ABELIAN DIFFERENTIALS AND THEIR PERIODS: THE BI-ALGEBRAIC POINT OF VIEW

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Abstract. We study the transcendence of periods of abelian differentials, both at the arithmetic and functional level, from the point of view of the natural bi-algebraic structure on strata of abelian differentials. We characterise geometrically the arithmetic points, study their distribution, and prove that in many cases the bi-algebraic curves are the linear ones.

Contents

1. Introduction

1.1. Abelian differentials, their periods, and their bi-algebraic geometry.

An abelian differential (or translation surface) is a pair (C, ω), where C denotes a smooth irreducible complex projective curve and $\omega \in H^0(C, \Omega^1_C) \setminus \{0\}$ is a non-zero algebraic one-form on C. Its periods are the complex numbers $\int_\gamma \omega$, for γ an element in the relative homology group $H_1(C^{\mathrm{an}}, Z(\omega); \mathbb{Z})$, where $Z(\omega)$ denotes the finite set of zeroes of ω in the compact Riemann surface C^{an} associated to C. Such a period is said to be *pure* if γ belongs to the subgroup $H_1(C^{\mathrm{an}}, \mathbb{Z})$ of $H_1(C^{\mathrm{an}}, Z(\omega); \mathbb{Z})$. The goal of this paper is to study the transcendence theory of periods of abelian differentials, both at the arithmetic and functional level.

At the arithmetic level: given an abelian differential (C, ω) defined over $\overline{\mathbb{Q}}$, we want to study the transcendence properties of its periods. This is a classical topic. A famous result of Schneider [Schn41], generalizing Siegel [Sie32], says that at least one pure period of any abelian differential defined over $\overline{\mathbb{Q}}$ (meaning that both C and ω are defined over $\overline{\mathbb{Q}}$) is a transcendental number. From this point of view, it is natural to consider abelian differentials, as well as their periods, only *up to scaling*: pairs $(C, [\omega])$ with $[\omega] \in \mathbf{P}H^0(C, \Omega^1_C)$ and their period lines $\left[\int_{\gamma_0} \omega, \dots, \int_{\gamma_d} \omega\right] \in \mathbf{P}^d(\mathbb{C})$, where $d = \dim H_1(C^{\mathrm{an}}, Z(\omega); \mathbb{Q}) - 1$. Our main arithmetic objects of interest are the "least transcendent" differentials: the $(C, [\omega])$s defined over $\overline{\mathbb{Q}}$ whose period lines $\left[\int_{\gamma_0} \omega, \dots, \int_{\gamma_d} \omega\right]$ belong to $\mathbf{P}^d(\overline{\mathbb{Q}})$. Such an abelian differential will be said to be *arithmetic*.

At the functional level: given a *family* of abelian differentials, we want to study the algebraic relations satisfied by their periods. To do so, we regroup abelian differentials according to their combinatorial type. Let g be a positive integer and let α be a partition of $2g-2$, of length n_α. If we ignore orbifold phenomena (see Section 3.1), the abelian differentials (C,ω) with zeroes of multiplicity α are parametrized by a smooth complex quasi-projective algebraic variety H_α (not necessarily irreducible), called a *stratum of abelian differentials*. It is naturally realized as a locally closed algebraic subvariety defined over $\mathbb{Q}$ of the Hodge bundle $\Omega^1\mathrm{M}_g$ over the coarse moduli space M_g of smooth complex projective curves of genus g. Our main geometric object of study will be the projectivization $\mathrm{S}_\alpha \subset \mathbf{P}\Omega^1\mathrm{M}_g$ of H_α. This is a smooth quasi-projective variety of dimension $d_\alpha := 2g-2+n_\alpha$, parametrizing pairs $(C,[\omega])$ of type α. The variety H_α is a principal $\mathbf{G}_m$-bundle over S_α.

According to a remarkable theorem of Veech [Ve90, Theor. 7.15], the local geometry of the complex manifolds $\mathrm{H}_\alpha^{\mathrm{an}}$ and $\mathrm{S}_\alpha^{\mathrm{an}}$, analytifications of H_α and S_α respectively, can be completely described in terms of periods. Let $x_0 := (C_0,\omega_0) \in \mathrm{H}_\alpha^{\mathrm{an}}$, let $\mathrm{V}_{\alpha,\mathbb{Z}} := H^1(C_0, Z(\omega_0);\mathbb{Z})$ and $\mathrm{V}_\alpha = \mathrm{V}_{\alpha,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. On a small open simply connected neighborhood $U \subset \mathrm{H}_\alpha^{\mathrm{an}}$ around x_0, the map $D_U : U \to \mathrm{V}_\alpha \simeq \mathbb{C}^{d_\alpha+1}$ which associates to $(C,\omega) \in U$ the parallel transport in V_α of the Betti cohomology class defined by ω in $H^1(C^{\mathrm{an}}, Z(\omega);\mathbb{C})$, identified with the vector of periods of ω, is a bi-holomorphism. These *period charts* define an *integral linear structure* on $\mathrm{H}_\alpha^{\mathrm{an}}$, namely an atlas of charts with value in the complex vector space V_α, whose transition functions are locally constant elements of the integral linear group $\mathbf{GL}(\mathrm{V}_{\alpha,\mathbb{Z}})$. This integral linear structure on $\mathrm{H}_\alpha^{\mathrm{an}}$ induces a *linear projective structure* on $\mathrm{S}_\alpha^{\mathrm{an}}$, namely an atlas of charts with value in the projective space $\mathbf{P}\mathrm{V}_\alpha$, whose transition functions are locally constant elements of the integral projective linear group $\mathbf{PGL}(\mathrm{V}_{\alpha,\mathbb{Z}})$. These period charts of $\mathrm{S}_\alpha^{\mathrm{an}}$ are highly transcendental with respect to the algebraic structure of S_α: the maps D_U are defined via the non-algebraic operations of parallel transport and integration. We would like to understand the algebraic subvarieties of S_α whose periods also satisfy many algebraic relations.

To study both the functional and arithmetic transcendence properties of S_α, we introduce the following subvarieties, which should encode most of its interesting geometric and arithmetic information:

Definition 1.1. *(1) A bi-algebraic subvariety $W \subset \mathrm{S}_\alpha$ is an irreducible closed algebraic subvariety W of S_α such that W^{an} is algebraic in the period charts: the relative periods of ω satisfy (up to scaling) exactly* $\mathrm{codim}_{\mathrm{S}_\alpha} W$ *independent algebraic relations over $\mathbb{C}$ when $(C,[\omega])$ ranges through W.*

(2) A $\overline{\mathbb{Q}}$-bi-algebraic subvariety of S_α is a bi-algebraic subvariety $W \subset \mathrm{S}_\alpha$ such that both W^{an} (in the period charts) and W are defined over $\overline{\mathbb{Q}}$.

Remark 1.2. In particular, the $\overline{\mathbb{Q}}$-bi-algebraic points of S_α are by definition the arithmetic differentials. For simplicity, from now we will call *arithmetic points* the $\overline{\mathbb{Q}}$-bi-algebraic points of S_α.

1.2. **Linear subvarieties.** A priori, the simplest bi-algebraic subvarieties of S_α are the *linear ones*: the ones for which the algebraic relations between their periods are linear.

Definition 1.3. *A linear subvariety $W \subset \mathrm{S}_\alpha$ is an irreducible closed algebraic subvariety such that W^{an} is (projectively) linear in the period charts. It is $\overline{\mathbb{Q}}$-linear if moreover both W and W^{an} (in the period charts) are defined over $\overline{\mathbb{Q}}$.*

1.2.1. *Invariant linear subvarieties.* A particular class of linear subvarieties of S_α has been studied in great depth by dynamicists in the last twenty years: the so-called *invariant* ones. To explain the terminology, notice that the integral linear structure on $\mathrm{H}_\alpha^{\mathrm{an}}$ endows it with a a real analytic, non-algebraic action of $\mathbf{GL}^+(2,\mathbb{R})$. Indeed, the group $\mathbf{GL}^+(2,\mathbb{R})$ acts naturally on $\mathrm{V}_{\alpha,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, by identifying $\mathbb{C}$ with $\mathbb{R}^2$ and extending the natural action of $\mathbf{GL}^+(2,\mathbb{R})$ on $\mathbb{R}^2$ coordinate-wise on $\mathrm{V}_{\alpha,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}^2$. This action in period charts commutes with the one of $\mathbf{GL}(\mathrm{V}_{\alpha,\mathbb{Z}})$, hence descends to $\mathrm{H}_\alpha^{\mathrm{an}}$ (but not to $\mathrm{S}_\alpha^{\mathrm{an}}$). A closed irreducible algebraic subvariety W of S_α is said to be *invariant* if the analytification of its preimage in H_α is $\mathbf{GL}^+(2,\mathbb{R})$-invariant. One easily shows that a closed irreducible subvariety $W \subset \mathrm{S}_\alpha$ is invariant if and only if W is linear and W^{an} is defined over $\mathbb{R}$ (in the period charts). Prominent examples of invariant linear subvarieties of S_α are the famous *Teichmüller curves*: the projection in S_α of closed $\mathbf{GL}^+(2,\mathbb{R})$-orbits in $\mathrm{H}_\alpha^{\mathrm{an}}$. The Teichmüller curves are moreover $\overline{\mathbb{Q}}$-linear. Abelian differentials $(C,[\omega])$ belonging to a Teichmüller curve are called *Veech surfaces*. We refer, for instance, to [McM03a], [Cal04], [Mö06b], [McM07], [Wr15], [Mö18] for more details on Veech surfaces and Teichmüller curves. Generalizing Ratner's theory to this non-homogeneous setting, dynamicists proved the following beautiful result, see [McM07], [EM18], [EMM15], [Wr14], [Fil16b]:

Theorem 1.4 (McMullen in genus 2; Eskin-Mirzakhani-Mohammadi; Wright; Filip)**.** *The topological closure of any $\mathbf{GL}^+(2,\mathbb{R})$-orbit in $\mathrm{H}_\alpha^{\mathrm{an}}$ is the cone over an invariant $\overline{\mathbb{Q}}$-linear algebraic subvariety of S_α.*

1.2.2. *Linear Hurwitz spaces.* Let us describe nice examples of linear subvarieties of S_α which are not invariant: the linear Hurwitz spaces. The complex manifold $\mathrm{S}_\alpha^{\mathrm{an}}$ is naturally endowed with a codimension $2g$ foliation: the *isoperiodic foliation*, see Section 6.1. The isoperiodic leaf of a point $(C_0,[\omega_0]) \in \mathrm{S}_\alpha(\mathbb{C})$ consists locally of the nearby points $(C,[\omega])$ whose vectors of *pure* periods coincide with the one of $(C_0,[\omega_0])$. The dynamics of the isoperiodic foliation on $\mathrm{S}_\alpha^{\mathrm{an}}$, in particular its algebraic leaves, has been recently studied in depth, see [CDF15]. Let E be an elliptic curve and let $H_{g,d}(E)$ be the coarse Hurwitz moduli space classifying smooth projective curves C of genus g together with a degree d branched cover $C \to E$. The pullback, along the branched cover, of the 1-form on E provides a 1-form on C, hence a morphism $H_{g,d}(E) \to \mathbf{P}\Omega^1\mathrm{M}_g$. It is not hard to see that it is an immersion and that its image is contained in an isoperiodic leaf. One easily checks that the connected components of $(H_{g,d}(E) \cap \mathrm{S}_\alpha)^{\mathrm{an}}$ are saturated with respect to the foliation, hence coincide with closed isoperiodic leaves, and thus are linear subvarieties. It is proved in [CDF15] that for $g > 2$ these are the only algebraic leaves of the isoperiodic foliation. If the elliptic curve E is not defined over $\overline{\mathbb{Q}}$, these linear subvarieties are not defined over $\overline{\mathbb{Q}}$. In particular they are not invariant.

1.2.3. *Other examples.* In [Mö08, Section 6], Möller exhibits finitely many non-isoperiodic linear subvarieties not defined over $\mathbb{R}$: they are constructed from the families of cyclic covers of $\mathbf{P}^1$ studied by Deligne-Mostow [DM86].

2. Results and conjectures

The bi-algebraic format for S_α introduced in Definition 1.1 is a special instance of the general bi-algebraic format described for instance in [KUY18]. Given a complex algebraic variety S with an infinite topological fundamental group, this format proposes to emulate an algebraic structure on the universal cover $\widetilde{S^{\mathrm{an}}}$ of S^{an} using periods of algebraic differential forms on S; to analyse the bi-algebraic and $\overline{\mathbb{Q}}$-bi-algebraic subvarieties of S; to study the trancendence properties of the uniformizing map $\pi : \widetilde{S^{\mathrm{an}}} \to S^{\mathrm{an}}$ with respect to the emulated algebraic structure on $\widetilde{S^{\mathrm{an}}}$ and the algebraic structure of S (Ax-Lindemann and Ax-Schanuel heuristic); and to analyse the distribution of the arithmetic ($=\overline{\mathbb{Q}}$-bi-algebraic) points (Zilber-Pink heuristic). This format has first been successfully applied to tori, abelian varieties and Shimura varieties, see [KUY18] for a survey and references; then to general varieties S endowed with a variation of (possibly mixed) Hodge structures [K17], [BT19], [BKT20], [KO21], [BKU21]. In this paper we go one step further, moving to an even less homogeneous context which creates considerable new difficulties.

2.1. **Arithmetic points.**

2.1.1. *Geometric characterization of the arithmetic points.* Our first main result in this paper is the geometric elucidation of the arithmetic points in S_α. We will use the following

Definition 2.1. *Let $(C, [\omega])$ be an abelian differential. We define:*

(1) the abelian variety $A_{[\omega]}$, as the smallest factor of the Albanese $\mathrm{Alb}(C)$ (canonically identified with the Jacobian $\mathrm{Jac}\,(C)$) whose tangent bundle contains ω. The degree *$d_{[\omega]}$ of $(C, [\omega])$ is the dimension $\dim_{\mathbb{C}} A_{[\omega]}$. Thus $1 \leq d_{[\omega]} \leq g(C)$, where $g(C)$ is the genus of C.*

(2) the line $[\omega_{A_{[\omega]}}] \in \mathbf{P}H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$, as the unique line such that $[\omega] = [\mathrm{alb}^*_{[\omega]}\, \omega_{A_{[\omega]}}]$. Here $\mathrm{alb}_{[\omega]} : C \to A_{[\omega]}$ is the composition of the Albanese map $\mathrm{alb} : C \to \mathrm{Alb}(C)$ with the projection of $\mathrm{Alb}(C)$ onto $A_{[\omega]}$ (the maps alb and $\mathrm{alb}_{[\omega]}$ are uniquely defined up to a translation).

We also refer to Section 3.8 for our (standard) terminology concerning complex multiplication (CM).

Theorem 2.2. *A point $(C, [\omega]) \in \mathrm{S}_\alpha(\overline{\mathbb{Q}})$ is arithmetic if and only if the following conditions are satisfied:*

(1) The complex curve C is defined over $\overline{\mathbb{Q}}$: $C = C_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

(2) "The differential $[\omega]$ is an eigenform for complex multiplication"; namely:
   (a) The abelian variety $A_{[\omega]}$ has complex multiplication and is isotypic.
   (b) The line $[\omega_{A_{[\omega]}}] \in \mathbf{P}H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$ is an eigenline for the K-action on $H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$, where K denotes the CM field center of $\mathrm{End}_{\mathbb{Q}} A_{[\omega]}$.

(3) Given any two points x and y in $Z([\omega])$, the difference $\mathrm{alb}_{[\omega]}(x) - alb_{[\omega]}(y)$ is a torsion point of $A_{[\omega]}$.

The main tool in the proof of Theorem 2.2 is, as for most results nowadays in arithmetic transcendence theory, Wüstholz' Analytic Subgroup Theorem [Wus87].

2.1.2. *Arithmetic points and Veech surfaces.* There is a striking similarity between Theorem 2.2 characterising arithmetic points, and properties of the Veech surfaces. In [Mö06b, Theorem 2.7] and [Mö06a, Theorem 3.3], Möller shows that Veech surfaces satisfy conditions similar to, but essentially weaker than, the ones of Theorem 2.2: if an abelian differential $(C, [\omega])$ is a Veech surface, then:

(1V) The factor $A_{[\omega]}$ of $\mathrm{Alb}(C)$ has *real* multiplication by a totally real field K_0 satisfying $[K_0 : \mathbb{Q}] = d_{[\omega]}$ (in particular $A_{[\omega]}$ is isotypic);

(2V) The line $[\omega_{A_{[\omega]}}]$ is an eigenline for the K_0-action on $H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$;

(3V) Condition (3) of Theorem 2.2 holds.

Moreover the degree $d_{[\omega]}$ of a Veech surface $(C, [\omega])$ coincides with the degree of the Teichmüller curve $\Gamma\backslash\mathfrak{H} \subset \mathrm{S}^{\mathrm{an}}_\alpha$ it generates, defined as the degree of its trace field $\mathbb{Q}[\mathrm{tr}\,\gamma,\ \gamma \in \Gamma]$.

It follows immediately from Theorem 2.2 that any arithmetic point $(C, [\omega])$ with $A_{[\omega]}$ simple also satisfy the conditions $(1V)$, $(2V)$ and $(3V)$ (for K_0 the maximal totally real subfield of the CM field K). On the other hand, these conditions are in general not sufficient for an abelian differential (C, ω) to be a Veech surface. The following proposition clarifies the relation between arithmetic points and Veech surfaces, showing that arithmetic points of degree 1 are familiar, while arithmetic points of higher degree are mysterious and do not seem related to the $\mathbf{GL}^+(2, \mathbb{R})$-action on $\mathrm{H}^{\mathrm{an}}_\alpha$:

Proposition 2.3. *The arithmetic points of $\mathrm{S}^{\mathrm{an}}_\alpha$ of degree 1 are Veech surfaces. On the other hand there exist arithmetic points of degree at least 2 which are not Veech surfaces.*

2.1.3. *Distribution of arithmetic points.* The arithmetic points are many:

Proposition 2.4. *Arithmetic points of degree 1 are (analytically) dense in $\mathrm{S}_\alpha^{\mathrm{an}}$ for all α.*

Remark 2.5. This is where considering abelian differentials *up to scaling* is crucial. If we were to consider strata H_α of abelian differentials without scaling, with their natural integral linear structure, Schneider's theorem implies that there are no arithmetic points at all in H_α!

On the other hand, in stark contrast with the bi-algebraic geometry of tori, abelian varieties or Shimura varieties, we show that the arithmetic points are in general not Zariski dense in the $\overline{\mathbb{Q}}$-bi-algebraic subvarieties of S_α. The situation here is thus similar with the one for general variations of Hodge structures. Our second main result in this paper is the following (see Section 5.5 for the definition of the rank and the degree of a general invariant linear subvariety; any Teichmúller curve has rank 1):

Theorem 2.6. *Any invariant linear subvariety of S_α of rank 1 and degree at least 2 does not contain a Zariski-dense set of arithmetic points. In particular any Teichmüller curve of degree at least 2 contains only finitely many arithmetic points.*

In general, an invariant linear subvariety of S_α of rank $k \geq 1$ and degree $d > 1$ does not contain a Zariski-dense set of arithmetic points of degrees at least kd.

In addition to Theorem 2.2 and the results of [Mö06b], [Fil16a], [Fil17] and [EFW18] on invariant linear varieties, the main tool in the proof of Theorem 2.6 is the André-Oort conjecture for mixed Shimura varieties whose pure part is of abelian type [Ts18], [Gao16].

2.2. **Bi-algebraic subvarieties.** Let us now turn to the purely geometric aspects of the bi-algebraic geometry of S_α. For tori, abelian varieties or Shimura varieties, its bi-algebraic subvarieties are its weakly special subvarieties: they are defined by group-theoretic conditions and may be thought as the most "linear" algebraic subvarieties. We propose the following:

Conjecture 2.7. *The bi-algebraic subvarieties of S_α are the linear ones.*

In words: if the periods of an algebraic family of abelian differentials satisfy as many algebraic relations as possible, then these algebraic relations are linear.

The absence of homogeneity for $\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ makes Conjecture 2.7 of a completely different order of difficulty than for tori, abelian varieties or Shimura varieties. Our third main result is a proof of Conjecture 2.7 in two cases: for bi-algebraic curves on which ω does not vary in a constant local subsystem (see the condition ($\star$) of Definition 6.5); and for bi-algebraic subvarieties of S_α contained in an isoperiodic leaf.

Theorem 2.8. *The bi-algebraic curves in S_α satisfying condition ($\star$) are linear.*

Theorem 2.9. *The bi-algebraic subvarieties of S_α contained in an isoperiodic leaf are linear.*

The main tools for proving Theorem 2.8 and Theorem 2.9 are a detailed analysis of the monodromy action, using the classical results of Satake [Sa65], as well as the Ax-Schanuel conjecture for abelian varieties [Ax72].

Thanks to Theorem 2.8 and Theorem 2.9, Conjecture 2.7 holds true in genus 2:

Theorem 2.10. *Any bi-algebraic curve of $\mathrm{S}_{1,1}$ or S_2 either coincide with a linear Hurwitz space, or satisfy condition ($\star$). In particular it is linear.*

Remark 2.11. Notice that [Mö08, Theorem 7.1] fully describes linear curves in genus 2. Thus Theorem 2.10, in combination with this description, provides a complete classification of the bi-algebraic curves in genus 2.

In genus 3, we show in Section 6.4 an example of a bi-algebraic curve in S_4 which is not a linear Hurwitz space, nor satisfies condition ($\star$), but is still linear.

2.3. **Conjectures.** Let us propose some questions and conjectures for S_α, which are suggested by the bi-algebraic format and which place the previous results in their proper context.

First of all, in view of Theorem 2.6, it would be interesting to decide the following

Question 2.12. Does any $\overline{\mathbb{Q}}$-bi-algebraic subvariety S_α contain at least one arithmetic point?

We check in Remark 5.8 that this holds true for the unique known series of primitive Teichmüller curves generated by Veech surfaces of unbounded genera: the triangle group series of Bouw-Möller [BM10]. One also easily checks this is true for the $\overline{\mathbb{Q}}$-linear Hurwitz spaces.

Even if not every $\overline{\mathbb{Q}}$-bi-algebraic subvariety of S_α does contain a Zariski-dense set of arithmetic points, we still conjecture that the converse holds true:

Conjecture 2.13 (André-Oort for S_α)**.** *Let $S \subset \mathrm{S}_\alpha$ be an irreducible algebraic subvariety of S_α containing a Zariski-dense set of arithmetic points. Then S is $\overline{\mathbb{Q}}$-bi-algebraic (hence $\overline{\mathbb{Q}}$-linear in view of Conjecture 2.7).*

Both Theorem 2.6 and Conjecture 2.13, as well as the main result of [EFW18] stating that all but finitely many linear invariant subvarieties of S_α have degree at most 2, are instances of a general *Zilber-Pink conjecture* for *atypical intersections* in S_α. We won't write about it in detail here and will come back to it in a future article. Let us just state the following, which we see as the core of this conjecture:

Conjecture 2.14. *Any stratum S_α contains only finitely many arithmetic points of degree at least* 3*.*

In the same way as for other bi-algebraic structures (see [UY14], [PT14], [KUY16], [K17]), the main geometric step towards the Zilber-Pink conjecture is an *Ax-Schanuel conjecture* for S_α, of which we will describe here only the following particular case:

Conjecture 2.15 (Ax-Lindemann for S_α)**.** *Let $V \subset \widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ be an irreducible algebraic subvariety of the universal cover $\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ of $\mathrm{S}_\alpha^{\mathrm{an}}$, namely a closed irreducible analytic subvariety of $\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ which is algebraic in the period charts. Then the Zariski-closure $\overline{\pi(V)}^{\mathrm{Zar}}$ of its projection in S_α is bi-algebraic in S_α (hence linear in view of Conjecture 2.7).*

2.4. **Organization of the paper.** Section 3 fixes the notations and complements the introduction with the details we need later, including the mixed Hodge theory of strata and its relation to their bi-algebraic geometry. In Section 4 we prove the geometric characterization Theorem 2.2 of the arithmetic points. Section 5.1 discusses the distribution of the arithmetic points and Theorem 2.6. Section 6 proves Theorem 2.8 and Theorem 2.9.

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3. Preliminaries

3.1. **Strata.** Let $(C, [\omega])$ be an abelian differential (up to scaling) of genus $g \geq 1$. The divisor $\mathrm{div}([\omega])$ on C of zeroes of $[\omega]$ can be uniquely written $\sum_{i=1}^{n} \alpha_i x_i$, with $\alpha_i \in \mathbb{N}^*$ and the x_is are pairwise distinct in C. The set of integers $\alpha \coloneqq \{\alpha_1, \dots, \alpha_n\}$ is called the type of $(C, [\omega])$. As the weight $|\alpha| := \sum_{i=1}^{n} \alpha_i$ of α coincides with the degree of $\mathrm{div}(\omega)$, it satisfies $|\alpha| = 2g - 2$.

Remark 3.1. The algebro-geometric notion of abelian differential is equivalent to the differential geometric notion of *translation surface*, i.e. a compact Riemann surface Σ, a finite union of points $Z \subset \Sigma$, and an atlas of charts for $\Sigma \setminus Z$ with value in $\mathbb{C}$ whose transition functions are locally constant translations (a so-called *translation structure* on $\Sigma \setminus Z$) such that the cone angle at each point of Σ is a positive integral multiple of 2π. The equivalence is obtained by associating to (C, ω) the pair $(\Sigma := C^{\mathrm{an}}, Z := Z(\omega))$, the translation structure being given by locally integrating ω on $C^{\mathrm{an}} \setminus Z(\omega)$. This differential geometric point of view will play no role in this paper.

Let M_g denote the coarse moduli space of smooth projective curves of genus g. The coarse moduli space of non-zero abelian differentials (up to scaling) of genus g is the projectivization $\mathbf{P}\Omega^1\mathrm{M}_g$ of the Hodge bundle $\Omega^1\mathrm{M}_g$ on M_g (whose fiber at a closed point $C \in \mathrm{M}_g$ is the space of algebraic one-forms on C). The algebraic variety $\mathbf{P}\Omega^1\mathrm{M}_g$ is defined over $\mathbb{Q}$ and has dimension $4g-4$. It is naturally stratified according to the type of the one-forms:

$$\mathbf{P}\Omega^1\mathrm{M}_g = \coprod_{\substack{\alpha = \{\alpha_1, \dots, \alpha_n\} \\ \sum_i \alpha_i = 2g-2}} \mathrm{S}_\alpha$$

where the stratum S_α parametrizing abelian differentials of type α is defined set-theoretically as the abelian differentials $(C, [\omega]) \in \mathbf{P}\Omega^1\mathrm{M}_g$ for which there are pairwise distinct points $x_i \in C$ with $\mathrm{div}([\omega]) = \sum_{i=1}^{n} \alpha_i x_i$. This is a quasi-projective variety defined over $\mathbb{Q}$ and has dimension $2g+n-2$. It is not connected in general, see [KZ03]. All the coarse moduli spaces introduced above have orbifold singularities. To define the period coordinates around every point we will need to work with fine moduli spaces: this can be achieved by passing to a finite ramified cover, for instance introducing a level-ℓ structure, with $\ell \geq 3$, on the curves underlying the abelian differentials. *For the rest of the text, we fix a level $\ell \geq 3$ and, by abuse of notations, we call stratum and denote by S_α any connected component of the level-ℓ ramified cover of the S_α defined above.* With this convention, S_α is a smooth, connected, quasi-projective variety defined over $\overline{\mathbb{Q}}$. In the same way, we suppress the integer ℓ from the notation of the moduli space of level-ℓ curves: we denote by $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,n}$) the fine moduli space of level-ℓ smooth projective curves of genus g (resp. with n distinct marked points). Our results do not depend on the choice of the level.

Let us mention two variants of the above definitions which we will use. First, it will be sometimes convenient to work with the moduli space H_α of abelian differentials *without scaling*, with zeros of type α (a $\mathbf{G}_m$-bundle over S_α). It will also be useful to consider a version of S_α where the zeroes of the forms are marked. Let $\mathcal{M}_{g,n}$ be the fine moduli space of level-ℓ smooth projective curves of genus g with n distinct marked points. The moduli space of canonical divisors of type α is the locally closed subvariety of $\mathcal{M}_{g,n}$ defined set theoretically by

$$\mathcal{S}_\alpha := \left\{ [C, x_1, \dots, x_n] \in \mathcal{M}_{g,n} \mid \mathcal{O}_C(\sum_{i=1}^{n} \alpha_i \, x_i) \simeq \Omega^1_C \right\} \overset{\iota}{\hookrightarrow} \mathcal{M}_{g,n} \quad .$$

Notice that if $\alpha = (m_1, \dots, m_1, m_2, \dots, m_2, \dots, m_r, \dots, m_r)$ with $m_i \neq m_j$ for $i \neq j$, then the product of symmetric groups $\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$ acts freely on $\mathcal{S}_\alpha$; and $\mathcal{S}_\alpha$ is a finite étale cover of S_α.

3.2. **Hodge theory of n-pointed curves.** We first recall the classical mixed Hodge theory of curves with marked points. Let C be a smooth complex projective curve of genus g. The Betti cohomology group $H^1(C, \mathbb{Z})$ is a pure $\mathbb{Z}$-Hodge structure of weight 1. If $Z = \{x_1, \dots, x_n\}$ is a set of n distinct points on C, the long exact sequence of relative cohomology for the pair (C, Z)

$$\cdots \to H^0(C, \mathbb{Z}) \to H^0(Z, \mathbb{Z}) \to H^1(C, Z, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to H^1(Z, \mathbb{Z}) = 0$$

defines a short exact sequence in the category of $\mathbb{Z}$-mixed Hodge structures (abbreviated $\mathbb{Z}$MHS)

$$0 \to \tilde{H}^0(Z, \mathbb{Z}) \simeq \mathbb{Z}(0)^{n-1} \to H^1(C, Z, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to 0 \ , \tag{3.1}$$

where $\tilde{H}^\bullet$ denotes the reduced cohomology. The $\mathbb{Z}$-mixed Hodge structure on $H^1(C, Z, \mathbb{Z})$ is of type $(0,0)$, $(1,0)$ and $(0,1)$:

- $\mathrm{Gr}_0^W H^1(C, Z; \mathbb{Q}) = \mathbb{Q}(0)^{n-1}$ and $\mathrm{Gr}_1^W H^1(C, Z; \mathbb{Q}) = H^1(C, \mathbb{Z})$.
- the only non-trivial piece of the Hodge filtration is $F^1 H^1(C, Z; \mathbb{C})$, defined as the subspace $H^0(C, \Omega^1_C)$ of $H^1(C, Z; \mathbb{C})$.

Remark 3.2. As C is a smooth projective curve, Poincaré duality provides various identifications of Hodge structures $H^1(C, Z, \mathbb{Z}) \simeq H_1(C - Z; \mathbb{Z})(-1) \simeq H^1_c(C - Z, \mathbb{Z})$, while the dual Hodge structure $H^1(C, Z, \mathbb{Z})^\vee$ identifies with $H^1(C - Z, \mathbb{Z})(1) \simeq H_1(C, Z, \mathbb{Z})$. As a result, any natural $\mathbb{Z}$-mixed Hodge structure associated with the pair (C, Z) coincides, up to twist and duality, with (3.1). For example the $\mathbb{Z}$-mixed Hodge structure

$$0 \to H^1(C, \mathbb{Z})(1) \to H^1(C - Z, \mathbb{Z})(1) \to \tilde{H}^0(Z, \mathbb{Z}) \to 0$$

is dual to (3.1). The only non-trivial piece of the Hodge filtration on $H^1(C - Z, \mathbb{C})$ is $F^1 H^1(C - Z, \mathbb{C})$, defined as the space $H^0(C, \Omega^1_C(\log Z))$ of logarithmic one-forms on C with poles at Z, which is the natural annihilator of $F^0 H^1(C, Z; \mathbb{C})(1) \simeq H^0(C, \Omega^1_C)$.

3.3. **The variation of $\mathbb{Z}$-mixed Hodge structure $\mathbb{V}_\alpha$ on S_α.**

3.3.1. When C varies through $\mathcal{M}_g$ the cohomology $H^1(C, \mathbb{Z})$ defines a $\mathbb{Z}$-variation of Hodge structure ($\mathbb{Z}$VHS) $\mathbb{V}_{g,\mathbb{Z}}$ on $\mathcal{M}_g$ of weight one. If $f : \mathcal{C}_g \to \mathcal{M}_g$ denotes the universal smooth projective curve of genus g and level-ℓ then $\mathbb{V}_{g,\mathbb{Z}} = R^1 f_* \mathbb{Z}$. Varying $(C, [\omega])$ through S_α, one obtains similarly that $\mathbb{V}_{\alpha,\mathbb{Z}}$ is an admissible, graded polarized, variation of $\mathbb{Z}$-mixed Hodge structure ($\mathbb{Z}$VMHS). The weight filtration has two steps:

$$0 \to W_0 \mathbb{V}_{\alpha,\mathbb{Z}} \simeq \mathbb{Z}(0)^{n-1} \to \mathbb{V}_{\alpha,\mathbb{Z}} \to \mathrm{Gr}_1^W \mathbb{V}_{\alpha,\mathbb{Z}} = p^* \mathbb{V}_{g,\mathbb{Z}} \to 0 \ , \tag{3.2}$$

where $p : \mathrm{S}_\alpha \to \mathcal{M}_g$ denotes the canonical projection.

Remark 3.3. If one replaces S_α by its finite étale cover $\mathcal{S}_\alpha \overset{\iota}{\hookrightarrow} \mathcal{M}_{g,n}$, then the $\mathbb{Z}$VMHS $\mathbb{V}_{\alpha,\mathbb{Z}}$ on $\mathcal{S}_\alpha$ coincides with $\iota^{-1} R^1 f^0{}_! \mathbb{Z}_{\mathcal{C}^0_{g,n}}$, where $f^0 : \mathcal{C}^0_{g,n} \to \mathcal{M}_{g,n}$ is the open curve complement in $f : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ (the universal smooth projective n-pointed curve of genus g and level-ℓ) of the canonical sections $x_i : \mathcal{M}_{g,n} \to \mathcal{C}_{g,n}$, $1 \leq i \leq n$.

Remark 3.4. It is interesting to notice that the variety $\mathcal{S}_\alpha$ defined set-theoretically by (3.1) can itself be defined in a purely Hodge theoretic way. Let $\mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$ denotes the relative Picard scheme whose fiber at $C \in \mathcal{M}_g$ is the abelian variety $\mathrm{Pic}^0(C)$ parametrizing degree zero line bundles on C, and $p : \mathcal{M}_{g,n} \to \mathcal{M}_g$ the natural map forgetting the marking. The abelian scheme $p^* \mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_{g,n}$ has two natural sections: the identity section e and the section s_α defined by associating to $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ the degree zero line bundle $\Omega^1_C(-\sum_{i=1}^n \alpha_i x_i)$ on C. The variety $\mathcal{S}_\alpha$ is the subvariety of $\mathcal{M}_{g,n}$ defined by the Cartesian diagram

$$\begin{tikzcd} \mathcal{S}_\alpha \arrow[r] \arrow[d] & \mathcal{M}_{g,n} \arrow[d, "e"] \\ \mathcal{M}_{g,n} \arrow[r, "s_\alpha"'] & p^*\mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g} \end{tikzcd} .$$

The variety $\mathcal{M}_{g,n}$ has dimension $3g-3+n$, thus $p^*\mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g}$ has dimension $4g-3+n$. The two sections $s_\alpha(\mathcal{M}_{g,n})$ and $e(\mathcal{M}_{g,n})$ are thus of codimension g. On the other hand their intersection $\mathcal{S}_\alpha$ has dimension $2g-2+n$ hence is of codimension $2g-1$ in $p^*\mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g}$. This shows that $s_\alpha(\mathcal{M}_{g,n})$ and $e(\mathcal{M}_{g,n})$ are not transverse. Over $\mathbb{C}$ the Picard variety $\mathrm{Pic}^0(C)$ is canonically isomorphic to the group of extensions $\mathrm{Ext}^1_{\mathbb{Z}\mathrm{MHS}}(H^1(C;\mathbb{Z}), \mathbb{Z}(0))$ in the abelian category of $\mathbb{Z}$-mixed Hodge structures. The section $s_\alpha : \mathcal{M}_{g,n} \to p^*\mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g}$ can be thought as an element of the group of extensions $\mathrm{Ext}^1_{\mathbb{Z}\mathrm{VMHS}^{\mathrm{adm}}_{\mathcal{M}_{g,n}}}(p^*(R^1 f_*\mathbb{Z}), \mathbb{Z}(0))$ in the abelian category of admissible variations of $\mathbb{Z}$MHS on $\mathcal{M}_{g,n}$ i.e. a normal function; and $\mathcal{S}_\alpha$ as the zero locus of this normal function.

3.4. **The period map $\Phi : \mathrm{S}_\alpha \to \mathfrak{A}_g^{(n-1)}$.** The classifying space for weight one $\mathbb{Z}$VHS of dimension $2g$ is the Shimura variety $\mathcal{A}_g$ moduli space of principally polarized abelian varieties of dimension g. The classifying space for the $\mathbb{Z}$VMHS extensions of a weight one $\mathbb{Z}$VHS of dimension $2g$ by $\mathbb{Z}(0)$ is the mixed Shimura variety $\mathfrak{A}_g$, the universal principally polarized abelian variety of dimension g over $\mathcal{A}_g$. Hence the classifying space for the $\mathbb{Z}$VMHS extensions of a weight one $\mathbb{Z}$VHS of dimension $2g$ by $\mathbb{Z}(0)^{n-1}$ is the mixed Shimura variety $\mathfrak{A}_g^{(n-1)} := \mathfrak{A}_g \times_{\mathcal{A}_g} \mathfrak{A}_g \times_{\mathcal{A}_g} \cdots \times_{\mathcal{A}_g} \mathfrak{A}_g$ (product of $(n-1)$ factors). The $\mathbb{Z}$VMHS $\mathbb{V}_{\alpha,\mathbb{Z}}$ on S_α is classified by the period map

$$\Phi : \mathrm{S}_\alpha \to \mathfrak{A}_g^{(n-1)} \quad . \tag{3.3}$$

By a refined version of the classical Abel-Jacobi theorem (see [ArOh97, Theor. 7.2]), the period map is quasi-finite.

3.5. **Integral projective structure on S_α.** Recall that an *integral projective structure* on a complex manifold M can be defined equivalently

(i) as a maximal atlas of charts for M with value in $\mathbf{P}(\mathrm{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C})$ whose transition functions are locally constant elements of the integral projective linear group $\mathbf{PGL}(\mathrm{V}_\mathbb{Z})$; here $V_\mathbb{Z}$ denotes a finite free $\mathbb{Z}$-module of rank the (complex) dimension of M.

(ii) as a local biholomorphism $D : \tilde{M} \to \mathbf{P}(\mathrm{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C})$, called the developing map of the projective structure, which is equivariant under a monodromy representation $\rho : \pi_1(M, x_0) \to \mathbf{PGL}(\mathrm{V}_\mathbb{Z})$. Here $\pi : \widetilde{M} \to M$ is the universal cover of M at x_0, and D is obtained by glueing the local projective charts.

In the case of $\mathrm{S}_\alpha^{\mathrm{an}}$, let $\mathbb{V}_{\alpha,\mathbb{Z}}$ be the $\mathbb{Z}$-local system on $\mathrm{S}_\alpha^{\mathrm{an}}$ whose fiber at any point $(C, [\omega]) \in \mathrm{S}_\alpha^{\mathrm{an}}$ is the relative cohomology group $H^1(C, Z([\omega]); \mathbb{Z})$. Let $\mathbb{V}_\alpha$ be its complexification and $(\mathcal{V}_\alpha^{\mathrm{an}}, \nabla^{\mathrm{an}})$ the associated complex analytic integrable connection on $\mathrm{S}_\alpha^{\mathrm{an}}$. Given $U \subset \mathrm{S}_\alpha^{\mathrm{an}}$ a simply connected neighbourhood of a point $x_0 := (C_0, [\omega_0])$ in $\mathrm{S}_\alpha^{\mathrm{an}}$, the local period chart of the introduction is the local biholomorphism

$$D_U : U \stackrel{[\omega]}{\to} \mathbf{P}\mathcal{V}^{\mathrm{an}}_{\alpha\ |U} \stackrel{\varphi}{\to} \mathbf{P}\mathrm{V}_{\alpha,x_0} := \mathbf{P}H^1(C_0, Z([\omega_0]); \mathbb{C}) \stackrel{\gamma}{\simeq} \mathbf{P}^{2g+n-2}\mathbb{C}$$

which to a point $x \in U$ associates the vector of periods

$$\left[\int_{\gamma_1} \varphi(\omega_x), \ldots, \int_{\gamma_{2g+n-1}} \varphi(\omega_x)\right] \in \mathbf{P}^{2g+n-2}\mathbb{C} \ .$$

Here $[\underline{\omega}]$ is the tautological section of $\mathbf{P}\mathcal{V}_\alpha$ which to a point $(C, [\omega]) \in \mathrm{S}_\alpha^{\mathrm{an}}$ associates the class of $[\omega]_{\mathrm{Betti}}$ in $\mathbf{P}H^1(C, Z([\omega]), \mathbb{C})$; the map φ is the parallel transport with respect to ∇^{an} of $\mathbf{P}\mathcal{V}^{\mathrm{an}}_{\alpha\ |U}$ on its central fiber $\mathbf{P}\mathrm{V}_{\alpha,x_0}$; and γ is the identification of $\mathbf{P}\mathrm{V}_{\alpha,x_0}$ with $\mathbf{P}^{2g+n-2}\mathbb{C}$ provided by the choice of an integral basis $(\gamma_i)_{1\leq i\leq 2g+n-1}$ of $H_1(C_0, Z([\omega_0]); \mathbb{Z})$. Changing the base point (C_0, ω_0) results in a locally constant change of coordinates with integral coefficients between the two corresponding charts.

More globally, if $\pi : \widetilde{\mathrm{S}_\alpha^{\mathrm{an}}} \to \mathrm{S}_\alpha^{\mathrm{an}}$ denotes the universal cover of $\mathrm{S}_\alpha^{\mathrm{an}}$, the canonical trivialization of the local system $\pi^*\mathbb{V}_{\alpha,\mathbb{Z}}$ on the simply connected space $\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ induces a decomposition of the associated vector bundle $\widetilde{\mathcal{V}_\alpha^{\mathrm{an}}}$ into a product $\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}} \times \mathrm{V}_\alpha$, where $\mathrm{V}_\alpha := \mathrm{V}_{\alpha,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathrm{V}_{\alpha,\mathbb{Z}} := H^0(\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}, \pi^*\mathbb{V}_{\alpha,\mathbb{Z}})$ is isomorphic to any of the fibers $\mathrm{V}_{\alpha,\mathbb{Z},x_0}$. The developing map

$$D : \widetilde{\mathrm{S}_\alpha^{\mathrm{an}}} \to \mathbf{P}\mathrm{V}_\alpha \tag{3.4}$$

of the projective structure on S_α is obtained as the composition of the section $\widetilde{[\underline{\omega}]} : \widetilde{\mathrm{S}_\alpha^{\mathrm{an}}} \to \widetilde{\mathbf{P}\mathcal{V}_\alpha^{\mathrm{an}}}$ lifting $[\underline{\omega}]$, followed by the parallel transport projection $\widetilde{\mathbf{P}\mathcal{V}_\alpha^{\mathrm{an}}} \simeq \widetilde{\mathrm{S}_\alpha^{\mathrm{an}}} \times \mathbf{P}\mathrm{V}_\alpha \to \mathbf{P}\mathrm{V}_\alpha$. It is naturally equivariant under $\pi_1(\mathrm{S}_\alpha^{\mathrm{an}})$, we denote by $\rho : \pi_1(\mathrm{S}_\alpha^{\mathrm{an}}) \to \mathbf{PGL}(\mathrm{V}_{\alpha,\mathbb{Z}})$ the associated monodromy.

3.6. **The bi-algebraic geometry of S_α.** Recall from [KUY16, Section 4] that:

Definition 3.5. *A bi-algebraic structure on a connected complex algebraic variety S is a pair*

$$(D : \widetilde{S^{\mathrm{an}}} \to X, \quad \rho : \pi_1(S^{\mathrm{an}}) \to \mathrm{Aut}(X))$$

where $\widetilde{S^{\mathrm{an}}}$ denotes the topological universal cover of S, X is a complex algebraic variety, $\mathrm{Aut}(X)$ its group of algebraic automorphisms, $\rho : \pi_1(S^{\mathrm{an}}) \to \mathrm{Aut}(X)$ is a group morphism (called the holonomy representation) and D is a ρ-equivariant holomorphic map (called the developing map).

Definition 3.6. *A $\overline{\mathbb{Q}}$-bi-algebraic structure on a complex algebraic variety S is a complex bi-algebraic structure $(D : \widetilde{S^{\mathrm{an}}} \to X, \rho : \pi_1(S^{\mathrm{an}}) \to \mathrm{Aut}(X))$ such that:*

(1) S is defined over $\overline{\mathbb{Q}}$.

(2) $X = X_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ and the homomorphism ρ takes value in $\mathrm{Aut}_{\overline{\mathbb{Q}}} X_{\overline{\mathbb{Q}}}$.

Definition 3.7. *The* linear $\overline{\mathbb{Q}}$-bi-algebraic structure *on S_α is the one defined by (D, ρ) as in (3.4).*

Following the general format of [KUY16, Section 4], we define:

Definition 3.8. *A complex analytic subvariety $Y \subset \widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ is said to be algebraic if it is an irreducible complex analytic component of $D^{-1}(Z)$, for $Z \subset \mathbf{P}\mathrm{V}_\alpha$ a complex algebraic subvariety. It is said to be defined over $\overline{\mathbb{Q}}$ if Z is.*

Remark 3.9. Notice that in Definition 3.8 one can equivalently replace Z by the Zariski-closure $\overline{D(Y)}^{\mathrm{Zar}}$ of $D(Y)$ in the projective space $\mathbf{P}\mathrm{V}_\alpha$. We call $\overline{D(Y)}^{\mathrm{Zar}}$ *the algebraic model* of Y.

Definition 3.10. *A bi-algebraic subvariety $W \subset \mathrm{S}_\alpha$ is an irreducible algebraic subvariety W of S_α such that W^{an} is the projection $\pi(Y)$ of an algebraic subvariety Y of $\widetilde{\mathrm{S}_\alpha^{\mathrm{an}}}$ (in the sense of Definition 3.8). It is $\overline{\mathbb{Q}}$-bi-algebraic if moreover W and Y are defined over $\overline{\mathbb{Q}}$.*

Definition 3.11. *A linear subvariety $W \subset \mathrm{S}_\alpha$ is a bi-algebraic subvariety $W = \pi(Y)$ such that the algebraic model of Y is a (projectively) linear subspace Z of $\mathbf{P}\mathrm{V}_\alpha$. It is a $\overline{\mathbb{Q}}$-linear subvariety if moreover W and Y are defined over $\overline{\mathbb{Q}}$.*

Remark 3.12. Of course Definition 3.10 and Definition 3.11, given in terms of the developing map D, coincides with Definition 1.1 and Definition 1.3 of the introduction, given in terms of local period charts.

3.7. **Variants.** The content of Section 3.1 and Section 3.5 holds for H_α, with the difference that the developing map will have values in V_α, thus defining an *integral linear structure* on $\mathrm{H}_\alpha^{\mathrm{an}}$.

As $\mathcal{S}_\alpha$ is a finite étale cover of S_α, the integral projective structure on S_α induces one on $\mathcal{S}_\alpha$, thus defining a bi-algebraic structure on $\mathcal{S}_\alpha$ making the finite morphism $\mathcal{S}_\alpha \to \mathrm{S}_\alpha$ a morphism of bi-algebraic structures in the obvious sense.

3.8. **Abelian varieties with many endomorphisms.** For the convenience of the reader we recall in this section classical results on (complex) abelian varieties with many endomorphisms.

Let $A \simeq A_1^{n_1} \times \cdots \times A_k^{n_k}$ be a complex abelian variety (where $\simeq$ denotes an isogeny and the A_is are simple, pairwise non isogenous). Let $\mathrm{End}(A)$ be its ring of endomorphism, and $\mathrm{End}_{\mathbb{Q}}(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The $\mathbb{Q}$-algebra $\mathrm{End}_{\mathbb{Q}}(A)$ is semi-simple, isomorphic to $\mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k}(D_k)$, where $D_i := \mathrm{End}_{\mathbb{Q}}(A_i)$ is a central division algebra over a finite extension K_i of $\mathbb{Q}$. We define the reduced degree of $\mathrm{End}_{\mathbb{Q}}(A)$ by

$$[\mathrm{End}_{\mathbb{Q}}(A) : \mathbb{Q}]_{\mathrm{red}} := \sum_{i=1}^{k} n_i [D_i : K_i]^{\frac{1}{2}} [K_i : \mathbb{Q}] \quad , \tag{3.5}$$

it is also the degree over $\mathbb{Q}$ of a maximal étale subalgebra of $\mathrm{End}_{\mathbb{Q}}(A)$. As $\mathrm{End}_{\mathbb{Q}}(A)$ acts faithfully on $H^1(A^{\mathrm{an}}, \mathbb{C})$, the reduced degree satisfies trivially:

$$2 \dim_{\mathbb{C}} A \geq [\mathrm{End}_{\mathbb{Q}}(A) : \mathbb{Q}]_{\mathrm{red}} \quad , \tag{3.6}$$

Definition 3.13. *The abelian variety A is said to have complex multiplication (CM) if the inequality (3.6) is an equality, or equivalently if $\mathrm{End}_{\mathbb{Q}}(A)$ contains an étale subalgebra of dimension $2\dim_{\mathbb{C}} A$, see* [Mi20, Prop. 3.3].

It follows immediately that A has CM if and only if each A_i has CM, if and only if $D_i = K_i$ has degree $[K_i : \mathbb{Q}] = 2\dim_{\mathbb{C}} A_i$. With a little more effort, one shows that K_i is a CM field (i.e. a totally complex quadratic extension of a totally real field); moreover if A is isotypic then A has CM if and only if $\mathrm{End}_{\mathbb{Q}}(A)$ contains a CM field of degree $2\dim_{\mathbb{C}} A$, see [Mi20, Prop. 3.6].

Definition 3.14. *The abelian variety A is said to have real multiplication if* $\mathrm{End}_{\mathbb{Q}}(A)$ *contains a totally real field K_0 of degree $[K_0 : \mathbb{Q}] = \dim_{\mathbb{C}} A$.*

In particular, any abelian variety with complex multiplication has real multiplication. One has the following partial converse:

Lemma 3.15. *Let A be a complex abelian variety with real multiplication by K_0. Then A is isotypic, and* $\mathrm{End}_{\mathbb{Q}} A$ *has a maximal étale subalgebra which is either K_0, or a quadratic CM extension K of K_0 (in which case A has CM).*

Proof. Let $g = \dim A$. Assume that A is not isotypic. Then there exists an isogeny $A \simeq A_1 \times A_2$, where the A_is, $i = 1, 2$, are positive dimensional abelian varieties without common simple factor up to isogeny. This realizes K_0 as a subalgebra of $\mathrm{End}^0(A_i)$, $i = 1, 2$. The numerical condition (3.6) forces A_i, $i = 1, 2$, to have CM with $\mathrm{End}_{\mathbb{Q}} A_i = K_0$. As K_0 has a real place, it follows from [Mi20, Lemma 3.7] that $\dim_{\mathbb{C}} A = [K_0 : \mathbb{Q}]$ divides $\dim_{\mathbb{C}} A_1$. This is a contradiction, thus A is isotypic.

The field K_0 of degree $\dim_{\mathbb{C}} A$ is contained in a maximal étale subalgebra E of $\mathrm{End}_{\mathbb{Q}} A$, which is of degree at most $2\dim_{\mathbb{C}} A$. Thus either $E = K_0$, or $[E:\mathbb{Q}] = 2\dim_{\mathbb{C}} A$, in which case A has CM, and E can be chosen to be a CM quadratic extension of K_0. □

4. Characterization of arithmetic points

For $\alpha = 0$ (i.e. C is an elliptic curve) the stratum S_0 coincides with the modular curve. Theorem 2.2 says in this case that its arithmetic points are the CM points of the modular curve, a classical result of Schneider [Schn37]. Schneider's theorem was generalized by Shiga and Wolfart [ShWo95] to say that if A is a simple abelian variety over $\overline{\mathbb{Q}}$ admitting a non-zero algebraic one-form (defined over $\overline{\mathbb{Q}}$) all of whose periods are algebraic multiples one of each other, then A has complex multiplication. Theorem 2.2 can thus be thought as a generalisation "to the mixed case" of [ShWo95]. As for most results nowadays in transcendence theory, our main tool is Wüstholz' Analytic Subgroup Theorem [Wus87, Theor.1 and Corollary]:

Theorem 4.1. *(Wüstholz) Let $\mathbf{G}$ be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ with Lie algebra $\mathfrak{g}$. Let $\exp : \mathfrak{g} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to \mathbf{G}(\mathbb{C})$ be the exponential map for $\mathbf{G}(\mathbb{C})$.*

Let $\mathfrak{b} \subset \mathfrak{g}$ be a positive dimensional $\overline{\mathbb{Q}}$-vector subspace, $\mathbf{B} \coloneqq \exp(\mathfrak{b} \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$, and $u \in \mathfrak{b} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ such that $\exp(u) \in \mathbf{G}(\overline{\mathbb{Q}})$. Let $\mathfrak{h} \subset \mathfrak{b}$ be the smallest $\overline{\mathbb{Q}}$-vector subspace such that $u \in \mathfrak{h} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. Then $\mathfrak{h}$ is the Lie algebra of an algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$ defined over $\overline{\mathbb{Q}}$.

Let us now prove Theorem 2.2 Without loss of generality we can replace S_α by its finite étale cover $\mathcal{S}_\alpha$. Let us show that any arithmetic point $(C, [\omega]) \in \mathcal{S}_\alpha(\mathbb{C})$ satisfies necessarily the conditions stated in Theorem 2.2. Condition (1) is obviously satisfied. Let us choose $\omega \in H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}})$ representing $[\omega]$.

We first analyse the absolute periods of ω in order to prove (2). We denote by $\mathrm{Alb}(C)$ the Albanese of C (a principally polarized abelian variety defined over $\overline{\mathbb{Q}}$, isomorphic to the Jacobian of C). Poincaré's reduction theorem states that $\mathrm{Alb}(C)$ is isogenous over $\overline{\mathbb{Q}}$ to a product $A_1^{k_1} \times \ldots \times A_N^{k_N}$, with $A_\nu/\overline{\mathbb{Q}}$, $1 \leq \nu \leq N$, pairwise non-isogenous simple abelian varieties. Let

$$\mathrm{alb} : C \to \mathrm{Alb}(C) \simeq \prod_{\nu=1}^{N} A_\nu^{k_\nu}$$

be the Albanese map. It is uniquely defined up to fixing the image of a single point in $C(\overline{\mathbb{Q}})$. We normalize it by fixing $\mathrm{alb}(x_0)$ to be the identity of $\mathrm{Alb}(C)$, where x_0 is the first zero of $Z(\omega)$ (this is where we use that we work with the finite étale cover $\mathcal{S}_\alpha$ rather than with S_α). Let

$$\mathrm{alb}_\nu : C \to A_\nu^{k_\nu}, \quad 1 \leq \nu \leq N,$$

be the projections of alb on each isogeny factor $A_\nu^{k_\nu}$.

As the De Rham pull-back map

$$\prod_\nu \mathrm{alb}_\nu^* : \prod_{\nu=1}^{N} H^0(A_\nu^{k_\nu}, \Omega^1_{A_\nu^{k_\nu}/\overline{\mathbb{Q}}}) \to H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}})$$

is an isomorphism, there exists a unique $\omega_\nu \in H^0(A_\nu^{k_\nu}, \Omega^1_{A_\nu^{k_\nu}/\overline{\mathbb{Q}}})$, $1 \leq \nu \leq N$, such that $\omega = \sum_{n=1}^{N} \mathrm{alb}_\nu^* \omega_\nu$.

On the other hand the Betti pull-back map

$$\prod_\nu \mathrm{alb}_\nu^* : \prod_\nu H^1((A_\nu^{k_\nu})^{\mathrm{an}}, \mathbb{Q}) \to H^1(C^{\mathrm{an}}, \mathbb{Q})$$

is also an isomorphism and the comparison diagram between De Rham and Betti cohomologies

$$\begin{tikzcd} \prod_\nu \left(H^0(A_\nu^{k_\nu}, \Omega^1_{A_\nu^{k_\nu}/\overline{\mathbb{Q}}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}\right) \arrow[r, "\prod_\nu \mathrm{alb}_\nu^*"] \arrow[d] & H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \arrow[d] \\ \prod_\nu \left(H^1((A_\nu^{k_\nu})^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}\right) \arrow[r, "\prod_\nu \mathrm{alb}_\nu^*"'] & H^1(C^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \end{tikzcd}$$

commutes. Hence if the projectivized Betti class $[\omega]_{\mathrm{Betti}} \in \mathbf{P}H^1(C^{\mathrm{an}}, \mathbb{C})$ lies in $\mathbf{P}H^1(C^{\mathrm{an}}, \overline{\mathbb{Q}})$, all the classes $[\omega_\nu]_{\mathrm{Betti}} \in \mathbf{P}H^1((A_\nu^{k_\nu})^{\mathrm{an}}, \mathbb{C})$ corresponding to the non-zero factors ω_ν lie in $\mathbf{P}H^1((A_\nu^{k_\nu})^{\mathrm{an}}, \overline{\mathbb{Q}})$.

It follows from [ShWo95, Prop.3] that for each ν, $1 \le \nu \le N$, such that $\omega_\nu \neq 0$, the factor A_ν has complex multiplication by a CM field K_ν.

Moreover, it follows from [ShWo95, prop.1 (1)] that there exists at most one index ν such that $\omega_\nu \neq 0$. With the notation of Definition 2.1, $A_{[\omega]} := A_\nu^{k_\nu}$ has complex multiplication, thus, in view of Section 3.8, satisfies the condition $(2)(a)$ in Theorem 2.2.

Let us show Condition $(2)(b)$ of Theorem 2.2. Suppose first that $A_{[\omega]}$ is simple. Let K be the CM field of degree $2d_{[\omega]}$ acting on $A_{[\omega]}$. The homology $H_1(A_{[\omega]}^{\mathrm{an}}, \mathbb{Z})$ is a degree one module $\mathcal{O}_K \cdot \gamma_0$ under an order $\mathcal{O}_K$ of K. Let us choose a basis of $H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$ over $\overline{\mathbb{Q}}$ consisting of eigendifferentials $\omega_1, \ldots, \omega_{d_{[\omega]}}$ for the action of K, with CM-type $\Phi = (\Phi_1, \cdots \Phi_{d_{[\omega]}})$. For each $1 \le i \le d_{[\omega]}$, the periods $\int_\gamma \omega_i$, $\gamma \in H_1(A_{[\omega]}^{\mathrm{an}}, \mathbb{Z})$, generate a one dimensional $\overline{\mathbb{Q}}$-vector space $\overline{\mathbb{Q}} \cdot \int_{\gamma_0} \omega_i$. Moreover, these $\overline{\mathbb{Q}}$-lines $\overline{\mathbb{Q}} \cdot \int_{\gamma_0} \omega_j$, $1 \le j \le d_{[\omega]}$, are $\overline{\mathbb{Q}}$-linearly independent, see [ShWo95, Prop. 1(2)]. Let us decompose $\omega_{[\omega]} = \sum_{i \in I \subset \{1, \cdots, d_{[\omega]}\}} \alpha_i \omega_i$, with $0 \neq \alpha_i \in \overline{\mathbb{Q}}$. The condition that the periods $\int_\gamma \omega$, $\gamma \in H_1(A_{[\omega]}^{\mathrm{an}}, \mathbb{Z})$, generate a one dimensional $\overline{\mathbb{Q}}$-vector space is equivalent to saying that for all $k \in \mathcal{O}_K$, the number $\Phi_i(k)$ is independent of $i \in I$. This implies that $|I| = 1$, that is, $\omega_{[\omega]}$ is a K-eigendifferential. This finishes the proof of $(2)(b)$ in case $A_{[\omega]}$ is simple. In the general isotypic case $A_{[\omega]} = B^m$, choosing a K-eigenbasis for $H^0(B, \Omega^1_B)$ and taking for basis of $H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$ the one obtained by pullback of $H^0(B, \Omega^1_B)$ under the m projections, one easily checks that the periods of $\omega \in H^0(A_{[\omega]}, \Omega^1_{A_{[\omega]}})$ generate a $\overline{\mathbb{Q}}$-vector space of dimension 1 if and only if ω is a K-eigendifferential.

Let us now analyse the relative periods of ω for proving Condition (3) of Theorem 2.2, namely that $\mathrm{alb}_{[\omega]}(Z(\omega))$ is a set of torsion points of $A_{[\omega]}$. Let us assume for simplicity that $A_{[\omega]}$ is simple (the proof immediately adapts to the general case) and let us just write A for $A_{[\omega]}$, ω_A for $\omega_{[\omega]}$ and d for $d_{[\omega]}$. We denote by $\exp_A : \mathrm{Lie}(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to A^{\mathrm{an}}$ the exponential for A^{an}.

Choose $\omega_1 := \omega_A, \omega_2, \ldots, \omega_d$ a basis of $H^0(A, \Omega^1_{A/\overline{\mathbb{Q}}})$ completing ω_A. Let $\omega_1^*, \ldots, \omega_d^*$ be the corresponding dual basis of $\mathrm{Lie}(A)$. Let

$$p : H_1(A^{\mathrm{an}}; \mathbb{C}) \to H^{-1,0}(A) = \mathrm{Lie}(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

be the natural Hodge projection, it associates to any element $\gamma \in H_1(A^{\mathrm{an}}; \mathbb{C})$ the vector $p(\gamma)$ with coordinates $(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_d)$ in the basis (ω_i^*). The map p defines an isomorphism of $H_1(A^{\mathrm{an}}, \mathbb{Z})$ with the period lattice $\Gamma := p(H_1(A^{\mathrm{an}}, \mathbb{Z})) \simeq \mathbb{Z}^{2d} \subset \mathrm{Lie}(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. Moreover, as A has complex multiplication by a number field K, there exists a unique $\overline{\mathbb{Q}}$-vector subspace $V_{\overline{\mathbb{Q}}} \subset \mathrm{Lie}(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ of dimension d containing Γ.

Remark 4.2. In other words: $V_{\overline{\mathbb{Q}}}$ defines a new $\overline{\mathbb{Q}}$-structure on $\mathrm{Lie}(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

Let x be a point of $Z(\omega)$, different from x_0. Let β be a path in A joining x_0 to x, and $\tilde{x} \in \mathrm{Lie}(A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ the corresponding element. Hence $\exp(\tilde{x}) = \mathrm{alb}_{[\omega]}(x)$ and $\tilde{x} = (\int_\beta \omega_A, \int_\beta \omega_2, \dots, \int_\beta \omega_d)$.

Lemma 4.3. $\tilde{x} \in V_{\overline{\mathbb{Q}}}$.

Proof. We follow the strategy of the proof of [ShWo95, Prop.3].

Let us fix a non-zero $\gamma \in H_1(A^{\mathrm{an}}, \mathbb{Z})$. As noticed in [ShWo95, proof of Proposition 1], the simplicity of A and Theorem 4.1 imply that $\int_\gamma \omega \neq 0$. The condition $[\omega]_{\mathrm{Betti}} \in \mathbf{P}H^1(C, Z([\omega]); \overline{\mathbb{Q}})$ implies that there exists $\alpha \in \overline{\mathbb{Q}}$ such that $\int_\beta \omega_A = \alpha \cdot \int_\gamma \omega_A$.

Consider $\exp_{A\times A} : \mathrm{Lie}(A \times A) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to (A \times A)^{\mathrm{an}}$, the exponential for $(A \times A)^{\mathrm{an}}$. Let $\mathfrak{b} \subset \mathrm{Lie}(A \times A)$ the $\overline{\mathbb{Q}}$-hyperplane defined by

$$\mathfrak{b} := \{(z, w) \in \mathrm{Lie}(A) \times \mathrm{Lie}(A) \mid \omega_A(w) = \alpha \cdot \omega_A(z)\} \ .$$

The vector $u := (p(\gamma), \tilde{x}) \in \mathfrak{b} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is non-zero and satisfies $\exp_{A\times A}(u) = (1, x) \in (A \times A)(\overline{\mathbb{Q}})$. Applying Theorem 4.1 to $\mathbf{G} := A \times A$, we conclude that there exists a non-zero $\overline{\mathbb{Q}}$-subspace $\mathfrak{h} \subset \mathfrak{b}$ which is the Lie algebra of an algebraic subgroup $\mathbf{H}$ of $A \times A$ defined over $\overline{\mathbb{Q}}$. As A is simple, the group $\mathbf{H}$ is isogenous to A. By [ShWo95, Lemma 3] the Lie algebra $\mathfrak{h}$ is defined by a non-singular system of d linear equations $w = C \cdot z$ for some $C \in \mathrm{End}_0(A)$. As A is simple, with complex multiplication by a number field K, $\mathrm{End}_0(A) = K$. Hence $C = \alpha \in K$ and $\tilde{x} = \alpha \cdot p(\gamma) \in V_{\overline{\mathbb{Q}}}$. □

We conclude that x is a torsion point in A by applying the following result of Masser [Ma76] (a similar statement was obtained by Lang [Lang75]).

Theorem 4.4. *(Masser) Let A be an abelian variety of dimension r with CM (in particular A is defined over $\overline{\mathbb{Q}}$). Let $\exp_A : \mathrm{Lie}(A_{\mathbb{C}}) \to A^{\mathrm{an}}$ be the uniformization for A^{an}, with kernel Γ. Let $V_{\overline{\mathbb{Q}}} \subset \mathrm{Lie}(A_{\mathbb{C}})$ be the $\overline{\mathbb{Q}}$-vector subspace of dimension r containing Γ.*

Let $\tilde{x} \in V_{\overline{\mathbb{Q}}}$ such that $x := \exp_A(\tilde{x})$ belongs to $A(\overline{\mathbb{Q}})$. Then x is a torsion point in A and $\tilde{x} \in \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.

This finishes the proof that any arithmetic point $(C, [\omega]) \in \mathrm{S}_\alpha(\mathbb{C})$ satisfies (a), (b) and (c) of Theorem 2.2.

The same analysis shows easily that the converse is true. This finishes the proof of Theorem 2.2. □

Remark 4.5. Masser's Theorem 4.4 is itself an incarnation of the bi-algebraic format over $\overline{\mathbb{Q}}$ for abelian varieties with complex multiplication, the algebraic structure over $\overline{\mathbb{Q}}$ considered on $\mathrm{Lie}(A_{\mathbb{C}})$ being the one given by the period lattice Γ rather than the standard one $\mathrm{Lie}A$. See [U16, Section 2.2.2]. It is also a direct consequence of Theorem 4.1. See [U16, Theor. 2.9] for a generalization.

5. Distribution of arithmetic points

5.1. **Density of arithmetic points: proof of Proposition 2.4.**

For $\alpha = 0$, the stratum S_0 is the modular curve parametrizing complex elliptic curves. The arithmetic points of S_0 are precisely the CM elliptic curves, which are well known to be dense in $\mathrm{S}_0^{\mathrm{an}}$, and are obviously of degree 1.

For $\alpha \neq 0$, we construct a dense set of arithmetic points of degree 1 in $\mathrm{S}_\alpha^{\mathrm{an}}$ as covers of CM elliptic curves ramified precisely over torsion points. Let us notice we do not need Theorem 2.2 for this construction.

Let $U \subset \mathrm{S}_\alpha^{\mathrm{an}}$ be a simply-connected open subset and $\phi : U \to \mathbf{P}^{d_\alpha}(\mathbb{C})$ be a period chart. Let us show that all points in U with homogeneous ϕ-coordinates in $\mathbb{Q}(i) \subset \mathbb{C}$ are arithmetic of degree 1 (they are clearly dense in $\mathrm{S}_\alpha^{\mathrm{an}}$). Let $u \in U$ be such a point, corresponding to a pair $(C, [\omega])$. Choose a representative ω of $[\omega]$ such that its period coordinates lie in $\mathbb{Q}(i)$. Let $\gamma_0, \dots, \gamma_{d_\alpha}$ be a basis of $H_1(C^{\mathrm{an}}, Z(\omega); \mathbb{Z})$ and let $D \in \mathbb{N}$

be an integer such that $\int_{\gamma_k} \omega \in \frac{1}{D}(\mathbb{Z}+i\mathbb{Z})$, for $0 \leq k \leq d_\alpha$. If $x_0 \in Z(\omega)$ and E is the complex elliptic curve defined by $E^{\mathrm{an}} = \mathbb{C}/\frac{1}{D}(\mathbb{Z}+i\mathbb{Z})$, the map

$$f : C^{\mathrm{an}} \to E^{\mathrm{an}}$$
$$x \mapsto \int_{x_0}^{x} \omega$$

defines a ramified cover, whose ramification locus lies over $0 \in E^{\mathrm{an}}$. Moreover $f^*(dz) = \omega$, where dz is the 1-form on E deduced from dz on $\mathbb{C}$. The elliptic curve E is defined over $\mathbb{Q}$ (indeed its analytification is isomorphic to $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, which has j-invariant 1728, hence defined over $\mathbb{Q}$), in particular over $\overline{\mathbb{Q}}$. Since $0 \in E(\overline{\mathbb{Q}})$ and C is ramified only over 0, we deduce by [Gro71, XIII, Proposition 4.6] that C and f are defined over $\overline{\mathbb{Q}}$ (in the notation of *loc. cit.* one takes $k = \overline{\mathbb{Q}}$ and $Y = \mathrm{Spec}(\mathbb{C})$). Moreover, the line $\mathbb{C}\omega$ in $H^0(C, \Omega^1_C)$ is defined over $\overline{\mathbb{Q}}$: indeed, since $H^0(E, \Omega^1)$ is one dimensional, there exists $\alpha \in \mathbb{C}$ such that $\alpha\, dz \in H^0(E_{\overline{\mathbb{Q}}}, \Omega^1_{E/\overline{\mathbb{Q}}})$, and then $\alpha\,\omega$ is defined over $\overline{\mathbb{Q}}$. □

Remark 5.1. (1) An abelian differential (C,ω) with relative periods in $\mathbb{Q}(i)$ is known in the literature as a *square-tiled surface*.

(2) The above argument goes through with $\mathbb{Q}(i)$ replaced by any quadratic imaginary number field embedded in $\mathbb{C}$.

Let us describe an apparently more general construction of arithmetic points of degree 1, by allowing several branch torsion points in E rather than just one. Let E be a complex elliptic curve with complex multiplication (we refer to [Sil94, Chap.II] for a description of such curves). Hence E admits a model over $\overline{\mathbb{Q}}$, which we still denote E by abuse of notations. Let us write its complex uniformization $E^{\mathrm{an}} = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, where τ is imaginary quadratic. Let $\omega_E \in H^0(E, \omega^1_{E/\overline{\mathbb{Q}}})$ be a non-zero algebraic one-form on E defined over $\overline{\mathbb{Q}}$. Its pull-back to $\mathbb{C}$ under the uniformization map $\pi : \mathbb{C} \to E^{\mathrm{an}}$ can be written $\tilde{\omega} = \alpha \cdot dz$, where z is a natural holomorphic coordinate on $\mathbb{C}$. The absolute periods of ω_E on $H_1(E^{\mathrm{an}};\mathbb{Z}) = \mathbb{Z}+\tau\mathbb{Z}$ lie in $\alpha\cdot(\mathbb{Z}+\tau\mathbb{Z})$ hence in $\alpha\cdot\overline{\mathbb{Q}}$.

Let k be an integer and let $e_i \in E(\overline{\mathbb{Q}})$, $1 \leq i \leq k$, be distinct torsion points in E. Choose $\tau_i \in \mathbb{C}$, $1 \leq i \leq k$, some π-preimage of e_i under the uniformization map $\mathbb{C} \to E^{\mathrm{an}}$. If n_i denotes the order of e_i in E, the complex number $n_i \cdot \tau_i$ lies in the lattice $\mathbb{Z}+\tau\mathbb{Z}$. In particular

$$\forall i \in \{1,\dots,k\}, \qquad \int_0^{\tau_i} \tilde{\omega}_E \in \alpha\cdot\overline{\mathbb{Q}} \ . \tag{5.1}$$

This shows that

$$[\omega_E]_{\mathrm{Betti}} \in \mathbf{P}H^1(E^{\mathrm{an}}, \{e_1,\dots,e_k\}; \overline{\mathbb{Q}}) \ . \tag{5.2}$$

Let $g \geq 2$ and choose a type α for g. Given $f : C \to E$ a smooth projective curve over $\overline{\mathbb{Q}}$ ramified only over the e_i's, we define $\omega := f^*\omega_E \in H^0(C, \Omega^1_{C/\overline{\mathbb{Q}}})$ as the pull-back to C of ω_E. Classical ramification theory shows one can choose k and f such that the pair $(C,[\omega])$ defines a point in $\mathrm{S}^{\mathrm{an}}_\alpha$. As the point $[\omega] \in \mathbf{P}H^1(C^{\mathrm{an}}, Z(\omega); \mathbb{C})$ is the image of $[\omega_E]$ under the complexification of the natural Betti pull-back map

$$f^* : \mathbf{P}H^1(E^{\mathrm{an}}, \{e_1,\dots,e_k\}; \mathbb{Z}) \to \mathbf{P}H^1(C^{\mathrm{an}}, Z(\omega); \mathbb{Z}) \ ,$$

it follows from (5.2) that $[\omega]_{\mathrm{Betti}}$ belongs to $\mathbf{P}H^1(C, Z(\omega); \overline{\mathbb{Q}})$. Hence the point $(C,[\omega])$ is arithmetic in $\mathrm{S}_\alpha(\mathbb{C})$.

Let us notice that the construction of arithmetic points of degree 1 we just described does not produce more points in S_α than square-tiled surfaces. Indeed, if $f : C \to E$ is a ramified cover whose whose branch locus consists of torsion points, let $N \in \mathbb{N}$ be an exponent killing these torsion points and $[N] : E \to E$ be the multiplication-by-N map. Then $[N] \circ f : C \to E$ is a ramified cover, ramified only over $0 \in E$. The abelian differentials $(C, f^*\omega_E)$ and $(C, ([N] \circ f)^*\omega_E)$ are different points of H_α, but define the same point in S_α.

5.2. **Propagation of arithmetic points by ramified covers.** The construction Section 5.1 can be generalized to the following lemma, whose proof using Theorem 2.2 is immediate:

Lemma 5.2. *Let $(C, [\omega])$ be an arithmetic point in $\mathrm{S}_\alpha(\overline{\mathbb{Q}})$, and let $\mathrm{alb}_{A_{[\omega]}} : C \to A_{[\omega]}$ be as in Definition 2.1, normalised by sending a zero of ω to the origin in $A_{[\omega]}$. If $f : C' \to C$ is a ramified cover of curves whose branch locus Branch(f) in C satisfies $\mathrm{alb}_{A_{[\omega]}}(Branch(f)) \subset Tors(A_{[\omega]})$, then $(C', [f^*\omega])$ is also an arithmetic point (in the corresponding stratum $\mathrm{S}_{\alpha'}(\overline{\mathbb{Q}})$).*

Moreover, the degrees of $(C, [\omega])$ and $(C', [f^*\omega])$ are equal.

Proof. With our normalisation, points of $Z(\omega)$ are sent to torsion points in $A_{[\omega]}$, by condition Theorem 2.2(c). The variety $A_{[f^*\omega]}$ factor of $\mathrm{Alb}(C')$ coincides with $A_{[\omega]}$, the image of C' in $A_{[\omega]}$ is $\mathrm{alb}_{A_{[\omega]}}(C)$ and the zeros of $f^*\omega$ lie over torsion points. Hence $(C', [f^*\omega])$ is an arithmetic point in $\mathrm{S}_{\alpha'}(\overline{\mathbb{Q}})$ by Theorem 2.2, of the same degree as $(C, [\omega])$. $\square$

5.3. **Primitive arithmetic points.**

Definition 5.3. *An arithmetic point $(C, [\omega]) \in \mathrm{S}_\alpha(\overline{\mathbb{Q}})$ is said to be geometrically primitive if it cannot be obtained from an arithmetic point in a stratum of smaller genus via a ramified cover as in Lemma 5.2.*

Remark 5.4. Notice that any geometrically primitive arithmetic point $(C, [\omega])$ in $\mathrm{S}_\alpha(\overline{\mathbb{Q}})$ has degree $d_{[\omega]}$ at least 2 as soon as $\alpha \neq 0$.

Definition 5.5. *An arithmetic point $(C, [\omega]) \in \mathrm{S}_\alpha(\overline{\mathbb{Q}})$ is said to be algebraically primitive if its associated CM factor $A_{[\omega]}$ coincides with $\mathrm{Alb}(C)$, i.e. $d_{[\omega]} = g(C)$.*

Remark 5.6. Any algebraically primitive arithmetic point $(C, [\omega]) \in \mathrm{S}_\alpha(\overline{\mathbb{Q}})$ is obviously geometrically primitive.

Example 5.7. Let us provide examples of algebraically primitive arithmetic points of degree at least 2 which are Veech surfaces. The simplest examples of algebraically primitive Teichmüller curves belong to the original Veech's family [Ve89, Theorem 1.1]. For all $n \geq 1$, these Teichmüller curves are the projection in $\mathbf{P}\Omega\mathcal{M}_{\lfloor \frac{n-1}{2} \rfloor}$ of the $\mathbf{GL}^+(2, \mathbb{R})$-orbit of $(C_n, \frac{dx}{y})$ where C_n is the (smooth projective model of the) plane hyperelliptic curve $y^2 = x^n - 1$. When n is odd, they belong to the minimal stratum S_{n-3} of $\mathbf{P}\Omega\mathcal{M}_{\frac{n-1}{2}}$ parametrizing abelian differentials with a single zero. When $n = p$ is prime, it is well-known that $\mathrm{Jac}\,(C_p)$ is CM and simple; since ω_p has a single zero, Theorem 2.2 implies that $(C_p, [\omega_p])$ is an algebraically primitive arithmetic point in S_{p-3} which is a Veech surface.

Similarly, let $C_{1,1}$ be the genus 2 hyperelliptic curve $y^2 = x^6 - x$. By [LB92, Remark 13.3.8] its Jacobian surface $\mathrm{Jac}\,(C_{1,1})$ has complex multiplication. Moreover it is simple: according to Bolza [Bol87] (see also [LB92, p.340]) its reduced group of automorphism is $\mathbb{Z}/5\mathbb{Z}$, hence C does not admit any involution with quotient an elliptic curve. The abelian differential $(C_{1,1}, [\frac{dx}{y}])$ is thus an algebraically primitive arithmetic point in $\mathrm{S}_{1,1}(\overline{\mathbb{Q}})$ which is a Veech surface. The algebraically primitive Teichmüller curves it generates is the famous decagon family, see [McM06, Theor. 1.1] (we thank M. Möller for mentioning this example to us).

Remark 5.8. Veech's examples are a particular case of a more general family of Teichmüller curves discovered by Bouw and Möller [BM10], whose Veech groups are triangle groups with parameters (m, n, ∞), $m, n \in \mathbb{N} \cup \{\infty\}$. The Bouw-Möller family also contains algebraically primitive Teichmüller curves in unbounded genera, different from Veech's. It is at present the only known family containing primitive Teichmüller curves in unbounded genera.

We claim that every Teichmüller curve in the Bouw-Möller family contains an arithmetic point. Indeed let $\mathscr{H} \to \overline{C}$ be the family of curves constructed in [BM10, Section 6]. Thus $\overline{C} \xrightarrow{\pi} \mathbf{P}^1$ is a suitable cover ramified over $\{0, 1, \infty\}$, such that the image of $\overline{C} \setminus \pi^{-1}(\infty)$ in $\mathcal{M}_g$ is a Teichmüller curve. If $c \in \pi^{-1}(0)$, one shows that the fiber $\mathscr{H}_c$ has a Jacobian with CM. This follows by inspection of the construction in [BM10, Lemma 6.8] together with the fact that (a connected component of) the curve $z^N = x^a(x-1)^b$ has Jacobian with CM. By Theorem 2.2 and the fact that $\mathscr{H}_c$ together with its 1-form up to scaling $[\omega]$ is a Veech surface, we deduce that $(\mathscr{H}_c, [\omega])$ is an arithmetic point.

5.4. Proof of Proposition 2.3.

Notice that the statement of Lemma 5.2 remains valid if one replaces "arithmetic point" by "Veech surface". As any arithmetic point $(C, [\omega]) \in \mathrm{S}_\alpha$ of degree 1 is obtained as in Lemma 5.2 from a CM elliptic curve $(E, [\omega_E]) \in \mathrm{S}_0$, and as $(E, [\omega_E])$ is trivially a Veech surface in S_0, it follows that any arithmetic point of degree 1 is a Veech surface.

Let us exhibit an arithmetic point of higher degree which is not a Veech surface. The curve $C_7 : y^2 = x^7 - 1$ together with the differential $\left[\frac{x\,dx}{y}\right]$ defines an arithmetic point of degree 3. To see this we check the conditions of Theorem 2.2. As remarked above, $\mathrm{Jac}\,(C_7)$ is simple of dimension 3 and has complex multiplication. Denote by ∞ the unique point of C_7 that lies above $\infty \in \mathbf{P}^1$ for the projection $(x, y) \mapsto x$. The divisor of zeros of $\left[\frac{x\,dx}{y}\right]$ is given by $(0, i) + (0, -i) + 2 \cdot \infty$. The classes $[(0, \pm i) - \infty]$ are 7-torsion in the Jacobian (as $7(0, i) - 7\infty$ is the divisor of the function $y - i$), hence condition of (c) of Theorem 2.2 is fulfilled. However, by [Mö06b, Remark 2.9] and [McM03a, Theorem 7.5], this abelian differential does not generate a Teichmüller curve. □

5.5. Linear invariant subvarieties with only few arithmetic points: proof of Theorem 2.6.

Let us first give the proof of Theorem 2.6 in the case of a Teichmüller curve, where the geometry is simpler. Recall the following fundamental result:

Theorem 5.9 ([Mö06b]). *Let $Z \subset \mathbf{P}\Omega^1\mathcal{M}_g$ be a Teichmüller curve of degree r. Then the restriction of $\mathbb{V}_\alpha$ to Z satisfies*

$$\mathrm{Gr}_1^W \mathbb{V}_{\alpha|Z} = (\bigoplus_{i=1}^{d} \mathbb{L}_i) \oplus \mathbb{M} \quad ,$$

where $\mathbb{M}$ is defined over $\mathbb{Q}$, the $\mathbb{L}_i$ are rank 2 local subsystems, maximal Higgs for $i = 1$, and non-unitary but not maximal Higgs for $i \neq 1$.

In particular, the image Z'^{p} of Z in $\mathcal{A}_g$ is contained in the locus of abelian varieties that split up to isogeny as $A_1 \times A_2$, $\dim A_1 = d$ and A_1 has real multiplication.

Let $Z \subset \mathrm{S}_\alpha \subset \mathbf{P}\Omega^1\mathcal{M}_g$ be a Teichmüller curve of degree d. Consider the following diagram provided by (3.3) and Theorem 5.9:

$$
\begin{tikzcd}
\mathrm{S}_\alpha \arrow[r, hook, "\Phi"] & \mathfrak{A}_g^{(n-1)} \arrow[r, two heads] & \mathcal{A}_g \\
Z \arrow[u, hook] \arrow[r, hook] \arrow[d] & \mathfrak{A}_{d,g-d}^{(n-1)} \arrow[u, hook] \arrow[r, two heads] \arrow[d] & \mathcal{A}_r \times \mathcal{A}_{g-d} \arrow[u, hook] \arrow[d] \\
Z' \arrow[r, hook] & \mathfrak{A}_d^{(n-1)} \arrow[r, two heads] & \mathcal{A}_d .
\end{tikzcd}
\tag{5.3}
$$

Here $\mathfrak{A}_{d,g-d}^{(n-1)} \subset \mathfrak{A}_g^{(n-1)}$ is the special subvariety of $\mathfrak{A}_g^{(n-1)}$ containing Z and parametrizing abelian varieties that split up to isogeny as $A_1 \times A_2$, $\dim A_1 = d$. It is isomorphic to $\mathfrak{A}_{d,g-d} \times_{(\mathcal{A}_d \times \mathcal{A}_{g-d})} \cdots \times_{(\mathcal{A}_d \times \mathcal{A}_{g-d})} \mathfrak{A}_{d,g-d}$, and $\mathfrak{A}_d^{(n-1)} := \mathfrak{A}_d \times_{\mathcal{A}_d} \cdots \times_{\mathcal{A}_d} \mathfrak{A}_d$ is its natural quotient.

Let $Z'^{\mathrm{p}} \subset \mathcal{A}_d$ be the projection of Z' in $\mathcal{A}_d$ (its pure part). If $(C, [\omega]) \in Z(\mathbb{C})$ is an arithmetic point, it follows from Theorem 2.2 that the image of $(C, [\omega])$ in $Z''(\mathbb{C}) \subset \mathcal{A}_d(\mathbb{C})$ is a CM point. Therefore, if Z contains infinitely many arithmetic points, then $\overline{Z'^{\mathrm{p}}}$ contains infinitely CM points. By the André-Oort conjecture for $\mathcal{A}_d$ [Ts18], $\overline{Z'^{\mathrm{p}}} \subset \mathcal{A}_d$ is a Shimura curve in $\mathcal{A}_d$.
It then follows from [Mö11, Theor.1.2] that the $\mathbb{L}_i$s, $2 \leq i \leq d$, which comes from the Shimura curve $\overline{Z'^{\mathrm{p}}}$, are also maximal Higgs. This is a contradiction to Theorem 5.9 if $d \geq 2$. This proves Theorem 2.6 for Teichmüller curves.

Let us now turn to the general case. Let $Z \subset \mathrm{S}_\alpha$ be a linear invariant subvariety. Recall the variation (3.2) of mixed $\mathbb{Z}$-Hodge structures ($\mathbb{Z}$VMHS) on S_α:

$$
0 \to W_0\mathbb{V}_{\alpha,\mathbb{Z}} \simeq \mathbb{Z}(0)^{n-1} \to \mathbb{V}_{\alpha,\mathbb{Z}} \to \mathrm{Gr}_1^W \mathbb{V}_{\alpha,\mathbb{Z}} \to 0 \ . \tag{5.4}
$$

The real tangent bundle $T_\mathbb{R}Z$ is a sub-$\mathbb{R}$VMHS of the restriction to Z of $\mathbb{V}_{\alpha,\mathbb{R}} := \mathbb{V}_{\alpha,\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{R}$, of the form:

$$
0 \to W_0T_\mathbb{R}Z \to T_\mathbb{R}Z \to \mathbb{H}_1 := \mathrm{Gr}_1^W T_\mathbb{R}Z \to 0 \ . \tag{5.5}
$$

We denote by u the rank of $W_0T_\mathbb{R}Z$ and by $2k$ the rank of $\mathbb{H}_1$. The integer k is called the rank of Z. In particular:

$$
\dim_\mathbb{C} Z := u + 2k - 1 \ . \tag{5.6}
$$

According to [Fil16a], the $\mathbb{R}$VMHS $T_\mathbb{R}Z$ is defined over a totally real number field $K_0 \overset{\iota_1}{\hookrightarrow} \mathbb{R}$:

$$
T_\mathbb{R}Z = T_{K_0}Z \otimes_{K_0,\iota_1} \mathbb{R} \ .
$$

We write $d := [K_0 : \mathbb{Q}]$. In particular

$$
\mathrm{Gr}_1^W \mathbb{V}_{\alpha,\mathbb{R}|Z} = (\mathbb{H}_1 \oplus \cdots \oplus \mathbb{H}_d) \oplus \mathbb{M} \ , \tag{5.7}
$$

where the $\mathbb{H}_i$, $1 \leq i \leq d$, are the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of $\mathbb{H}_1$ and the factor $\mathbb{M}$ is defined over $\mathbb{Q}$. We thus obtain the following generalization of (5.3) (Teichmüller curves correspond to the case $(k = 1, u = 0)$):

$$
\begin{tikzcd}
\mathrm{S}_\alpha \arrow[r, hook, "\Phi"] & \mathfrak{A}_g^{(n-1)} \arrow[r, two heads] & \mathcal{A}_g \\
Z \arrow[u, hook] \arrow[d] \arrow[r, hook] & \mathfrak{A}^{(n-1)}_{kd,g-kd} = \mathfrak{A}^{(n-1)}_{kd} \times \mathfrak{A}^{(n-1)}_{g-kd} \arrow[u, hook] \arrow[d] \arrow[r, two heads] & \mathcal{A}_{kd} \times \mathcal{A}_{g-kd} \arrow[u, hook] \arrow[d] \\
Z' \arrow[r, hook] & \mathfrak{A}^{(n-1)}_{kd} \arrow[r, two heads] & \mathcal{A}_{kd}.
\end{tikzcd}
\tag{5.8}
$$

Lemma 5.10. *The projection $Z \to Z'$ is an isomorphism. In particular* $\dim_{\mathbb{C}} Z = \dim_{\mathbb{C}} Z'$.

Proof. Let $(C, [\omega]) \in Z$. Seen in $\mathfrak{A}_g^{(n-1)}$, this point corresponds to the mixed Hodge structure

$$0 \to \mathbb{Z}(0)^{n-1} \to H^1(C, Z([\omega]); \mathbb{Z}) \to H^1(\mathrm{Alb}(C), \mathbb{Z}) \to 0 \quad . \tag{5.9}$$

Its image in $Z' \subset \mathfrak{A}^{(n-1)}_{kd}$ is the mixed Hodge structure

$$0 \to \mathbb{Z}(0)^{n-1} \to E \to H^1(A_{[\omega]}, \mathbb{Z}) \to 0 \tag{5.10}$$

corresponding to the projection of the zeroes of $[\omega]$ in $A_{[\omega]}$. But (5.9) is nothing else than the pull-back of (5.10) under the natural projection $\mathrm{Alb}(C) \to A_{[\omega]}$. Hence the result. $\square$

Suppose that $k = 1$ and $(C, [\omega]) \in Z(\mathbb{C})$ is any arithmetic point, or that $k > 1$ and $(C, [\omega]) \in Z(\mathbb{C})$ is an arithmetic point of degree at least kd. In all these cases the containment $A_{[\omega]} \in \mathcal{A}_{kd}$ implies that $(C, [\omega])$ has degree exactly kd. It then follows from Theorem 2.2 that the image of $(C, [\omega])$ in $Z'(\mathbb{C}) \subset \mathfrak{A}^{(n-1)}_{kd}(\mathbb{C})$ is a CM point. Therefore, if Z contains a Zariski-dense set of arithmetic points of such points, then $\overline{Z'}^{\mathrm{Zar}} \subset \mathfrak{A}^{(n-1)}_{kd}$ contains a Zariski-dense set of CM points. By the André-Oort conjecture for the mixed Shimura variety of Kuga type $\mathfrak{A}^{(n-1)}_{kd}$ [Ts18], [Gao16], the closed subvariety $\overline{Z'}^{\mathrm{Zar}} \subset \mathfrak{A}^{(n-1)}_{kd}$, which contains Z' as a Zariski-open dense subset, is a special subvariety of $\mathfrak{A}^{(n-1)}_{kd}$.

In [Fil17, Cor. 1.7], extended in [EFW18, Prop. 4.7], it is shown that the real Zariski-closure of the monodromy of the flat bundle $T_{\mathbb{R}}Z$ at a point $z_0 \in Z$ is as big as it can be, namely equal to $\mathbf{Sp}(\mathbb{H}_{1,z_0}) \ltimes \mathrm{Hom}(\mathbb{H}_{1,z_0}, (W_0 T_{\mathbb{R}} Z)_{z_0})$; and that the same holds true for the Galois conjugates bundles: the real Zariski closure of $(T_{\mathbb{R}}Z)_{\iota_i}$, $1 \leq i \leq d$, is $\mathbf{Sp}(\mathbb{H}_{i,z_0}) \ltimes \mathrm{Hom}(\mathbb{H}_{i,z_0}, (W_0 T_{\mathbb{R}} Z)_{z_0})$. If we denote by $T_{\mathbb{Q}}Z$ the $\mathbb{Q}$-sub-VMHS $\mathrm{Res}_{K_0/\mathbb{Q}} T_{K_0} Z$ of $\mathbb{V}_{\alpha,\mathbb{Q}}$, it follows that the ($\mathbb{Q}$-)algebraic monodromy group of $T_{\mathbb{Q}}Z$ is the group $\mathrm{Res}_{K_0/\mathbb{Q}}(\mathbf{Sp}_{2k,K_0} \ltimes \mathrm{Hom}_{K_0}(\mathbb{H}_{1,K_0,z_0}, (W_0 T_{\mathbb{R}} Z)_{K_0,z_0}))$. In particular the $\mathbb{Q}$-algebraic monodromy group defining the special subvariety $\overline{Z'}^{\mathrm{Zar}}$ of $\mathfrak{A}^{(n-1)}_{kd}$ equals $\mathrm{Res}_{K_0/\mathbb{Q}}(\mathbf{Sp}_{2k,K_0} \ltimes \mathrm{Hom}_{K_0}(\mathbb{H}_{1,K_0,z_0}, (W_0 T_{\mathbb{R}} Z)_{K_0,z_0}))$. It follows that:

$$\dim \overline{Z'}^{\mathrm{Zar}} = d \times \left(\frac{k(k+1)}{2} + uk\right) \quad . \tag{5.11}$$

Comparing (5.6) and (5.11) using Lemma 5.10, one obtains:

$$u + 2k - 1 = d \times \left(\frac{k(k+1)}{2} + uk\right) \quad . \tag{5.12}$$

One easily checks that this inequality can be satisfied only for $(d = 1, k = 2, u = 0)$ or $(d = 1, k = 1, \text{any } u)$. This finishes the proof of Theorem 2.6.

$\square$

6. Bi-algebraic subvarieties of S_α and linearity

Let us introduce a few notations we will use in this section. Working over H_α or S_α, we write:

- $\mathbb{V}^{\mathrm{p}}_{\alpha,\mathbb{Z}}$, resp. $\mathbb{V}^{\mathrm{p}}_{\alpha}$, for the pure piece of weight one $\mathrm{Gr}^W_1\mathbb{V}_{\alpha,\mathbb{Z}}$, resp. $\mathrm{Gr}^W_1\mathbb{V}_\alpha$ (either on H_α or S_α, depending on the context); $\mathcal{V}^{\mathrm{p}}_\alpha$ for the holomorphic vector bundle defined by $\mathbb{V}^{\mathrm{p}}_\alpha$; and $\mathrm{V}^{\mathrm{p}}_\alpha$ for the fiber of $\mathbb{V}^{\mathrm{p}}_\alpha$ over a point.
- $\mathcal{C}_g \to \mathcal{M}_g$ for the universal curve.
- $\mathcal{C}_\alpha := \pi^*_\alpha\mathcal{C}_g$ and $\mathcal{J}_\alpha := \mathrm{Jac}\,(\mathcal{C}_\alpha/\mathrm{H}_\alpha)$ the relative Jacobian of $\mathcal{C}_\alpha$ over H_α or S_α (where π_α denotes the projection from H_α or S_α to $\mathcal{M}_g$).

Let $Z \subset \mathrm{S}_\alpha$ be an algebraic subvariety. Let $\mathbb{T}$ be the largest constant local sub-system of $\mathrm{Gr}^W_1\mathbb{V}_\alpha|_Z$, $\mathcal{T} := \mathbb{T}\otimes\mathcal{O}_Z$ and $\omega^{\mathrm{p}}|_Z$ the projection of $\omega|_Z$ to $\mathbf{P}\mathcal{V}^{\mathrm{p}}_\alpha|_Z$. Our analysis of the bi-algebraic structure on S_α depends on the position of $\omega^{\mathrm{p}}|_Z$ with respect to $\mathbf{P}\mathcal{T}$. We deal with two "opposite" cases:

- $\omega^{\mathrm{p}}|_Z$ is a constant section of $\mathbf{P}\mathcal{T}$.
- $\omega^{\mathrm{p}}|_Z$ is not generically contained in $\mathbf{P}\mathcal{T}$.

We do not deal with the intermediate case where $\omega^{\mathrm{p}}|_Z$ is a non-constant section of $\mathbf{P}\mathcal{T}$. The first case corresponds to the case where Z is contained in an isoperiodic leaf. We treat it in Section 6.1. The second case is treated in Section 6.2. In Section 6.3 we prove that the intermediate case mentioned above does not occur in genus 2 and in Section 6.4 that it does occur in genus 3.

6.1. The isoperiodic foliation and proof of Theorem 2.8.

The isoperiodic foliation is a foliation on H_α by complex submanifolds of codimension $2g$ defined locally as follows. Over a chart $U \subset \mathrm{H}^{\mathrm{an}}_\alpha$ with period coordinates $U \to \mathrm{V}_\alpha$, the isoperiodic foliation on U is given as pullback of the foliation on V_α given by the fibers of the projection $\mathrm{V}_\alpha \to \mathrm{V}^{\mathrm{p}}_\alpha$. In other words, a leaf around a point (C_0, ω_0) consists of nearby points (C, ω) such that the absolute periods of ω are the same as the absolute periods of ω_0 (the periods are computed on a flat frame of homology). By passing to the projectivisations, one obtains the isoperiodic foliation on S_α.

Lemma 6.1. *Let $Z \subset \mathrm{H}_\alpha$ be an algebraic subvariety of an isoperiodic leaf. There exists an isogeny of abelian schemes*

$$f : \mathcal{J}_\alpha|_Z \to \mathcal{A}\times\mathcal{A}',$$

where $\mathcal{A}$ is an isotrivial abelian scheme with constant fiber A_0 and $\omega|_Z \in H^0(\mathcal{J}_\alpha|_Z, \Omega^1_{\mathcal{J}_\alpha|_Z/\mathrm{H}_\alpha|_Z})$ is the pullback of a constant 1-form on $\mathcal{A}$.

Remark 6.2. This lemma generalises the construction of Section 1.2.2 corresponding to the case $\dim A_0 = 1$.

Proof of Lemma 6.1. Translating the statement in terms of $\mathbb{Q}$VHS, it is equivalent to prove a splitting

$$\mathbb{V}_{\alpha,\mathbb{Q}}|_Z \cong \mathbb{V}'\times\mathbb{V}'',$$

where $\mathbb{V}'$ is a constant $\mathbb{Q}$VHS and such that $\omega|_Z$, which is a global section of $\mathcal{V}_\alpha|_Z$, is the pullback of a constant global section of $\mathbb{V}'\otimes\mathcal{O}_Z$.

The fact the Z is contained in an isoperiodic leaf is equivalent to the fact that $\omega|_Z$ is locally flat for the Gauss-Manin connection on $\mathcal{V}_\alpha|_Z$. Since it is a global section, it is globally flat. In particular, the largest constant local sub-system, call it $\mathbb{V}'$, of $\mathbb{V}_{\alpha,\mathbb{Q}}|_Z$ is non-zero. By the theorem of the fixed part, $\mathbb{V}'$ underlies a polarizable $\mathbb{Q}$VHS of weight 1 and the orthogonal to $\mathbb{V}'$ with respect to the polarization gives the required splitting. $\square$

Proposition 6.3. *Let $Z \subset \mathrm{H}_\alpha$ be bi-algebraic subvariety contained in an isoperiodic leaf. Then Z is affine in the period charts.*

Proof. Up to passing to a finite cover of the stratum, we may assume that there are sections $\sigma_1, \dots, \sigma_n : \mathrm{H}_\alpha \to \mathcal{C}_\alpha$, which label the zeros, where $n := \mathrm{length}(\alpha)$. If $n = 1$, then H_α is a minimal stratum and the irreducible components of isoperiodic leaves are singletons. Henceforth, we assume $n \geq 2$. By Lemma 6.1, the tautological section $\omega|_Z$ is the pullback of a constant non-zero 1-form, that we keep calling ω, on a constant factor A of $\mathcal{J}_\alpha|_Z$ of dimension $r \geq 1$. Complete ω to a basis $\omega = \omega_1, \dots, \omega_r$ of $H^0(A, \Omega^1)$. The dual basis gives an identification $\mathrm{Lie}(A) \cong \mathbb{C}^r$, and we denote by $\pi : \mathbb{C}^r \to A^{\mathrm{an}}$ the universal cover, given by the exponential map. Its multivalued inverse is given by

$$\begin{aligned} A^{\mathrm{an}} &\to \mathbb{C}^r, \\ x &\mapsto \left(\int_0^x \omega_1, \dots, \int_0^x \omega_r \right). \end{aligned}$$

Let $\mathrm{pr}_1 : \mathbb{C}^r \to \mathbb{C}$ be the projection to the first factor. Consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\tilde{\sigma}"] \arrow[d] \arrow[rr, bend left, "f"] & \mathbb{C}^r \times \dots \times \mathbb{C}^r \arrow[r, "\mathrm{pr}"] \arrow[d, "\Pi"] & \mathbb{C} \times \dots \times \mathbb{C} \\ Z^{\mathrm{an}} \arrow[r, "\sigma"] & A^{\mathrm{an}} \times \dots \times A^{\mathrm{an}}\,. & \end{tikzcd}$$

where $\mathrm{pr} := (\mathrm{pr}_1, \dots, \mathrm{pr}_1)$, $\Pi := (\pi, \dots, \pi)$, $\sigma := (\sigma_2 - \sigma_1, \dots, \sigma_n - \sigma_1)$, and the products all consist of $n-1$ factors. Let $V := \overline{f(\widetilde{Z})}^{\mathrm{Zar}} \subset \mathbb{C}^{n-1}$ and $W := \mathrm{pr}^{-1}(V)$. We want to show that V is an affine subspace. If $V = \mathbb{C}^{n-1}$, we are done. Assume $V \subsetneq \mathbb{C}^{n-1}$, so that $\mathrm{codim}_{\mathbb{C}^{r(n-1)}} W \geq 1$. Consider the subvariety $\sigma(Z) \times W \subset A^{n-1} \times (\mathbb{C}^r)^{n-1}$, let $\Delta \subset A^{n-1} \times (\mathbb{C}^r)^{n-1}$ be the graph of Π and let U be the analytic irreducible component of $(\sigma(Z) \times W) \cap \Delta$ containing the image of $\widetilde{Z}$. Clearly $\dim U \geq \dim \sigma(Z)$. The following computation shows that U is an unlikely intersection (the codimensions are calculated in $A^{n-1} \times (\mathbb{C}^r)^{n-1}$ and $N := r(n-1)$):

$$\begin{aligned} &\mathrm{codim}\, \sigma(Z) \times W = 2N - \dim \sigma(Z) - \dim W \geq N - \dim \sigma(Z) + 1, \\ &\mathrm{codim}\, \Delta = N, \\ &\mathrm{codim}\, U \leq 2N - \dim \sigma(Z). \end{aligned}$$

According to the Ax-Schanuel theorem (see Theorem 6.4 below), $\mathrm{pr}_{A^{n-1}}(U)$, hence $\sigma(Z)$, is contained in a translate of a strict abelian subvariety B of A^{n-1}. Thus W is contained in a translate $x + L$ of a strict linear subspace $L \subset \mathbb{C}^{r(n-1)}$ and $V \subset \mathrm{pr}(x) + \mathrm{pr}(L)$ (the latter is not necessarily a strict subset of $\mathbb{C}^{n-1}$). After translating back to the origin in A^{n-1}, $\mathbb{C}^{r(n-1)}$ and $\mathbb{C}^{n-1}$, we obtain a new commutative diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\tilde{\sigma}"] \arrow[d] \arrow[rr, bend left, "f"] & L \arrow[r, "\mathrm{pr}|_L"] \arrow[d] & \mathrm{pr}(L) \\ Z \arrow[r, "\sigma"] & B & \end{tikzcd}$$

and we repeat the previous argument. At each step of the process the dimension of L drops at least by one so the process terminates. In the final step, the equality $V = \mathrm{pr}(L)$ holds. $\square$

Theorem 6.4 (Ax-Schanuel for abelian varieties [Ax72])**.** *Let A be a complex abelian variety and $\pi : \mathrm{Lie}(A) \to A^{\mathrm{an}}$ its universal cover. Let $\Delta \subset A \times \mathrm{Lie}(A)$ be the graph of π, V an algebraic subvariety of $A \times \mathrm{Lie}(A)$ and U an analytic irreducible component of $V \cap \Delta$. If*

$$\mathrm{codim}_{A\times \mathrm{Lie}(A)} U < \mathrm{codim}_{A\times \mathrm{Lie}(A)} V + \mathrm{codim}_{A\times \mathrm{Lie}(A)} \Delta \ ,$$

then $\mathrm{pr}_A(U)$ is contained in a translate of a strict abelian subvariety of A (here $\mathrm{pr}_A : A \times \mathrm{Lie}(A) \to A$ is the projection to the first factor).

Proof of Theorem 2.9. Let $Z \subset \mathrm{S}_\alpha$ be a bi-algebraic subvariety contained in an isoperiodic leaf and let Z' be the preimage of Z under the quotient map $\mathrm{H}_\alpha \to \mathrm{S}_\alpha$. It is enough to prove that Z' is linear.

The model of Z' in the absolute periods is a linear subspace $L \subset \mathrm{V}^{\mathrm{p}}_\alpha$ of dimension 1, stable under the algebraic monodromy group $\mathbf{H}_{Z'}$ of $\mathbb{V}^{\mathrm{p}}_\alpha|_{Z'}$. Since $\mathbf{H}_{Z'}$ is semisimple, it has no non-trivial characters, hence it stabilizes L pointwise. This implies that the projection of $\omega|_{Z'}$ to $\mathcal{V}^{\mathrm{p}}_\alpha|_{Z'}$ is stable under the action of $\pi_1(Z')$. By the theorem of the fixed part, we deduce that $\omega|_{Z'}$ is the pullback of a (non-constant) 1-form $\omega_A \in H^0(A \times Z', \Omega^1_{A/Z'})$ on a constant factor A of the relative Jacobian $\mathcal{J}_\alpha|_{Z'}$. The assignment $Z'(\mathbb{C}) \ni z \mapsto \omega_{A,z} \in H^0(A, \Omega^1)$ is algebraic, and the fibers are bi-algebraic subvarieties contained in an isoperiodic leaf. By Proposition 6.3, they are affine. Therefore, the algebraic model M of Z' in the relative periods, is fibered over L by affine subspaces (i.e. the preimage M_x of $x \in L$ is an affine subspace of V_α). But M is stable under $\mathbb{C}^*$ and multiplication by $\lambda \in \mathbb{C}^*$ defines a linear isomorphism $M_x \to M_{\lambda x}$ for $x \in L$. We conclude that Z' is linear. $\square$

6.2. **The condition ($\star$) and proof of Theorem 2.8.**

Definition 6.5. *Let $Z \subset \mathrm{S}_\alpha$ be an irreducible algebraic subvariety. Let $\mathbb{T}$ be the largest constant local sub-system of $\mathrm{Gr}^W_1 \mathbb{V}_\alpha|_Z$. We call condition ($\star$) the following:*

$$(\star) : \textit{There exists } z \in Z(\mathbb{C}) \textit{ such that } \omega_z \notin \mathbf{P}\mathbb{T}_z. \tag{6.1}$$

Proof of Theorem 2.8. We first introduce some notation. Let $Z \subset \mathrm{S}_\alpha$ be a bi-algebraic curve satisfying ($\star$). We fix a base point $z_0 \in Z(\mathbb{C})$ corresponding to a pair $(C_0, [\omega_0])$. Projection to the pure periods induces a rational map $\mathrm{pr}_\alpha : \mathbf{P}\mathrm{V}_\alpha \dashrightarrow \mathbf{P}\mathrm{V}^{\mathrm{p}}_\alpha$. Let $\widetilde{Z} \subset \widetilde{\mathrm{S}^{\mathrm{an}}_\alpha}$ be an analytic irreducible component of $\pi^{-1}(Z^{\mathrm{an}})$, where $\pi : \widetilde{\mathrm{S}^{\mathrm{an}}_\alpha} \to \mathrm{S}^{\mathrm{an}}_\alpha$ is the universal cover at z_0. Let $Y := \overline{D(\widetilde{Z})}^{\mathrm{Zar}} \subset \mathbf{P}\mathrm{V}_\alpha$ be its algebraic model, where D is the developing map, and $Y^{\mathrm{p}} = \overline{\mathrm{pr}_\alpha(Y)} \subset \mathbf{P}\mathrm{V}^{\mathrm{p}}_\alpha$ (note that the image of the developing map does not meet the indeterminacy locus of pr_α). Since Z is bi-algebraic, Y is a closed irreducible curve in $\mathbf{P}\mathrm{V}_\alpha$. Note that Y^{p} is a curve as well: otherwise Y^{p} is a point, thus Z is contained in an isoperiodic leaf, which is not possible since Z satisfies condition ($\star$). We need to prove that Y is a linear subvariety. We divide the proof in two steps.

Step 1: Y^{p} is a linear subvariety.

Let $\bar{v}$ be the image of z_0 in Y^{p} and v a lift of $\bar{v}$ to $\mathrm{V}^{\mathrm{p}}_\alpha$. Let $\mathbf{H}$ be the algebraic monodromy group of $\mathrm{Gr}^W_1 \mathbb{V}_{\alpha,\mathbb{Q}}|_Z$. Since $\mathbf{H}_\mathbb{C}$ is semisimple, we can uniquely write the $\mathbf{H}_\mathbb{C}$-module $\mathrm{V}^{\mathrm{p}}_\alpha$ as $\mathrm{V}^{\mathrm{p}}_\alpha = T \oplus N$, where T is the largest trivial sub-representation and N its complement. Decompose N into its isotypical components

$$N \cong N_1 \oplus \ldots \oplus N_\ell.$$

Observe that the action of $\mathbf{H}(\mathbb{C})$ on $\mathbf{P}N_i$ has no fixed points. Indeed, let $u \in N_i$, $u \neq 0$ with image $\overline{u} \in \mathbf{P}N_i$ and assume for the sake of contradiction that $\mathbf{H}(\mathbb{C})\overline{u} = \overline{u}$. Then $\mathbf{H}(\mathbb{C})$ acts through a character on $\mathbb{C}u$. Since $\mathbf{H}_\mathbb{C}$ is semisimple it has no nontrivial characters, hence u is fixed. However the fixed vectors in $\mathrm{V}^{\mathrm{p}}_\alpha$ belong to T.

By hypothesis ($\star$), up to changing base points, we can assume $v \notin T$. Write $v = v_T + \sum_{i=1}^{\ell} v_i$, with $v_T \in T, v_i \in N_i$. Up to permutation, we can assume that for a certain $\ell' \in \{1, \dots, \ell\}$, we have $v_i \neq 0$, for $1 \leq i \leq \ell'$ and $v_i = 0$ for $\ell' < i \leq \ell$. For $1 \leq i \leq \ell'$, denote $\overline{v_i}$ the image of v_i in $\mathbf{P}N_i$. Then $\dim \mathbf{H}(\mathbb{C})\overline{v_i} = 1$. Indeed consider the rational map

$$p_i : \mathbf{P}\mathrm{V}_\alpha^{\mathrm{p}} \dashrightarrow \mathbf{P}N_i.$$

Since $\overline{v}$ belongs to the domain of definition of this map, it makes sense to consider the image of Y^{p} under p_i and we have $\dim p_i(Y^{\mathrm{p}}) \leq \dim Y^{\mathrm{p}} = 1$. If $\dim p_i(Y^{\mathrm{p}}) = 0$ then $\overline{v_i}$ is a fixed point under $\mathbf{H}(\mathbb{C})$ and we saw that it is not possible. We conclude that $\dim p_i(Y^{\mathrm{p}}) = 1$. Since $\mathbf{H}(\mathbb{C})\overline{v_i} \subset p_i(Y^{\mathrm{p}})$ and $\dim \mathbf{H}(\mathbb{C})\overline{v_i} \geq 1$, we deduce $\dim \mathbf{H}(\mathbb{C})\overline{v_i} = 1$. Moreover, the orbit $\mathbf{H}(\mathbb{C})\overline{v_i} \subset \mathbf{P}N_i$ is closed (otherwise, the boundary of the orbit would consist of fixed points). Therefore, the stabilizer of $\overline{v_i}$ is a parabolic subgroup of $\mathbf{H}(\mathbb{C})$. In particular, $\mathbb{C}v_i$ is stabilized by a Borel subgroup: this can only happen if v_i is a highest weight vector of the $\mathbf{H}_{\mathbb{C}}$-module N_i.

For each $i \in \{1, \dots, \ell'\}$, let M_i be the $\mathbf{H}_{\mathbb{C}}$-submodule of N_i generated by the orbit $\mathbf{H}(\mathbb{C})v_i$. It is clearly irreducible and v_i is a highest-weight vector. We now appeal to the classification, due to Satake [Sa65, Table p.461] and recalled in Theorem 6.6 below, of the possible irreducible representations that can appear in the $\mathbf{H}_{\mathbb{C}}$-module $\mathrm{V}_\alpha^{\mathrm{p}}$. We compute the dimension of the orbit of a highest weight vector in each case, following the list of Theorem 6.6. The more involved cases are taken care in the representation-theoretic lemma Lemma 6.7. In the end, we will be able to exclude all the cases of Satake's theorem except $\mathbf{SL}_2$ acting on its standard representation. For an algebraic group $\mathbf{G}$ we denote by $\widetilde{\mathbf{G}}$ its simply connected cover, in the sense of algebraic groups.

(1) Type A_n. Let $\mathbf{SL}_{n+1}$ act on $\bigwedge^r \mathbb{C}^{n+1}$, the r-th exterior power of its standard representation, with $1 \leq r \leq n$. Let v be a highest weight vector of $\bigwedge^r \mathbb{C}^{n+1}$ and $\bar{v}$ its image in $\mathbf{P}\left(\bigwedge^r \mathbb{C}^{n+1}\right)$. Then $\mathbf{SL}_{n+1}(\mathbb{C})\bar{v}$ is isomorphic to the Grassmannian $G(r, n+1)$ parametrizing r-planes in $\mathbb{C}^{n+1}$ and we have $\dim G(r, n+1) = r(n+1-r)$. It has dimension one if and only if $r = 1, n = 1$, corresponding to $\mathbf{SL}_2$ acting on its standard representation.
(2) Type B_n. We can assume $n \geq 2$ since $B_1 \cong A_1$. Let $\widetilde{\mathbf{SO}}_{2n+1}$ act on its spin representation. By Lemma 6.7(1), the dimension of the orbit of a highest weight vector is $\binom{n+1}{2}$, which is ≥ 3 for $n \geq 2$.
(3) Type C_n. Again we can assume $n \geq 2$ since $C_1 \cong A_1$. Let $\mathbf{Sp}_{2n}$ act on its standard representation $\mathbb{C}^{2n}$. The orbits are $\{0\}$ and its complement. Hence the orbit of a highest weight vector is the whole space and has dimension ≥ 4.
(4) Type D_n. We can assume $n \geq 3$ because of the isomorphisms of Dynkin diagrams $D_1 \cong A_1$ and $D_2 \cong A_1 \times A_1$. Let $\widetilde{\mathbf{SO}}_{2n}$ act on one of its half-spin representations. By Lemma 6.7(2), the dimension of the orbit of a highest weight vector is $\binom{n}{2}$. It never equals 1 for $n \geq 3$. This takes care of the half-spin representations.

   The orbits of $\mathbf{SO}_{2n}$ on the standard representation are the level sets $\{Q(v) = \alpha\}$, where Q is the bilinear form on $\mathbb{C}^{2n}$ given by $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Since $n \geq 3$, the level sets have complex dimension at least 5.

Write $\widetilde{\mathbf{H}}_{\mathbb{C}} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_s \times \mathbf{K}$ as a direct product where $\mathbf{H}_j = \mathbf{SL}_2$ and $\mathbf{K}$ is semisimple without factors isomorphic to $\mathbf{SL}_2$. By Theorem 6.6 and the previous dimension count, the action of $\widetilde{\mathbf{H}}_{\mathbb{C}}$ on M_i must factor through a group $\mathbf{H}_{\sigma(i)}$ where $\sigma : \{1, \dots, \ell'\} \to \{1, \dots, s\}$ is a set-theoretic function, and M_i is the standard representation of $\mathbf{H}_{\sigma(i)}$. We claim that σ maps all indices to the same one: otherwise if say $\sigma(1) = 1, \sigma(2) = 2$, then the action of $\widetilde{\mathbf{H}}_{\mathbb{C}}$ on $M_1 \oplus M_2$ is equivalent to the action of $\mathbf{SL}_2 \times \mathbf{SL}_2$ on $\mathbb{C}^2 \times \mathbb{C}^2$ coordinatewise, which is transitive. This is not possible since the projection of Y^{p} under $\mathbf{P}\mathrm{V}_\alpha^{\mathrm{p}} \dashrightarrow \mathbf{P}(M_1 \oplus M_2)$ has dimension one and is not the whole of $\mathbf{P}(M_1 \oplus M_2)$. Finally, since each M_i lives in a different isotypic component, we deduce that $\ell' = 1$ and $\mathbf{H}(\mathbb{C})v_N = M_1 \setminus \{0\}$.

We now conclude the proof of Step 1. Recall that $v = v_T + v_N$, with $v_T \in T$. If $v_T \neq 0$, then the subset

$$\{\lambda(v_T, w) : \lambda \in \mathbb{C}, w \in L\} \subset T \oplus N$$

has dimension 3, and its image in $\mathbf{P}\mathrm{V}_\alpha^{\mathrm{p}}$, which is $\mathbf{H}(\mathbb{C})\overline{v}$, would have dimension ≥ 2. Therefore $v_T = 0$ and we conclude that $\mathbf{H}(\mathbb{C})v = M_1 \setminus \{0\}$, i.e. Y^{p} is a projective line.

Step 2: Y is a linear subvariety.

Let $\mathbf{H}^{\mathrm{rel}}$ be the algebraic monodromy group of $\mathbb{V}_{\alpha,\mathbb{Q}}|_Z$. The map $\mathbf{H}^{\mathrm{rel}} \to \mathbf{H}$ is surjective. Let $\mathcal{Y}$ (resp. $\mathcal{Y}^{\mathrm{p}}$) the closure of the lifts of Y (resp. Y^{p}) to V_α (resp. to $\mathrm{V}_\alpha^{\mathrm{p}}$). They are closed, two dimensional irreducible subvarieties of their respective affine spaces. By Step 1, $\mathcal{Y}^{\mathrm{p}}$ is a linear subspace on which $\mathbf{H}_\mathbb{C}$ acts through a map $\mathbf{H}_\mathbb{C} \to \mathbf{SL}_{2,\mathbb{C}}$ and $\mathcal{Y}^{\mathrm{p}}$ is isomorphic to the standard representation of $\mathbf{SL}_{2,\mathbb{C}}$. Note that the complex points of $\mathbf{U} := \ker(\mathbf{H}^{\mathrm{rel}} \to \mathbf{H})^\circ$ act trivially on $\mathcal{Y}$: indeed, the fibers $\mathcal{Y} \to \mathcal{Y}^{\mathrm{p}}$ are union of orbits under $\mathbf{U}(\mathbb{C})$ and the fibers are zero-dimensional over a dense open of $\mathcal{Y}^{\mathrm{p}}$ (because $\dim \mathcal{Y} = \dim \mathcal{Y}^{\mathrm{p}}$). Therefore, since $\mathbf{U}(\mathbb{C})$ is connected, it acts trivially on those fibers, hence everywhere.

Let $W \subset \mathrm{V}_\alpha$ be the smallest linear subspace containing $\mathcal{Y}$. Then W maps surjectively onto $\mathcal{Y}^{\mathrm{p}}$. Since $\mathbf{U}$ acts trivially on $\mathcal{Y}$, it acts trivially on W. In particular, the action of $\mathbf{H}_\mathbb{C}^{\mathrm{rel}}$ on W factors through (the complex points of) $\mathbf{H}^{\mathrm{rel}}/\mathbf{U}$, a not necessarily connected group whose identity component $\mathbf{H}'$ is isogenous to $\mathbf{H}$. The group $\mathbf{H}'_\mathbb{C}$ acts on $\mathcal{Y}^{\mathrm{p}}$ through $\mathbf{H}_\mathbb{C}$. Since $\mathcal{Y}^{\mathrm{p}}$ contains a dense orbit under $\mathbf{H}(\mathbb{C})$ and since $\dim \mathcal{Y} = \dim \mathcal{Y}^{\mathrm{p}}$, it follows that also $\mathcal{Y}$ contains a dense orbit under $\mathbf{H}'(\mathbb{C})$. This implies that W is an irreducible $\mathbf{H}'(\mathbb{C})$-module. But W surjects onto $\mathcal{Y}^{\mathrm{p}}$ hence they are isomorphic as $\mathbf{H}'(\mathbb{C})$-modules. In particular, they have the same dimension. This implies $\mathcal{Y} = W$ and that $\mathbf{H}_\mathbb{C}^{\mathrm{rel}}$ acts on W through the standard representation of $\mathbf{SL}_{2,\mathbb{C}}$. □

Theorem 6.6 ([Sa65])**.** *Let V be a $\mathbb{Q}$-Hodge structure of weight 1, $\mathbf{MT} \subset \mathbf{GL}(V)$ its Mumford-Tate group and $\mathbf{G}$ a connected normal subgroup of $\mathbf{MT}^{\mathrm{der}}$. Let $\widetilde{\mathbf{G}}$ be its simply connected cover and $\widetilde{\mathbf{G}}_\mathbb{C} = \mathbf{G}_1 \times \ldots \times \mathbf{G}_n$ a decomposition into a product of almost simple, simply connected $\mathbb{C}$-groups. Let W be an irreducible $\mathbf{G}_\mathbb{C}$-submodule of $V_\mathbb{C}$, and consider it as $\widetilde{\mathbf{G}}_\mathbb{C}$-module. Write $W \cong W_1 \otimes \ldots \otimes W_j$, for some $j \geq 1$, where W_k is an irreducible representation of a factor $\mathbf{G}_{i_k}$ of $\widetilde{\mathbf{G}}_\mathbb{C}$, $1 \leq k \leq j$. Then the possible couples $(\mathbf{G}_{i_k}, W_k)$, given at the level of the Lie algebras, are as follows (for $n \geq 1$):*

- $\mathfrak{sl}_{n+1}(\mathbb{C})$*, acting on an exterior power $\bigwedge^r \mathbb{C}^{n+1}$ of the standard representation, $1 \leq r \leq n$.*
- $\mathfrak{so}_{2n+1}(\mathbb{C})$*, acting on the spin representation, of dimension 2^n;*
- $\mathfrak{sp}_{2n}(\mathbb{C})$*, acting on the standard representation, of dimension $2n$;*
- $\mathfrak{so}_{2n}(\mathbb{C})$*, acting on one of the two half-spin representations, of dimension 2^{n-1}, or on the standard representation.*

Lemma 6.7. *(1) Let V be the spin representation of $\widetilde{\mathbf{SO}}_{2n+1}$, $v \in V$ a highest weight vector and P the parabolic subgroup that stabilizes the line $\mathbb{C}v$. Then $\dim \widetilde{\mathbf{SO}}_{2n+1} - \dim P = \binom{n+1}{2}$.*

(2) Let V be one of the half-spin representations of $\widetilde{\mathbf{SO}}_{2n}$, $v \in V$ a highest weight vector and P the parabolic subgroup that stabilizes the line $\mathbb{C}v$. Then $\dim \widetilde{\mathbf{SO}}_{2n+1} - \dim P = \binom{n}{2}$.

Proof. Left to the reader. □

6.3. **The case of genus 2: proof of Theorem 2.10.** In genus 2, there are only two strata $\mathrm{S}_{1,1}$ and S_2, of dimension 4 and 3 respectively.

Proposition 6.8. *Let $Z \subset \mathrm{S}_\alpha$ be a bi-algebraic curve, where $\alpha = (1,1)$ or $\alpha = (2)$. Then either Z is contained in an isoperiodic leaf or it satisfies condition $(\star)$.*

Proof. Assume that Z does not satisfy $(\star)$. In particular, the fixed part of $\mathrm{Gr}_1^W \mathbb{V}_{\alpha,\mathbb{Q}}|_Z$ is non-zero. There are two possibilities: $\mathcal{J}_\alpha|_Z$ is isogenous either to $\mathcal{E}_1 \times \mathcal{E}_2$, where $\mathcal{E}_1$ is an isotrivial elliptic curve over Z; or to an isotrivial abelian surface $\mathcal{A}$ over Z (and, in these two cases, the tautological section $\omega|_Z$ comes from pullback from the isotrivial factor). In the first case, Z is contained in the isoperiodic foliation of S_α : indeed an elliptic curve has a unique non-zero 1-form up to multiplication. (Note that since the isoperiodic foliation on S_2 has zero-dimensional leaves, we have necessarily $\alpha = (1,1)$.)

In the latter case, by Torelli's theorem and the fact that the locus of abelian varieties isogenous to a fixed abelian variety is countable, the fibers of $\mathcal{C}_\alpha|_Z$ are isomorphic, i.e. Z is contained in a fiber of the forgetful morphism $\mathrm{S}_\alpha \to \mathcal{M}_2$. Since the fibers $\mathrm{S}_2 \to \mathcal{M}_2$ are zero-dimensional, we must have $\alpha = (1,1)$. The fibers of $\mathrm{S}_{1,1} \to \mathcal{M}_2$ are of dimension 1, so Z actually coincides with a fiber. We prove that such components are not bi-algebraic. Clearly it is enough to show the corresponding statement with S_α replaced by H_α and $\mathbf{P}\mathrm{V}_\alpha$ by V_α. Proposition 6.8 is thus a consequence of Proposition 6.9 below. □

Proposition 6.9. *Let C be a smooth projective curve of genus 2. Let $\mathcal{U} \subset H^0(C, \Omega^1)$ be the Zariski open subset (complement of a finite number of lines) consisting of holomorphic 1-forms with two distinct simple zeros. Consider the multivalued function*

$$T : \mathcal{U} \to \mathbb{C}$$
$$\omega \mapsto \int_{\gamma_\omega} \omega$$

where $\gamma_\omega \in H_1(C, Z(\omega); \mathbb{Z})$ is a path joining the distinct zeros, and let $\tilde{T} : \widetilde{\mathcal{U}} \to \mathbb{C}$ be a lift to the universal cover $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$. Then the image of $(\pi, \widetilde{T}) : \widetilde{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}$ is not "algebraic", i.e. it is not contained in a Zariski closed set of dimension $= \dim \mathcal{U} = 2$.

Proof. Consider the degree two étale cover $p : \mathcal{U}' \to \mathcal{U}$ on which we can label the zeros of the 1-form, i.e. there are two morphisms $\sigma_1, \sigma_2 : \mathcal{U}' \to C$, such that $\mathrm{div}(p(u)) = \sigma_1(u) + \sigma_2(u)$, for $u \in \mathcal{U}'$. Consider the $\mathbb{Z}$VMHS whose fiber over a point u is $H_1(C, \{\sigma_1(u), \sigma_2(u)\}, \mathbb{Z})$ (it is the dual of the "restriction" of $\mathbb{V}_{\alpha,\mathbb{Z}}$ to $\mathcal{U}'$). This $\mathbb{Z}$VMHS is unipotent. Fix a base point $u_0 \in \mathcal{U}'$ and consider the monodromy representation

$$\rho : \pi_1(\mathcal{U}', u_0) \to \mathrm{GL}(H_1(C, \{\sigma_1(u_0), \sigma_2(u_0)\}, \mathbb{Z})).$$

If $\mathrm{im}(\rho)$ is finite then the local system underlying the $\mathbb{Z}$VMHS is constant after a finite étale cover. By [HZ87] a unipotent $\mathbb{Z}$VMHS is essentially classified by its monodromy representation, thus our $\mathbb{Z}$VMHS is constant. But this contradicts the remark after (3.3): indeed $\mathcal{U}$ would map to a point in $\mathfrak{A}_g$ under the period map, contradicting quasi-finiteness.

We deduce that $\mathrm{im}(\rho)$ contains a non-trivial element A. If γ is a path connecting $\sigma_1(u_0)$ to $\sigma_2(u_0)$ then $A\gamma = \gamma + c$, where $c \in H_1(C, \mathbb{Z})$ is non-zero. The set of 1-forms in $H^0(C, \Omega^1)$ that pair to zero against c is a strict linear subspace. Let $\omega \in \mathcal{U}$ be a 1-form that pairs non-trivially against c. Then the fiber of the image of $(\pi, \tilde{T})$ over ω contains an infinite coset $x + \mathbb{Z}\langle \omega, c\rangle \subset \mathbb{C}$ (the number $x \in \mathbb{C}$ depends on the choice of the lift $\tilde{T}$). Assume, for the sake of contradiction, that the image of $(\pi, \tilde{T})$ is contained in a Zariski closed subset V of dimension 2. Since $\dim V = \dim \mathcal{U}$, the (surjective) projection $V \to \mathcal{U}$ is generically finite. But the elements $\omega \in \mathcal{U}$ with $\langle \omega, c\rangle \neq 0$ are generic in $\mathcal{U}$, and by our analysis above, the fiber of V over ω is infinite. This contradicts generic finiteness. $\square$

6.4. **An interesting example in genus 3.** To conclude this paper let us study the bi-algebraic curves in the minimal stratum S_4, not satisfying ($\star$) and not isoperiodic. Let $Z \subset S_\alpha$ be such a curve. Up to isogeny, the relative Jacobian $\mathcal{J}_\alpha|_Z$ factors as $\mathcal{A} \times \mathcal{A}'$ where $\mathcal{A}$ is an isotrivial abelian scheme over Z corresponding to the fixed part, and $\mathcal{A}'$ is a complement. By hypothesis, $\omega|_Z$ comes by pullback from a non-zero relative one-form on $\mathcal{A}$. We suppose that $\mathcal{A}$ is a surface, and call A the abelian surface which is the fiber of $\mathcal{A}$ above every point. We then obtain linearity just by the dimension constraints:

Lemma 6.10. *Z is linear.*

Sketch of proof. Choose a base point $z_0 \in Z(\mathbb{C})$ and let $(C_0, [\omega_0])$ be the corresponding differential. Let $V \subset H^0(C_0, \Omega^1)$ be the 2-dimensional subspace of differentials coming from A. Identify V with the corresponding 2-dimensional subspace of $H^1(C_0, \mathbb{C})$. Since $\omega|_Z$ comes by pullback from A, its projectivized periods are constrained on the 1-dimensional line $\mathbf{P}V \subset \mathbf{P}H^1(C_0, \mathbb{C})$. $\square$

Let us now exhibit examples of such curves. The reference here is [HS08], and we mostly use their notations. Consider the family of smooth projective curves parametrized by $\lambda \in \mathbb{C} - \{0, 1\}$ given by

$$C_\lambda : Y^4 = XZ(X - Z)(X - \lambda Z) \subset \mathbf{P}^2.$$

In the affine chart $\{Z \neq 0\}$ the curve is given by the equation $y^4 = x(x-1)(x-\lambda)$. This is the total family of curves that underlies the unique Shimura-Teichmüller curve in genus 3, see [Mö11].

The curves C_λ have genus 3 and are not hyperelliptic. We call $P_0, P_1, P_\lambda, P_\infty$ the four points of C_λ lying on the line $\{Y = 0\}$, corresponding to $[0, 0, 1]$, $[1, 0, 1]$, $[\lambda, 0, 1]$ and $[1, 0, 0]$ respectively. On C_λ, we construct four abelian differentials up to scaling, with a single zero at P_i, $i = 0, 1, \lambda, \infty$, respectively.

We first state a trivial but useful fact. Let C be a smooth projective curve of genus g, non-hyperelliptic, and $i : C \to \mathbf{P}H^0(C, \Omega^1)^*$ be the canonical embedding. Tautologically, differential forms up to scaling correspond to hyperplanes in $\mathbf{P}H^0(C, \Omega^1)^*$ and the order of vanishing at $x \in C(\mathbb{C})$ of a differential form ω, is the order of tangency at $i(x)$ of $i(C)$ with the hyperplane $\omega = 0$.

The embedding in $\mathbf{P}^2$ coming from the definition of C_λ coincides with the canonical embedding (there is only one linear series $\mathfrak{g}^{g-1}_{2g-2}$ on a given curve of genus g). Hence to determine our differentials we will consider lines in $\mathbf{P}^2$.

We collect here the facts that we need on the family C_λ.

(1) There is an obvious map f_λ to the elliptic curve $E_\lambda = \{y^2 = x(x-1)(x-\lambda)\}$ given by $(x, y) \mapsto (x, y^2)$.
(2) There are involutions σ_λ and $-\sigma_\lambda$ of C_λ, such that $C_\lambda/\sigma_\lambda \cong C_\lambda/{-\sigma_\lambda} \cong E_{-1}$. We call the quotient maps κ_{σ_λ} and $\kappa_{-\sigma_\lambda}$ respectively. [*loc. cit.*, Section 1.3 and Proposition 6]
(3) The map $Jac(C_\lambda) \to E_\lambda \times E_{-1} \times E_{-1}$ induced by $f_\lambda \times \kappa_{\sigma_\lambda} \times \kappa_{-\sigma_\lambda}$ is an isogeny. [*loc. cit.*, Proposition 7]

We compute the lines corresponding to the unique differential up to scaling coming from the three factors. This is equivalent to knowing the ramification points of each of the three maps $f_\lambda, \kappa_{\sigma_\lambda}, \kappa_{-\sigma_\lambda}$; for f_λ they are the four points P_i, while for the other two morphisms they are listed in [HS08, Proposition 14]. Choose a square root ζ of $1 - \lambda$ (here we move to the degree 2 unramified cover of $\mathbb{C} - \{0, 1\}$ where the square root is well-defined). We find the following: the differential coming from E_λ corresponds to the line $\{Y = 0\}$; the differential coming from E_{-1} by pullback along κ_{σ_λ} is the line $L_+ := \{X - (1+\zeta)Z = 0\}$; and the differential coming from E_{-1} by pullback along $\kappa_{-\sigma_\lambda}$ is the line $L_- := \{X - (1-\zeta)Z = 0\}$. Therefore the differentials up to scaling coming from the factor $E_{-1} \times E_{-1}$ correspond to the pencil of lines $\{\alpha X + \beta Z = 0 : \alpha, \beta \in \mathbb{C}\}$, with homogenous coordinates $[\alpha, \beta]$ (in the chart $\{Z \neq 0\}$ the pencil is the pencil of vertical lines).

One easily checks there are exactly four points on C_λ such that the tangent line at the point belongs to the pencil, i.e. the points $P_0, P_1, P_\lambda, P_\infty$. Moreover at each of these points the tangent line has order of tangency equal to 4. The corresponding differentials up to scaling provide our four examples.

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