_*\mathbb{Z}_\eta)_{\overline{x}}$ is an isomorphism.

On the one hand $\mathbb{Z}_{X,\overline{x}} = \mathrm{colim}_{(V,\overline{v})}\, \mathbb{Z}_X(V) = \mathbb{Z}$, where the colimit can be taken over *connected* étale neighbourhoods $(V, \overline{v})$ of $(X, \overline{x})$ as X is irreducible.

On the other hand $(j_*\mathbb{Z}_\eta)_{\overline{x}} = \mathrm{colim}_{(V,\overline{v})}\, \mathbb{Z}_\eta(\eta \times_X V)$ where the colimit can be taken over the connected étale neighbourhoods $(V, \overline{v})$ of $(X, \overline{x})$. As $V \to X$ is étale, the scheme $\eta \times_X V$ is the disjoint union of the generic points of $\eta \times_X V$. As X is regular, V is regular too. As it is connected it is irreducible. Hence $\eta \times_X V$ is one point, $\mathbb{Z}_\eta(\eta \times_X V) = \mathbb{Z}$ and $(j_*\mathbb{Z}_\eta)_{\overline{x}} = \mathbb{Z}$.

One easily checks that the map $\mathbb{Z}_{X,\overline{x}} = \mathbb{Z} \to (j_*\mathbb{Z}_\eta)_{\overline{x}} = \mathbb{Z}$ is the identity, hence the result. □

□

In view of the proof of Sub-lemma 15.1.3, it is natural to consider only torsion étale sheaves.

Definition 15.1.4. *Let X be a scheme. An étale sheaf $F \in \mathbf{Ab}(X_{\text{ét}})$ is said to be a torsion sheaf if any local section of F is killed by a positive integer n, i.e. $F = \mathrm{colim}_n F_n$, where $F_n = \ker(F \xrightarrow{n\times\cdot} F)$.*

15.2. **Locally constant constructible sheaves.**

Definition 15.2.1. *Let S be a scheme. An étale sheaf $F \in \mathbf{Sh}(S_{\text{ét}})$ is said to be constant constructible (or constant finite) if it the étale sheafification of the constant presheaf associated to a finite set.*

We saw that any such constant sheaf is representable by $\Sigma \times S$, Σ finite set.

Definition 15.2.2. *Let S be a scheme. An étale sheaf $F \in \mathbf{Sh}(S_{\text{ét}})$ is said to be locally constant constructible (lcc), or locally constant finite, if there exists an étale covering family $(U_i \to S)_{i \in I}$ with $F_{|U_i} \in \mathbf{Sh}((U_i)_{\text{ét}})$ constant finite.*

The representability of constant sheaves generalizes to locally constant sheaves:

Lemma 15.2.3. *Let S be a scheme and $F \in \mathbf{Sh}(S_{\text{ét}})$. The following conditions are equivalent:*

(1) *F is lcc.*
(2) *$F \simeq h_U$ where $U \to S$ is a* finite *étale morphism.*

Proof. We start with the easy direction (2) $\Rightarrow$ (1). One has to show that for any $U \to S$ finite étale there exists an étale covering $(S_i \to S)_{i \in I}$ such that for any $i \in I$, $U \times_S S_i$ is isomorphic to a disjoint union of copies of S_i.

Write $S = \coprod_{n \in \mathbb{N}^*} S_n$, where S_n is defined by the condition $U_{|S_n} \to S_n$ is finite of degree n. Without loss of generality we can thus assume that $U \to S$ is of fixed degree $n > 0$.

If $n = 1$ the étale morphism $U \to S$ is an isomorphism and the conclusion holds true trivially. Suppose $n > 1$. Consider the second projection $p_2 : U \times_S U \to U$ obtained by base change to U from $U \to S$. It is an étale morphism of degree n and admits a section $\Delta_U : U \to U \times_S U$. Hence $U \times_S U = \Delta_U \coprod U'$ where $U' \to U$ is étale of degree $n-1$. By induction on n there exists an étale covering $(U_i \to U)_{i \in I}$ such that for any $i \in I$, $U' \times_U U_i$ is isomorphic to a disjoint union of copies of U_i. But then $U \times_S U_i$ is also isomorphic to such a disjoint union.

Conversely let us show that (1) $\Rightarrow$ (2). This is an application of fpqc descent for schemes.

Let $F \in \mathbf{Sh}(S_{\text{ét}})$ and $(f_i : S_i \to S)_{i \in I}$ be an étale covering family such that $F_{S_i} \simeq \Sigma_i \times h_{S_i}$ for some finite sets $(\Sigma_i)_{i \in I}$. We want to show that F is representable by some $X \to S$ finite étale.

One can work Zariski-locally on S: it is enough to prove the statement for each open subset S_n of an open Zariski cover $(S_n)_{n \in \mathbb{N}}$ of S. For $i \in I$ let $n_i := |\Sigma_i|$. For every positive integer n let us define $U_n := \coprod_{n_i = n} S_i$ and by S_n the image of U_n in S. As the f_i's are open, S_n is an open subscheme of S. Hence without loss of generality replacing S by S_n we can assume that $n_i = n$ for all $i \in I$.

We are thus reduced to considering the étale covering $S' := \coprod_{i \in I} S_i \to S$ with $\xi' : F_{|S'} \simeq \Sigma \times h_{S'}$, Σ a finite set of cardinality n.

Restricting S and replacing S' by a finite disjoint union of open subschemes if necessary, we can assume that S and S' are affine, hence $S' \to S$ is an fpqc morphism.

Consider the two projections $p_1, p_2 : S' \times_S S' \to S'$. Denoting by p_j^* the corresponding base change, one obtains an isomorphism:

$$\varphi : p_1^*(\Sigma \times S') \overset{p_1^*(\xi')^{-1}}{\simeq} p_1^*(F_{|S'_{\text{ét}}}) = F_{|(S' \times_S S')_{\text{ét}}} = p_2^*(F_{|S'_{\text{ét}}}) \overset{p_2^*(\xi')}{\simeq} p_2^*(\Sigma \times S') \ .$$

It obviously satisfies the cocycle condition

$$p_{23}^*(\varphi)\, p_{12}^*(\varphi) = p_{13}^*(\varphi) \ .$$

The effectivity of fpqc descent for affine morphisms implies that there exists an affine morphism $X \to S$ such that $\Sigma \times S' \simeq X \times_S S'$ (inducing an isomorphism of descent datas).

As the morphism $X \times_S S' \to S'$ is finite étale and $S' \to S$ is étale, the morphism $X \to S$ is finite étale too. Hence $\xi' : F_{|S'} \simeq h_{X \times_S S'}$ in $\mathbf{Sh}(S'_{\text{ét}})$ satisfies $p_1^* \xi' = p_2 \xi'$ in $\mathbf{Sh}((S' \times_S S')_{\text{ét}})$. Thus ξ' is a section of $\mathcal{H}om(F_{S'}, h_X)$ on S' whose two restrictions to $S' \times_S S'$ coincide. By the sheaf condition it descends to $\xi \in \mathcal{H}om(F, h_X)(S)$. Similarly for ξ'^{-1}, hence ξ is an isomorphism. □

15.3. **Constructible sheaves.** One checks easily that:

- if $f : X \to Y$ is a morphism of schemes and $G \in \mathbf{Sh}(Y_{\text{ét}})$ is lcc then $f^*G \in \mathbf{Sh}(X_{\text{ét}})$ is lcc.
- if $f : X \to Y$ is finite étale and $F \in \mathbf{Sh}(X_{\text{ét}})$ is lcc then $f_*F \in \mathbf{Sh}(Y_{\text{ét}})$ is lcc.

However, as in classical topology, the class of lcc sheaves is not stable under more general push-forward. The class of constructible sheaves will remedy this problem.

For the sake of generality let us start with a purely topological definition.

Definition 15.3.1. *Let X be a topological space. A subspace $Z \subset X$ is said to be retrocompact if the inclusion $i : Z \to X$ is quasi-compact, in other words: if the intersection of any quasi-compact open subset of X with Z is quasi-compact.*

Example 15.3.2. If X is a Noetherian scheme, any open subspace of $|X|$ is quasi-compact, hence retrocompact.

Definition 15.3.3. *A subspace $Z \subset X$ of a topological space X is said to be constructible if $Z = \bigcup_{i \in I} U_i \cap V_i^c$, where I is a finite set, and for any $i \in I$, U_i and V_i are retrocompact open subsets of X.*

It follows easily from this definition that if X is a Noetherian topological space then the constructible subsets of X are exactly the finite unions of locally closed subspaces.

Definition 15.3.4. *Let X be a scheme. A subscheme $T \subset X$ is said to be locally closed constructible if T is a locally closed subscheme of X such that the topological space $|T|$ is a constructible subspace of $|X|$.*

Definition 15.3.5. *Let X be a scheme. An étale sheaf $F \in Sh(X_{\text{ét}})$ is said to be constructible if for any open affine subscheme $U \subset X$, there exists a decomposition $U = \coprod_i U_I$ (called a partition of U) such that U_I is a localy closed constructible subscheme of U and $F_{|U_i} \in \mathbf{Sh}(U_{i\text{ét}})$ is lcc.*

Remarks 15.3.6. (1) Notice that the condition in Definition 15.3.5 depends only on the topological structure of the U_i's, not on their schematic structure. Indeed if $T, T' \subset X$ are two locally closed subscheme of X with $|T| = |T'|$ then $T_{\text{ét}} \simeq T'_{\text{ét}}$ c1

c1 to be added

(2) When X is quasi-compact quasi-separated an étale sheaf $F \in \mathbf{Sh}(X_{\text{ét}})$ is constructible if and only if there exists a *global* partition $X = \coprod_i X_i$ by locally closed constructible subschemes $X_i \subset X$ such that $F_{|X_i}$ is lcc.

15.4. **Properties of constructible sheaves on Noetherian schemes.** Let us start by stating a few easy properties of constructible sheaves on general schemes.

- If $X = \bigcup_{i \in I} U_i$ with $U_i \subset X$ open subschemes and $F \in \mathbf{Sh}(X_{\text{ét}})$ satisfies that for all $i \in I$, $F_{|U_i} \in \mathbf{Sh}((U_i)_{\text{ét}})$ is constructible then F is constructible.
- If $f : X \to Y$ is a morphism of schemes and $F \in \mathbf{Sh}(Y_{et})$ is constructible then $f^*F \in \mathbf{Sh}(X_{\text{ét}})$ is constructible.
- For Abelian sheaves the property of being constructible is stable under kernel, cokernel, image and extension. Hence the full subcategory $\mathbf{Ab}_c(X_{\text{ét}})$ of $\mathbf{Ab}(X_{\text{ét}})$ whose objects are the constructible Abelian sheaves is an Abelian subcategory.
- If X is a locally Noetherian scheme then F is constructible if and only if for all $x \in X$ there exists an open subscheme $U \subset \overline{\{x\}}$ such that $F_{|U}$ is lcc.

From now on we concentrate on Noetherian schemes.

Proposition 15.4.1. *Let X be a Noetherian scheme. Let $F \in \mathbf{Sh}(X_{\text{ét}})$. The following conditions are equivalent:*

(1) *F is lcc.*
(2) *F satisfies the following two properties:*
   (a) *For any geometric point $\overline{x} \to X$ the stalk $F_{\overline{x}}$ is finite.*
   (b) *If $\overline{y}$ is a specialization of $\overline{x}$ (meaning that $y \in \overline{\{x\}}$ and denoted $\overline{x} \rightsquigarrow \overline{y}$) the specialization morphism $F_{\overline{y}} \to F_{\overline{x}}$ is a bijection.*

Proof. The fact that (1) implies (2) is trivial, let us prove that (2) implies (1). Let $\overline{x} \to X$ be any geometric point of X. As $F_{\overline{x}} = \operatorname{colim}_{(V,\overline{v})} F(V)$ is a finite set (where $(V, \overline{v})$ runs through the étale neighborhoods of $(X, \overline{x})$) there exists an étale neighborhood $(V, \overline{v})$ of $(X, \overline{x})$ such that $F(V) \overset{f}{\twoheadrightarrow} F_{\overline{x}}$. Let us choose a finite set $E \subset F(V)$ with $f_{|E} : E \simeq F_{\overline{x}}$. This defines a sheaf morphism $E_V \to F_{|V}$ satisfying $(E_V)_{\overline{x}} \simeq F_{\overline{x}}$.

As V is Noetherian it follows that any geometric point $\overline{y}$ of V is related to $\overline{x}$ through a chain of specializations:

$$\overline{x} \rightsquigarrow \overline{p_1} \leftsquigarrow \overline{p_2} \rightsquigarrow \overline{p_3} \leftsquigarrow \cdots \rightsquigarrow \overline{p_n} \leftsquigarrow \overline{y} \ .$$

As $(E_V)_{\overline{y}} \simeq (E_V)_{\overline{x}}$ the condition (b) then implies:

$$(E_V)_{\overline{y}} \simeq F_{\overline{y}} \ .$$

Hence $E_V \simeq F_{|V}$. This proves that F is lcc. $\square$

Proposition 15.4.2. *Let X be a Noetherian scheme. Let $F \in \mathbf{Sh}(X_{\text{ét}})$. The following conditions are equivalent:*

(1) *F is constructible.*

(2) *The function $c : X \to \mathbb{N} \cup \{\infty\}$ which to $x \in X$ associates the cardinality of $F_{\overline{x}}$ is bounded and constructible (i.e. for all $n \in \mathbb{N}$ the preimage $c^1(n)$ is a constructible subset of $|X|$).*

Proof. Once more (1) $\Rightarrow$ (2) is clear, let us prove (2) $\Rightarrow$ (1). As X is Noetherian, the function c take only finitely many values. Hence without loss of generality one can assume that c is constant.

Without loss of generality we can assume that X is irreducible. Let $\overline{\eta}$ be a geometric point over the generic point η of X. As $F_{\overline{\eta}}$ is finite there exists an étale neighbourhood $(V, \overline{v})$ if $(X, \overline{\eta})$ such that $F(V) \twoheadrightarrow F_{\overline{\eta}}$. Any geometric point $\overline{x}$ of V is a specialization of $\overline{\eta}$, hence gives rise to a commutative diagram:

$$\begin{tikzcd}
F(V) \arrow[r] \arrow[dr] & F_{\overline{x}} \arrow[d] \\
& F_{\overline{\eta}} .
\end{tikzcd}$$

As $F_{\overline{x}} = F_{\overline{\eta}}$, it follows that $F_{\overline{x}} \simeq F_{\overline{\eta}}$. Let U be the image of V in X, this is a non-empty open subset of X and $F_{|U} \in \mathbf{Sh}(U_{\text{ét}})$ is lcc by Proposition 15.4.1.

By Noetherian induction one can assume that $F_{|X \setminus U}$ is constructible, hence the conclusion. □

Corollary 15.4.3. *Let $f : Y \to X$ be a surjective morphism of finite type between Noetherian schemes and $F \in \mathbf{Sh}(X_{\text{ét}})$. The following conditions are equivalent:*

(1) *F is constructible.*
(2) *f^*F is constructible.*

Proof. Let us prove the non-trivial implication (2) $\Rightarrow$ (1). The result is clear if the morphism f is moreover étale. We reduce to this case using Noetherian induction.

Without loss of generality we can assume that X is irreducible. Let $\eta = \operatorname{Spec} K$ be the generic point of X. The base change $Y_\eta := \eta \times_X Y$ is a K-scheme of finite type hence admits a closed point, with residue field L a finite extension of K. Let E denote the separable closure of K in L. Consider the commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[rr] \arrow[dr, "h"'] & & Y_\eta \arrow[r] \arrow[d] & Y \arrow[d] \\
& \operatorname{Spec} E \arrow[r, "g"'] & \eta \arrow[r] & X .
\end{tikzcd}$$

The morphism h is radicial finite surjective while the morphism g is finite étale surjective

All these data are of finite presentation hence lift to an open neighbourhood V of η in X:

$$\begin{tikzcd}
V_L \arrow[rr] \arrow[dr, "h"'] & & Y_V \arrow[r] \arrow[dd] & Y \arrow[dd] \\
& V_E \arrow[dr, "g"'] & & \\
& & V \arrow[r] & X ,
\end{tikzcd}$$

where the morphisms h and g have the same properties as above.

The fact that f^*F is constructible implies that $h^*g^*F_{|V}$ is constructible. As h is radicial $h^* : (V_E)_{\text{ét}} \to (V_L)_{\text{ét}}$ is an equivalence of categories, hence $g^*F_{|V}$ is constructible. But g is finite étale surjective hence $F_{|V}$ is constructible (easy case above).

We conclude by Noetherian induction. $\square$

Corollary 15.4.4. *Lte $f : V \to X$ be étale of finite type between Noetherian scheme. Then $h_V \in \mathbf{Sh}(X_{\text{ét}})$ is constructible.*

Proof. We apply Proposition 15.4.2 to the fibers of V/X. The result follows from the fact that the cardinality of the geometric fibers of an étale separated morphism of finite type varies lower semi-continuously on X, see [EGAIV, 18.2.8]. $\square$

From now on we denote by Λ the ring $\mathbb{Z}/n\mathbb{Z}$.

Proposition 15.4.5. *Let X be a Noetherian scheme. Let $F \in \mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$. The following conditions are equivalent:*

(1) *F is constructible.*
(2) *F is a Noetherian object in $\mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$ (recall that an object A in an Abelian category is Noetherian if any increasing sequence $A_0 \subset A_1 \subset .... \subset A$ is stationnary).*
(3) *There exists $f : V \to U$ in $X_{\text{ét}}$ of finite type over X such that $F \simeq \operatorname{Coker}(\Lambda_X(V) \xrightarrow{f_*} \Lambda_X(U)$.*

Proof. Without loss of generality we can assume that X is irreducible.

We first show that (1) $\Rightarrow$ (2) by Noetherian induction. Let $F_0 \subset F_1 \subset \cdots F$ be an increasing sequence. Let $U \subset X$ be a non-empty open subset such that $F_{|U}$ is locally constant and consider the restriction of $F_0 \subset F_1 \subset \cdots F$ to U.

Let $\overline{\eta}$ be a geometric point over the generic point η of X. The stalk $F_{\overline{\eta}}$ is finite hence the sequence $(F_i)_{\overline{\eta}}$ is necessary stationnary. Without loss of generality we can thus assume that the sequence $(F_i)_{\overline{\eta}}$ is constant.

As $F_{|U}$ is lcc, the specialization map $F_{\overline{x}} \to F_{\overline{\eta}}$ is an isomorphism for any $\overline{x}$ specialization of $\overline{\eta}$ in U. Hence the following diagram is commutative:

$$\begin{tikzcd}
(F_i)_{\overline{x}} \arrow[r, hook, dotted] \arrow[d, hook] & (F_i)_{\overline{\eta}} \arrow[d, equal] \\
F_{\overline{x}} \arrow[r] & F_{\overline{\eta}} .
\end{tikzcd}$$

Let $s_1, \cdots, s_n$ be generators of $(F_0)_{\overline{\eta}}$. Hence there exists an étale neighborhood of $\overline{\eta}$ such that the s_i's lift to $F_0(V)$. The diagram above implies that the germs of the s_i's generate $(F_i)_{\overline{x}}$ for any geometric point $\overline{x}$ of V.

It follows that the sequence $(F_i)_{i\in\mathbb{N}}$ is stationnary on V, hence on the image U_0 of V in U. By Noetherian induction the sequence $(F_i)_{X\setminus U_0}$ is stationnary. Finally the sequence $(F_i)_{i\in\mathbb{N}}$ is stationnary

Let us show (2) $\Rightarrow$ (3). Any $F \in \mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$ can be written as a quotient

$$\bigoplus_{i\in I} \Lambda_X(U_i) \stackrel{h}{\twoheadrightarrow} F$$

for an étale covering family $(U_i)_{i\in I}$. As F is Noetherian in $\mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$, there exists a finite subset $I_0 \in I$ such that

$$\bigoplus_{i\in I_0} \Lambda_X(U_i) \stackrel{h}{\twoheadrightarrow} F \quad .$$

Let us define $U = \coprod_{i\in I_0} U_i$, this is a separated étale X-scheme of finite type hence $\Lambda_X(U)$ is constructible by Corollary 15.4.4. Thus the kernel of h is constructible, hence Noetherian in $\mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$. Repeating the previous construction replacing F with Ker$h$, we obtain that F can be written Coker $(\Lambda_X(V) \stackrel{f_*}{\to} \Lambda_X(U)$ as required.

Finally we show tht (3) $\Rightarrow$ (1). By Corollary 15.4.4, both $\Lambda_X(V)$ and $\Lambda_X(U)$ are constructible, hence also $F \simeq$ Coker $(\Lambda_X(V) \stackrel{f_*}{\to} \Lambda_X(U)$.

□

Corollary 15.4.6. *The full subcategory $\mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})_c \subset \mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$ of constuctible Λ_X-module is a Serre subcategory.*

Proof. This is true for the full subcategory of Noetherian objects in any Abelain category> □

Corollary 15.4.7. *Any $F \in \mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$ is a filtered colimit of constructible $F_i \in \mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})_c$.*

Proof. The category $\mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$ admits as a generating family the $\Lambda_X(U)$, $U \to X$ affine étale, hence in particular constructible. Thus any $F \in \mathbf{\Lambda} - \mathbf{Mod}(X_{\text{ét}})$ is a filtered union of its constructible sub-modules. □

Corollary 15.4.8. *Any torsion sheaf in $\mathbf{Ab}(X_{\text{ét}})$ is a filtered colimit of constructible sheaves.*

Proof. Let F be an étale torsion sheaf. Hence F is a filtered colimit of $F_n := \ker(F \stackrel{\times n}{\to} F$. Each F_n belongs to $\mathbb{Z}/n\mathbb{Z} - \mathbf{Mod}(X_{\text{ét}})$, hence is a filtered colimit of constructible subsheaves by the previous corollary. Hence the result. □

16. Proper base change

The basic reference for this chapter is [SGA4, Exp. XII, XIII].

16.1. **The classical topological case.** Let $f : X \to S$ be a continuous map between topological spaces and $F \in \mathbf{Ab}(X)$. Given a point $s \in S$, let us denote by $i : f^{-1}(s) \to X$ the closed inclusion. Hence i_* is exact and the morphism of functors $1 \to i_* i^*$ induces a natural morphism of groups

$$(R^r f_* F)_s := \mathrm{colim}_{s \in V} H^r(f^{-1}(V), F_{|f^{-1}(V)}) \to H^r(f^{-1}(s), i^* F), \tag{30}$$

(where the colimit is taken over all open neighborhoods of s in S).

In general this morphism is not an isomorphism, even for $r = 0$. Suppose indeed that f is the inclusion of an open subset X of S. For a point $s \in \overline{X} \setminus X$ the stalk $(f_* F)_s$ is usually non-zero while $f^{-1}(s) = \emptyset$ hence the right hand side $H^0(f^{-1}(s), i^* F)$ is zero.

Notice that if f is closed and U is a neighborhood of $f^{-1}(s)$, the image $f(X \setminus U)$ is a closed subspace of S, the point s belongs to the open subspace $V := S \setminus f(X \setminus U)$, and $f^{-1}(V) \subset U$. Hence the open sets $f^{-1}(V)$ of X form a neighborhood basis of $f^{-1}(s)$. Thus $(R^r f_* F)_s = \mathrm{colim}_{U \supset f^{-1}(s)} H^r(U, F)$. In the case where X is locally compact one can go further thanks to the following result, whose elementary proof is left to the reader:

Lemma 16.1.1. *Let X be a locally compact space and $Z \overset{i}{\hookrightarrow} X$ a compact subspace. Then the natural map $\mathrm{colim}_{U \supset Z} H^r(U, F) \to H^r(Z, i^{-1} F)$ is an isomorphism.*

Recall that a continuous map $f : X \to S$ between topological spaces is said to be proper if it is separated and universally closed. When both X and S are locally compact (in particular Hausdorff) $f : X \to S$ is proper if and only if is universally closed, if and only if the preimage of a compact subset is compact.

Corollary 16.1.2. *Let $f : X \to S$ be a continuous* proper *map between topological spaces. For any $s \in S$ the natural morphism $(R^r f_* F)_s \to H^r(X_s, f)$ is an isomorphism.*

More generally:

Theorem 16.1.3. *(topological proper base change) Let $f : X \to S$ be a continuous proper map between topological spaces. Consider a Cartesian base change diagram of topological spaces:*

$$\begin{tikzcd} X_{S'} \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S . \end{tikzcd}$$

Then for any $F \in \mathbf{Ab}(X)$ the natural morphism of sheaves on S'

$$g^*(R^r f_* F) \to R^r f'_*(g'^* F)$$

is an isomorphism.

Remark 16.1.4. If $g := i_s : s \hookrightarrow S$ one recovers Corollary 16.1.2.

The morphism $g^*(R^r f_* F) \to R^r f'_*(g'^* F)$ is obtained as follows. By adjunction it is equivalent to construct a morphism of functors $R^r f_* \to g_*(R^r f'_*) g'^*$, which we define as the composition:

$$R^r f_* \to R^r f_* g'_* g'^* \to R^r (f \circ g')_* g'^* = R^r (g \circ f')_* g'^* \to g_*(R^r f'_*) g'^* .$$

The first map is given by the adjunction $1 \to g'_* g'^*$; the second and the last ones are special instance of the following: in the situation of Theorem 13.2.2 one has natural morphisms of functors $R^p G \circ F \to R^p(G \circ F)$ and $R^p(G \circ F) \to G \circ R^p F$, which are nothing else than the "border morphisms" of Grothendieck's spectral sequence.

16.2. **The étale case.** In étale topology the proper base change theorem still holds if one restricts oneself to torsion coefficients:

Theorem 16.2.1. *(étale Proper Base Change) Let S be a scheme and let $f : X \to S$ be a proper morphism (i.e. of finite type, separated and universally closed). Consider a Cartesian base change diagram*

$$\begin{tikzcd} X_{S'} \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S . \end{tikzcd}$$

Then for any Abelian torsion sheaf F on $X_{\text{ét}}$ the natural morphism of sheaves on $S'_{\text{ét}}$

$$g^*(R^r f_* F) \to R^r f'_*(g'^* F)$$

is an isomorphism.

Corollary 16.2.2. *Let $f : X \to S$ be a proper morphism of schemes and let F be an Abelian torsion sheaf on $X_{\text{ét}}$. For any geometric point $\overline{s} \to S$ the natural map*

$$(R^q f_* F)_{\overline{s}} \to H^q((X \times_S \overline{s})_{\text{ét}}, F_{|X \times_S \overline{s}})$$

is an isomorphism.

Proof. Apply Theorem 16.2.1 with $S' = \overline{s}$. ☐

Theorem 16.2.3. *Let A be a strictly henselian local ring and $S = \operatorname{Spec} A$. Let $f : X \to S$ be a proper morphism of schemes and X_0 the closed fiber of f. Then for any Abelian torsion sheaf F on $X_{\text{ét}}$ and any non-negative integer q, the natural restriction map $H^q(X_{\text{ét}}, F) \to H^q((X_0)_{\text{ét}}, F_{|X_0})$ is an isomorphism.*

Proposition 16.2.4. *Theorem 16.2.1 and Theorem 16.2.3 are equivalent.*

Proof. We first show that Theorem 16.2.1 implies Theorem 16.2.3. Let $s \in S$ be its closed point. As S is strictly henselian, $s = \overline{s}$ and $(R^q f_* F)_s \simeq H^q((X_0)_{\text{ét}}, F_{|X_0})$ by Corollary Corollary 16.2.2. On the other hand by the description of the stalks of étale sheaf given in Proposition Proposition 13.6.3,

$$(R^q f_* F)_{\overline{s}} \simeq H^q((X \times_S \mathcal{O}_{S,\overline{s}})_{\text{ét}}, F_{|X \times_S \mathcal{O}_{S,\overline{s}}}) .$$

But $\mathcal{O}_{S,\overline{s}} = A$ hence $X \times_S \mathcal{O}_{S,\overline{s}} = X$ and the conclusion follows.

Conversely let us show that Theorem 16.2.3 implies Theorem 16.2.1. Let $\overline{s'} \to S'$ be a geometric point of S' mapped to a geometric point $\overline{s} \to S$ of S. Then

$$(g^*(R^r f_* F))_{\overline{s'}} = (R^r f_* F)_{\overline{s}} = H^r((X \times_S \mathcal{O}_{S,\overline{s}})_{\text{ét}}, F)$$

while $(R^r f'_*(g'^* F))_{\overline{s'}} = H^r((X' \times_{S'} \mathcal{O}_{S',\overline{s'}})_{\text{ét}}, g'^* F)$. By Theorem 16.2.3 the natural map $H^r((X \times_S \mathcal{O}_{S,\overline{s}})_{\text{ét}}, F) \to H^r((X' \times_{S'} \mathcal{O}_{S',\overline{s'}})_{\text{ét}}, g'^* F)$ coincide with the identity map

$$H^r((X \times_S \overline{s})_{\text{ét}}, F) \to H^r((X' \times_{S'} \overline{s'})_{\text{ét}}, g'^* F) \ ,$$

hence the result. □

16.3. **Proof of Theorem 16.2.3.** The proof has three steps:

(a) Reduction to the case where F is a constant finite étale sheaf.
(b) Explicit computation of the cases $q = 0$ and $q = 1$ for $F = \mathbb{Z}/n\mathbb{Z}$.
(c) Reduction to the case where $f : X \to S$ is of relative dimension at most one; computation for $q = 2$.

16.3.1. *Notations.* If A denotes a local ring with maximal ideal $\mathfrak{m}$ and spectrum S and $f : X \to S$ a morphism, we denote by S_n the spectrum of $A/\mathfrak{m}^{n+1}$, by $X_n := X \times_S S_n$ the n-th infinitesimal neighborhood of the closed fiber X_0 in X and by $\hat{X} := \mathrm{colim}_n X_n \to \hat{S} := \mathrm{colim}_n S_n$ the formal scheme formal completion of X along X_0. Hence one has a commutative diagram:

(31)
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X_1 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & X_n \arrow[r, hook] \arrow[d] & \cdots \\ S_0 \arrow[r, hook] & S_1 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[r, hook] & \cdots \end{tikzcd}$$

16.3.2. *Reduction of Theorem 16.2.3 to the excellent case.* To prove Theorem 16.2.3 we will have to compare schemes over the stricty henselian ring A and schemes over its $\mathfrak{m}$-adic completion $\hat{A}$. For general A (even Noetherian) the flat map $A \to \hat{A}$ can have a pathological behaviour. The class of excellent rings was introduced by Grothendieck as a remedy to this problem. We recall the definition for completeness:

Definition 16.3.1. *A ring A is excellent if:*
- it is Noetherian,
- for every $\mathfrak{p} \in \mathrm{Spec}\, A$ the map $A_{\mathfrak{p}} \to \hat{A}_{\mathfrak{p}}$ is geometrically regular,
- for every finite A-algebra B the singular points of $\mathrm{Spec}\, B$ form a closed subset of $\mathrm{Spec}\, B$,
- A is universally catenary.

For us it will be sufficient to know that the strict henselization of a $\mathbb{Z}$-algebra of finite type is an excellent ring [c1]. [c1] reference?

Lemma 16.3.2. *If Theorem 16.2.3 is true for A excellent then it is true for all A.*

Proof. [c2] As any ring is a filtering colimit of its subrings which are of finite type as $\mathbb{Z}$-algebras and A is strictly henselian, A is a filtering colimit of A_i, $i \in I$, where A_i is the strict henselianization of a $\mathbb{Z}$-algebra of finite type. Hence $S = \mathrm{Spec}\, A$ is the projective limit of the S_i's, $S_i = \mathrm{Spec}\, A_i$. As $f : X \to S$ is of finite type, one can assume it is the limit of $f_i : X_i \to S_i$, $i \in I$ and F is the filtering colimit of constructible F_i on X_i. In the commutative diagram [c2] details!

$$\begin{tikzcd} \mathrm{colim}_i H^q((X_i)_{\acute{e}t}, F_i) \arrow[r] \arrow[d] & \mathrm{colim}_i H^q((X_{0,i})_{\acute{e}t}, F_i) \arrow[d] \\ H^q(X_{\acute{e}t}, F) \arrow[r] & H^q((X_0)_{\acute{e}t}, F) \end{tikzcd}$$

the vertical maps are isomorphism thanks to [SGA4, exp.VII, Th5.7]. As the A_i's are excellent, Theorem 16.2.3 in the excellent case implies that the top horizontal map is an isomorphism. Finally the bottom horizontal map is also an isomorphism and the result. □

Hence in the following we will be free to assume that A is excellent.

16.3.3. *Reduction to the case F constant.*

Proposition 16.3.3. *Under the hypotheses of Theorem 16.2.3, suppose that for any $n \geq 0$ and any finite morphism $X' \to X$, the restriction map $H^q(X'_{\text{ét}}, \mathbb{Z}/n) \to H^q((X'_0)_{\text{ét}}, \mathbb{Z}/n)$ is bijective for $q = 0$ and surjective for $q > 0$.*
Then for any abelian torsion sheaf F on $X_{\text{ét}}$ and any $q > 0$:

$$H^q(X_{\text{ét}}, F) \xrightarrow{\sim} H^q((X_0)_{\text{ét}}, F) \quad . \tag{32}$$

[c1] ici on utilise A noetherien?

Proof. First, any torsion sheaf F is a filtered colimit of constructible sheaves by Corollary 15.4.8 [c1]. As cohomology commutes with filtered colimits, it is enough to prove eq. (32) for F constructible.

The proof for F constructible works as follows:

(1) $H^q(X, \cdot) : \mathbf{Ab}_c(X_{\text{ét}}) \to \mathbf{Ab}$ and $H^q(X_0, \cdot) : \mathbf{Ab}_c(X_{\text{ét}}) \to \mathbf{Ab}$ are cohomological functors. Denote by $\varphi^q : H^q(X, \cdot) \to H^q(X_0, \cdot)$ the natural morphism.

(2) The functor $H^q(X, \cdot) : \mathbf{Ab}_c(X_{\text{ét}}) \to \mathbf{Ab}$ is effaceable for $q > 0$. Indeed, let $F \in \mathbf{Ab}_c(X_{\text{ét}})$. The sheaf $G' := \mathrm{God}^0(F) = \prod_{x \in X} i_{\overline{x}*} F_{\overline{x}}$ is an étale torsion sheaf on X which is flasque. Writing G' as a filtered colimit of constructible subscheaves, we see that there exists $F \subset G \subset G'$ with G constructible such that $H^q(X_{\text{ét}}, G) = 0$ for all $q > 0$.

(3) Every object of $\mathbf{Ab}_c(X_{\text{ét}})$ is a sub-object of

$$\mathcal{E} := \{\prod_i p_{i*} C_i, \quad p_i : X_i \to X \text{ finite}, \quad C_i \text{ constant}\} \quad .$$

The result then follows from the equivalence $(i) \Leftrightarrow (ii)$ in the following general homological lemma, whose proof by induction on q is left to the reader:

Lemma 16.3.4. *Let $\mathcal{A}$ be an Abelian category, $T^\bullet, T'^\bullet : \mathcal{A} \to \mathbf{Ab}$ be two cohomological functors, and $\mathcal{E} \subset \mathcal{A}$ a full subcategory such that any object of $\mathcal{A}$ is a sub-object of an object of $\mathcal{E}$. Suppose T^q is effaceable for all positive q.*
Let $\varphi : T^\bullet \to T'^\bullet$ be a morphism of cohomological functors. The following conditions are equivalent:

(i) *$\varphi^q(A)$ is a bijection for all $q \geq 0$ and all objects $A \in \mathcal{A}$.*
(ii) *$\varphi^0(M)$ is a bijection and $\varphi^q(M)$ is a surjection for all $q > 0$ and all objects $M \in \mathcal{E}$.*
(iii) *$\varphi^0(A)$ is an isomorphism for all $A \in \mathcal{A}$ and T'^q is effaceable for all $q > 0$.*

□

16.3.4. *The case $q = 0$, F constant (not necessarily finite).* If Y is a scheme and F a constant sheaf on $Y_{\text{ét}}$, $H^0(Y_{\text{ét}}, F) = F^{\pi_0(Y)}$. Hence Theorem 16.2.3 in this case follows from Zariski's connexity theorem:

Proposition 16.3.5. *Let A be a local henselian noetherian ring, $S = \operatorname{Spec} A$ and $f : X \to S$ a proper morphism. Then the natural morphism*

$$\pi_0(X_0) \to \pi_0(X)$$

is an isomorphism.

Proof. Equivalently we have to show that the set $OC(X)$ of clopen (closed and open) subsets of X are in bijection with the set $OC(X_0)$ of clopen subsets of X_0. As $OC(X)$ (resp. $OC(X_0)$) is in bijection with the set $\mathrm{Idem}\,\Gamma(X, \mathcal{O}_X)$ (resp. $\mathrm{Idem}\,\Gamma(X_0, \mathcal{O}_{X_0})$) we have to show that the natural map

$$\mathrm{Idem}\,\Gamma(X, \mathcal{O}_X) \to \mathrm{Idem}\,\Gamma(X_0, \mathcal{O}_{X_0})$$

is an isomorphism. Recall:

Theorem 16.3.6. *(Finiteness of proper morphisms, see* [EGAIII, 3.2]*) Let S be a locally Noetherian scheme and $f : X \to S$ a proper morphism. Then for any quasi-coherent $\mathcal{O}_X$-module F and any non-negative integer q the sheaf $R^q f_* F$ is $\mathcal{O}_S$-coherent.*

Applying this result for $q = 0$ gives in our case that $\Gamma(X, \mathcal{O}_X)$ is a finite A-algebra. As A is henselian, it follows that $\Gamma(X, \mathcal{O}_X)$ is a product of local rings, equivalently that the natural injection $\mathrm{Idem}\,\Gamma(X, \mathcal{O}_X) \to \mathrm{Idem}\,(\Gamma(\widehat{X, \mathcal{O}_X}))$ is a bijection [c1]. [c1] cf. Raynaud Prop.4 p.2]

On the other hand f proper also implies (see [EGAIII, 4.1]) that

$$\Gamma(\widehat{X, \mathcal{O}_X}) \xrightarrow{\sim} \lim_n \Gamma(X_n, \mathcal{O}_{X_n}),$$

hence

$$\mathrm{Idem}\,\Gamma(X, \mathcal{O}_X) \xrightarrow{\sim} \lim_n \mathrm{Idem}\,\Gamma(X_n, \mathcal{O}_{X_n}) \ .$$

But X_n and X_0 have the same underlying topological space thus the righthandside coincide with $\mathrm{Idem}\,\Gamma(X_0, \mathcal{O}_{X_0})$. □

16.3.5. *Case $q = 1$ and $F = \mathbb{Z}/n\mathbb{Z}$.* The group $H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ parametrizes isomorphism classes of étale Galois covers of X with Galois group $\mathbb{Z}/n\mathbb{Z}$ [c2]. Hence the result in this case follows from the more general. [c2] Démontré où?

Proposition 16.3.7. *Let A be an henselian excellent ring with spectrum S. Let $f : X \to S$ be a proper morphism. Then the natural functor*

$$\mathrm{FEt}(X) \to \mathrm{FEt}(X_0)$$

is an equivalence of categories (equivalently if X_0 is connected: the natural morphism $\pi_1(X_0) \to \pi_1(X)$ is an isomorphism).

Proof. If $X', X'' \in \mathrm{FEt}(X)$, an X-morphism from X' to X'' is defined by its graph, which is clopen in $X' \times_X X''$. Hence the full faithfullness of $\mathrm{FEt}(X) \to \mathrm{FEt}(X_0)$ follows from Proposition 16.3.5 applied to the proper morphism $X' \times_X X'' \to S$.

It remains to show that $\mathrm{FEt}(X) \to \mathrm{FEt}(X_0)$ is essentially surjective. Hence it is enough to show that any étale cover $h_0 : Y_0 \to X_0$ extends to an étale cover $h : Y \to X$.

Let us first assume $S = \hat{S}$. In this case let us consider the commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d] & \hat{X} \arrow[r, "j"] \arrow[d] & X \arrow[d] \\ s \arrow[r] & S \arrow[r, equal] & S, \end{tikzcd}$$

where the map j is a flat morphism in the category of locally ringed spaces. We want to show that the composite

$$\mathrm{FEt}(X) \xrightarrow{j^*} \mathrm{FEt}(\hat{X}) \xrightarrow{i^*} \mathrm{FEt}(X_0)$$

is essentially surjective.

As étale covers do not depend on nilpotents [c3], the finite étale cover $h_0 : Y_0 \to X_0$ can [c3] cite reference be uniquely extended to an étale cover $h_n : Y_n \to X_n$ for all $n \geq 0$, hence to an étale cover $\mathcal{Y} \to \hat{X}$ in the category of formal schemes [c4]. It remains to show that the formal étale scheme $\mathcal{Y} \to \hat{X}$ is the completion of an étale cover $h : Y \to X$ along Y_0. Recall:

[c4] Definition?

Theorem 16.3.8. *(Grothendieck's algebraization theorem, [EGAIII, 5]) Let S be a complete local ring and $f : X \to S$ a proper morphism. Then:*

(a) *The functor* $\mathrm{Coh}(\mathcal{O}_X) \xrightarrow{j^*} \mathrm{Coh}(\mathcal{O}_{X_0})$ *is an equivalence of categories.*
(b) *The module $M \in \mathrm{Coh}(\mathcal{O}_X)$ is locally free at any point of X_0 if and only if $M_n \in \mathrm{Coh}(\mathcal{O}_{X_n})$ is locally free for any non-negative integer n.*

This equivalence induces an equivalence between the category of finite X-schemes and the category of finite $\hat{X}$-schemes.

It follows from Theorem 16.3.8 that there exists a unique finite map $h : Y \to X$ such that $\mathcal{Y} \to \hat{X}$ is the completion of h. It remains to show that $h : Y \to X$ is étale.

On the one hand the locus of Y where $h : Y \to X$ is étale is open in Y [c1]. On the other hand any open subset of Y containing Y_0 is necessarily the all of Y as f is closed. Hence it is enough to show that $h : Y \to X$ is étale (i.e. flat and unramified) at every point y of Y_0.

[c1] reference

The sheaf $\mathcal{O}_Y$ is $\mathcal{O}_X$-flat if and only if it is a colimit of locally free $\mathcal{O}_X$-sheaves. It follows from Theorem 16.3.8(b) that being $\mathcal{O}_X$-free in restriction to X_0 is equivalent to being $\mathcal{O}_{\hat{X}}$-free on X_0. Hence the flatness of $h : Y \to X$ follows from the flatness of $\hat{Y} \to \hat{X}$, which holds true as it is a formal étale morphism.

Let us show that $\Omega^1_{Y/X_{|Y_0}}$ vanishes. By [Stacks Project, Lemma 28.32.10], $\Omega^1_{Y/X_{|Y_0}} = \Omega^1_{Y_0/X_0}$, hence vanishes as $h_0 : Y_0 \to X_0$ is étale.

In the general case, consider the commutative diagram:

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{i} & \hat{X} = \hat{\overline{X}} & \xrightarrow{j} & \overline{X} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s & \longrightarrow & \hat{S} & = & \hat{S} & \longrightarrow & S,
\end{array}$$

where the right hand square is Cartesian. Starting with the étale cover $h_0 : Y_0 \to X_0$, the previous case applied to the two left squares furnishes a finite étale cover $\overline{h} : \overline{Y} \to \overline{X}$ extending h_0. Recall:

Theorem 16.3.9. *(Artin's approximation theorem) Let $(A, \mathfrak{m}, k)$ be a local excellent ring and $F : A - \mathrm{Alg} \to$ **Sets** a functor locally of finite presentation. For every $\overline{\xi} \in F(\hat{A})$, there exists $\xi \in F(A)$ such that $\overline{\xi}$ and ξ have the same image in $F(k)$.*

Consider the functor $F : A - \mathrm{Alg} \to$ **Sets** which to an A-algebra B associates the set $\mathrm{FEt}(X \otimes_A B)/\sim$. One easily checks this is a functor of locally finite presentation (i.e. commutes with filtering colimits). It follows from Artin's Theorem 16.3.9 applied to $\overline{\xi} := [\overline{h} : \overline{Y} \to \overline{X}]$ that there exists $\xi = [h : Y \to X]$ a finite étale morphism whose restriction to Y_0 is h_0. $\square$

16.3.6. *Reduction to the case f projective of relative dimension at most 1.*

Proposition 16.3.10. *Suppose that the Proper Base Change theorem (PBC) holds true for $f : X \to S$ projective and S noetherian. Then it is true for general f.*

Proof. We admit the following:

Lemma 16.3.11. *(Chow's lemma, see* [EGAII, 6.5.1]*) Let $f : X \to S$ be a proper morphism. Then there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[dr, "f\circ h"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

such that $h : X' \to X$ is projective, surjective and an isomorphism over an open dense subset of X, and $f \circ h$ is projective.

Lemma 16.3.12. *Under the assumptions of Lemma 16.3.11, if (PBC) is true for h and $f \circ h$ then it is true for f.*

Proof. As f is projectif and surjectif let us first check that the natural adjunction morphism of sheaves $\varphi : F \to h_*h^*F$ is injective. If $\overline{x} \to X$ is a geometric point the (PBC) for h and $q = 0$ implies that $(h_*h^*F)_{\overline{x}} = \Gamma(X'_{\overline{x}}, F_{|X'_{\overline{x}}})$. Hence we can assume that $X = \overline{x}$ and $h : X' \to \overline{x}$ is projective. In this case the identity of the abelian group F factorizes

$$F \xrightarrow{\varphi} \Gamma(X'_{\overline{x}}, F_{|X'_{\overline{x}}}) \to (h^*F)_{\overline{x'}} \simeq F_{\overline{x}} \simeq F,$$

which shows that φ is injective.

Without loss of generality we can assume that $F = h_*L$, with L an étale torsion flasque sheaf on X'. Indeed, choose an injection $h^*F \hookrightarrow L^0$ with L^0 torsion flasque. Thus $F \hookrightarrow h_*L^0$. Replacing F by $\mathrm{Coker}\,(F \hookrightarrow h_*L^0)$ and iterating, one obtains a resolution

$$F \simeq L^\bullet \quad ,$$

where the L^i's are étale torsion flasque sheaves on X'. We want to show $R\Gamma(X, h_*L^\bullet) \xrightarrow{\sim} R\Gamma(X_0, (h_*L^\bullet)_{|X_0})$. Considering the hypercohomology spectral sequence, it is enough to show that for each i, the morphism $R\Gamma(X, h_*L^i) \to R\Gamma(X_0, (h_*L^i)_{|X_0})$ is an isomorphism.

Consider the commutative diagram

$$\begin{tikzcd} R\Gamma(X, h_*L) \arrow[r, "{[0]}"] \arrow[d, "{[1]}"] & R\Gamma(X_0, (h_*L)_{|X_0}) \arrow[d, "{[2]}"] \\ R\Gamma(X', L) \arrow[dr, "{[3]}"] & R\Gamma(X_0, Rh_*(L_{|X_0})) \arrow[d, "\sim"] \\ & R\Gamma(X'_0, L_{|X'_0}). \end{tikzcd}$$

To show that [0] is an isomorphism, it is enough to show that [1], [2], [3] are isomorphisms.

For [1]: as L is flasque, $h_*L = Rh_*L$ hence the result.

For [3]: this is (PBC) for the projective morphism $f \circ h$.

For [2]: apply (PBC) to the projective morphism $h : X' \to X$. It follows that

$$Rh_*(L_{|X'_0}) \simeq (Rh_*L)_{|X_0} \simeq (h_*L)_{|X_0} \quad ,$$

where the last equality follows from the fact that L is flasque. □

□

Proposition 16.3.13. *If (PBC) is true for $f : X \to S$ projective of relative dimension at most 1 then (PBC) is true.*

Proof. From Proposition 16.3.10, it is enough to prove that if one has a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & P := \mathbf{P}^n_S \arrow[dl] \\ S = \operatorname{Spec} A & \end{tikzcd}$$

then the morphism $R\Gamma(X, F) \to R\Gamma(X_0, F_{|X_0})$ is an isomorphism for any torsion étale sheaf F on X.

As the diagram

$$\begin{tikzcd} R\Gamma(X, F) \arrow[r] \arrow[d, "\sim"'] & R\Gamma(X_0, F_{|X_0}) \arrow[d, "\sim"] \\ R\Gamma(P, i_* F) \arrow[r, "{[1]}"] & R\Gamma(P_0, i_* F_{|P_0}) \end{tikzcd}$$

commutes, it is enough to prove that [1] is an isomorphism, i.e we are reduced to the case $X = \mathbf{P}^n_S$.

For $n = 1$ it follows from our hypothesis.

Let $n > 1$ and suppose by induction hat (PBC) is true for any projective morphism of relative dimension at most $n - 1$. Let $t_0, \cdots, t_n$ be homogeneous coordinates on $\mathbf{P}^n_S$. Consider the pencil of hypersurfaces $H_\lambda := \{\lambda t_0 + (1 - \lambda)t_1 = 0\}$, $\lambda \in \mathbf{P}^1_S$ with base locus $\Delta := H_0 \cap H_1$ and the blow-up diagram

$$\begin{tikzcd} X' := \mathrm{Bl}_\Delta X \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \mathbf{P}^1_S \arrow[r, "g"'] & S. \end{tikzcd}$$

By Lemma 16.3.12 (PBC) for f will follow from (PBC) for h and $f \circ h = g \circ f'$.

But h is projective of relative dimension at most one hence (PBC) holds for h by hypothesis.

For $g \circ f'$, consider the diagram

$$\begin{tikzcd} R\Gamma(X', F) \arrow[rr, "{[0]}"] \arrow[d, equal] & & R\Gamma(X'_0, F_{|X'_0}) \arrow[d, equal] \\ R\Gamma(\mathbf{P}^1_S, Rf'_* F)) \arrow[d, "{[1]}"] & & R\Gamma(\mathbf{P}^1_{S,0}, Rf'_{0*}(F_{|X'_0})) \\ R\Gamma(\mathbf{P}^1_{S,0}, (Rf'_* F)_{\mathbf{P}^1_{S,0}}). \arrow[urr, "{[2]}"] & & \end{tikzcd}$$

The morphism [1] is the (PBC) for g, hence is an isomorphism as g is projective of relative dimension 1. By induction on n, the morphism [2] is an isomorphism as f' is projective of relative dimension at most $n - 1$.

$\square$

16.3.7. *The case f projective of relative dimension at most 1: end of the proof of the Proper Base Change theorem.* If follows from the previous steps it is enough to show:

Proposition 16.3.14. *Let S be the spectrum of an excellent strictly henselian ring $(A, \mathfrak{m}, k)$ and $f : X \to S$ a proper morphism. Then for any integers $n > 1$ and $q \geq 0$ the restriction morphism $H^q(X_{\acute{e}t}, \mathbb{Z}/n) \to H^q((X_0)_{\acute{e}t}, \mathbb{Z}/n)$ is an isomorphism for $q = 0$ and a surjection for $q > 0$.*

Proof. The cases $q = 0$ and $q = 1$ are treated in Proposition 16.3.5 and Proposition 16.3.7 respectively for any X.

Under our assumptions X_0 is a point or a projective curve over the algebraically closed field k hence $H^q(X_0, \mathbb{Z}/n) = 0$ by [SGA4, IX 5.7] (we proved it for X_0 *smooth* projective and n invertible on X_0 in Corollary 14.0.3).

It remains to prove the statement for $q = 2$. Without loss of generality we can assume that $n = l^r$, l prime, then $n = l$. There are two cases:

Either $l = p = \mathrm{char} k$, in which case $H^2(X_0, \mathbb{Z}/p) = 0$. Indeed, the Artin-Schreier exact sequence of étale sheaves on X_0

$$0 \to \mathbb{Z}/p \to \mathcal{O}_{X_0} \stackrel{F-1}{\to} \mathcal{O}_{X_0} \to 0$$

induces an exact sequence of groups

$$H^1(X_0, \mathcal{O}_{X_0}) \stackrel{F-1}{\to} H^1(X_0, \mathcal{O}_{X_0}) \to H^2(X_0, \mathbb{Z}/p) \to 0 \ .$$

The result follows from the semi-linear algebra lemma:

Lemma 16.3.15. *c1 Let k be a separably closed field of positive characteristic p, V a finite dimensional k-vector space and $\varphi : V \to V$ and F-linear map. Then $F - 1 : V \to V$ is surjective.* c1 reference

In the case $l \neq p$, identify $\mathbb{Z}/n = \boldsymbol{\mu}_n$. The Kummer exact sequence of étale sheaves on X_0

$$0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

induces an exact sequence

$$\mathrm{Pic}\,(X_0) \to H^2(X_0, \boldsymbol{\mu}_n) \to H^2(X_0, \mathbb{G}_m) = 0$$

(once more we showed this exact sequence for X_0 smooth). c2 Consider the commutative diagram c2 reference

$$\begin{tikzcd} \mathrm{Pic}\,(X) \arrow[r, two heads, "{[1]}"] \arrow[d] & H^2(X_{\acute{e}t}, \boldsymbol{\mu}_n) \arrow[d] \\ \mathrm{Pic}\,(X_0) \arrow[r, two heads] & H^2((X_0)_{\acute{e}t}, \boldsymbol{\mu}_n). \end{tikzcd}$$

The surjectivity of [1] follows from the

Proposition 16.3.16. *Let $S = \mathrm{Spec}\, A$ with A a local noetherian henselian ring and $f : X \to S$ a proper morphism of relative dimension at most 1. Then the restriction map $\mathrm{Pic}\, X \to \mathrm{Pic}\, X_0$ is surjective.*

Proof. One can assume without loss of generality that S is excellent.

Consider the diagram eq. (31) Let L_0 be an invertible sheaf of X_0 and suppose that L_0 has been extended to an invertible sheaf L_n on X_n. The obstruction to extending L_n to X_{n+1} lies in $H^2(X_0, \mathfrak{m}^{n+1}/\mathfrak{m}_n) = H^2(X_0, \mathcal{O}_{X_0}) \otimes_A \mathfrak{m}^{n+1}/\mathfrak{m}^n$, which vanishes has $\dim X_0 \leq 1$.

Considering again the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d] & \hat{X} = \hat{\overline{X}} \arrow[r, "j"] \arrow[d] & \overline{X} \arrow[r] \arrow[d] & X \arrow[d] \\ s \arrow[r] & \hat{S} \arrow[r, equal] & \hat{S} \arrow[r] & S, \end{tikzcd}$$

it follows that there exists a formal invertible sheaf $\mathcal{L}$ on $\hat{\overline{X}}$ extending L_0.

By Grothendieck's Theorem 16.3.8, there exists a unique invertible sheaf $\overline{L}$ on $\overline{X}$ such that $\hat{\overline{L}} \simeq \mathcal{L}$.

Consider the functor $F : A - \mathrm{Alg} \to \mathbf{Sets}$ which to an A-algebra B associates the set $\mathrm{FEt}(X \otimes_A B)/\sim$. One easily checks this is a functor of locally finite presentation (i.e. commutes with filtering colimits). It follows from Artin's Theorem 16.3.9 applied to $\overline{\xi} := [\overline{h} : \overline{Y} \to \overline{X}]$ that there exists $\xi = [h : Y \to X]$ a finite étale morphism whose restriction to Y_0 is h_0. □

□

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