CHERN'S CONJECTURE FOR SPECIAL AFFINE MANIFOLDS

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Abstract. An affine manifold X in the sense of differential geometry is a differentiable manifold admitting an atlas of charts with value in an affine space, with locally constant affine change of coordinates. Equivalently, it is a manifold whose tangent bundle admits a flat torsion free connection. Around 1955 Chern conjectured that the Euler characteristic of any compact affine manifold has to vanish. In this paper we prove Chern's conjecture in the case where X moreover admits a parallel volume form.

Contents

1. Introduction

Let X be a connected topological manifold of dimension n. Following Klein's Erlangen program one may study X by asking if it supports a locally homogeneous geometric structure: a system of local coordinates modeled on a fixed homogeneous space G/H such that on overlapping coordinate patches, the coordinate changes are locally restrictions of transformations from the group G. Klein noticed that we recover as special cases all classical (i.e. euclidean, spherical or hyperbolic) geometries. This program, known as uniformization, has been highly successful for $n = 2$ and more recently for $n = 3$. We refer the reader to [Gol10] for a nice survey on locally homogeneous geometric structures.

1.1. **Affine structures.** In this paper we are interested in particularly natural locally homogeneous structures, namely *affine structures*, which are surprisingly very poorly understood. An affine structure on X is a maximal atlas of charts $(U_\alpha, \varphi_\alpha : U_\alpha \stackrel{\sim}{\to} \varphi_\alpha(U_\alpha) \subset V)$, where V denotes the real affine space of dimension n with underlying real vector space $\vec{V} \simeq \mathbb{R}^n$, such that:

- each U_α is open in $X = \bigcup_\alpha U_\alpha$;
- $\varphi_\alpha : U_\alpha \stackrel{\sim}{\longrightarrow} \varphi_\alpha(U_\alpha) \subset V$ is a homeomorphism;
- for every non-empty connected open set $U \subset U_\alpha \cap U_\beta$ the change of coordinates $(\varphi_\alpha \circ \varphi_\beta^{-1})_{|\varphi_\beta(U)} : \varphi_\beta(U) \stackrel{\sim}{\longrightarrow} \varphi_\alpha(U)$ is the restriction of an element $g(U, \alpha, \beta)$ of the affine group $\mathrm{Aff}(V) \simeq \mathrm{GL}(\vec{V}) \ltimes \vec{V}$.

Equivalently, fixing x_0 a point of X and using the monodromy principle, one easily shows that an affine structure on X is the datum of a group morphism $h : \pi_1(X, x_0) \longrightarrow \mathrm{Aff}(V)$ and a local homeomorphism $D : \tilde{X} \longrightarrow V$, where $\tilde{X}$ denotes the universal cover of X, which is h-equivariant: $\forall\, \gamma \in \pi_1(X, x_0),\ \forall x \in \tilde{X},\quad D(\gamma \cdot x) = h(\gamma) \cdot D(x)$. The developing map D is obtained by glueing the local charts (φ_α), and the holonomy h is obtained by piecing together the transition functions $(g(U, \alpha, \beta))$.

Notice that an affine structure on a topological manifold X defines canonically a C^∞-structure on X. Hence without loss of generality we will from now on assume that X is a C^∞-manifold (we will just say manifold) and work in the differentiable category. In this setting one obtains a third equivalent definition of an affine structure on X, namely a *flat torsion-free connection* on the tangent bundle TX.

1.2. **Chern's conjecture.** Around 1950 Chern tried to understand which topological constraints an affine structure imposes on a connected closed n-manifold. He apparently proposed the following:

Conjecture 1.1. *(Chern, 1955) Let X be a connected closed manifold. If X admits an affine structure then its Euler characteristic $\chi(X)$ vanishes.*

Notice that the Euler characteristic is multiplicative under passage to a finite covering space. Hence without loss of generality we can and will assume in Conjecture 1.1 that X is oriented. In this case, $\chi(X) := \chi(TX) =< e(TX), [X] >$, where $e(TX) \in H^{d_X}(X, \mathbb{Z})$ denotes the Euler class of the real oriented tangent bundle TX of X, the non-negative integer d_X denotes the dimension of X, and $[X] \in H_{d_X}(X, \mathbb{Z})$ denotes the fundamental class of X. As the Euler class of *any* odd rank real oriented vector bundle is killed by 2 (see [MS74, property 9.4]), Conjecture 1.1 trivially holds true for odd-dimensional manifolds.

We now explain why Conjecture 1.1 is non-trivial.

We first recall that *the existence of a flat connection on an oriented vector bundle does not imply the vanishing of its real Euler class*: while Chern-Weil theory (see [MS74, Appendix C]) says that the real Pontryagin classes $p_{i,\mathbb{R}}(E) \in H^{4i}(X, \mathbb{R})$ of an oriented real vector bundle E on X can be computed using the curvature form of any $\mathrm{GL}(r, \mathbb{R})^+$-connection ∇ on E, hence vanish if ∇ is flat, the real Euler class $e_\mathbb{R}(E) \in H^r(X, \mathbb{R})$ can be computed from the curvature form of ∇ only if ∇ is an $SO(r)$-connection on E. Indeed, the following easy construction provides oriented $\mathbb{R}^2$-bundles E admitting a flat connection but with non-zero Euler number on the closed oriented surface Σ of genus g, $g \geq 2$. Fix a complex structure on Σ and consider the induced hyperbolic uniformization $\Sigma \simeq \Gamma \backslash \mathbf{H}$, where $\mathbf{H}$ denotes the Poincaré upper half-plane and $\pi_1(\Sigma) \simeq \Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a cocompact torsion-free lattice. The group $\mathrm{PSL}(2, \mathbb{R})$ acts naturally on the projective line $\mathbf{P}^1(\mathbb{R})$ hence the S^1-bundle $\eta := \Gamma \backslash (\mathbf{H} \times \mathbf{P}^1(\mathbb{R}))$ over Σ can be seen as a bundle with structural group $\mathrm{PSL}(2, \mathbb{R})$ with the discrete topology. Notice that η is nothing else than the tangent circle bundle to Σ, hence has Euler number $2 - 2g$. It is easy to check that η admits a square root $\hat{\eta}$: namely the inclusion $\iota : \Gamma \hookrightarrow \mathrm{PSL}(2, \mathbb{R})$ lifts to $\hat{\iota} : \Gamma \hookrightarrow \mathrm{SL}(2, \mathbb{R})$. The 2-plane bundle E associated to $\hat{\iota}$ has structural group $\mathrm{SL}(2, \mathbb{R})$ with discrete topology (hence admits a flat connection) and has Euler number $1 - g$ (see [MS74, p.313-4] for details).

Remark 1.2. Notice however that the real Euler class of the oriented real vector bundle underlying a complex vector bundle of complex rank r identifies with its r-th real Chern class, hence vanish by Chern-Weil's theory. In particular Chern's conjecture holds true for *complex affine* manifolds.

Hence Conjecture 1.1 is not a general statement on flat vector bundles. One could nevertheless ask if it is a statement on *flat, not necessarily torsion-free, connection on tangent bundles*. In [Ben55] Benzécri proved Chern's conjecture for closed 2-manifolds: among them only tori admit affine structures. In [Mil58] Milnor proved his celebrated inequality:

Theorem 1.3 (Milnor)**.** *An oriented $\mathbb{R}^2$-bundle E over the closed oriented surface Σ_g of genus $g \geq 2$ admits a flat connection ∇ if and only if $|\chi(E)| < g$.*

This implies in particular the following stronger version of Benzécri's result: the tangent bundle of a closed connected surface X admits a flat, not necessarily torsion-free, connection if and only if $\chi(X) = 0$. Milnor asked if this result can be generalized in all dimensions. However in [Smi77] Smillie showed the following:

Theorem 1.4 (Smillie)**.** *For any $n \geq 2$ there are examples of closed $2n$-dimensional manifolds X with non-vanishing Euler characteristic $\chi(X)$ whose tangent bundle TX admits a flat connection with non-zero torsion.*

Hence Chern's conjecture is really a question on *affine structures*, not on flat connections on tangent bundles.

1.3. **Results.** The simplest examples of affine manifolds are the *complete* ones, namely the ones for which the developing map $D : \tilde{X} \longrightarrow V$ is a global diffeomorphism (equivalently, the ones which are geodesically complete). In this case X is a quotient of V by a subgroup $\Gamma \subset \mathrm{Aff}(V)$ acting freely discontinuously on V. After the results of Benzécri, Milnor and Smillie we mentioned, Kostant and Sullivan [KS75] proved Conjecture 1.1 in the case where the affine structure on X is moreover complete. Their proof in that case is an ingenious argument on the monodromy, which can not be generalized. In [HT75] Hirsch and Thurston proved Conjecture 1.1 when the image of the holonomy homomorphism $h : \pi_1(X, x_0) \longrightarrow \mathrm{Aff}(V)$ is built out of amenable groups by forming free products and taking finite extensions (a simpler proof when the holonomy is solvable was obtained by Goldman and Hirsch [GolHir81]). Recently, Bucher and Gelander proved Conjecture 1.1 for varieties which are locally a product of surfaces, see [BucGel11].

In this paper we deal with *special affine* structures, i.e. affine structures whose holonomy lies in the special affine subgroup $\mathrm{SAff}(V) \simeq \mathrm{SL}(\vec{V}) \ltimes \vec{V}$ of $\mathrm{Aff}(V) \simeq \mathrm{GL}(\vec{V}) \ltimes \vec{V}$. Equivalently, special affine structures are the affine structures admitting a parallel volume form. Markus conjectured in 1960 that a closed affine manifold is complete if and only if it is special affine (see [HT75]). This conjecture is largely opened, a significant step being Carrière's result [Car89] that any closed flat Lorentzian manifold is complete.

Our main result in this paper is the proof of Chern's conjecture for special affine manifolds:

Theorem 1.5. *If X is a connected closed special affine manifold then $\chi(X) = 0$.*

Notice that if X is a connected closed manifold with vanishing first Betti number, then any multiplicative character $\chi : \pi_1(X, x_0) \longrightarrow \mathbb{R}^*$ takes value in ± 1, thus any affine structure on (the oriented cover of) X is special affine. Hence:

Corollary 1.6. *Suppose X is a connected closed affine manifold with vanishing first Betti number. Then $\chi(X) = 0$.*

1.4. **Strategy of the proof of Theorem 1.5.** Let us now describe the strategy of the proof of Theorem 1.5. While most previous results build on group-theoretic arguments and generalized versions of the Milnor-Wood inequality, our approach relies on the geometry of the total space $\mathcal{E}$ of the tangent bundle of an affine manifold.

The following classical proposition, which we recall in the Appendix A, follows from the very definition of the (real) Euler class of an oriented real vector bundle and reduces its study to the study of the differential forms on the total space of the bundle:

Proposition 1.7. *Let X be a connected oriented closed n-manifold and E an oriented real vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$. The Euler class $e_{\mathbb{R}}(E) \in H^r(X, \mathbb{R})$ vanishes if and only if the natural morphism between cohomology with compact support and usual cohomology*

$$\mathbb{R} \simeq H^r_c(\mathcal{E}, \mathbb{R}) \longrightarrow H^r(\mathcal{E}, \mathbb{R}) \simeq H^r(X, \mathbb{R})$$

vanishes.

Remark 1.8. Proposition 1.7 is Poincaré dual to the following homological statement, which might be more intuitive: the Euler class $e_{\mathbb{R}}(E) \in H^r(X, \mathbb{R})$ is zero if and only if the natural morphism

$$H_n(\mathcal{E}, \mathbb{R}) = \mathbb{R} \cdot [X] \longrightarrow H^{BM}_n(\mathcal{E}, \mathbb{R})$$

vanishes (where $[X]$ denotes the fundamental class in $\mathcal{E}$ of the zero section of E, and $H^{BM}_\bullet$ denotes the Borel-Moore homology). Loosely speaking: if and only if the cycle X of $\mathcal{E}$ is a boundary in the complex of locally finite chain with non-compact support of $\mathcal{E}$.

We study differential forms on $\mathcal{E}$ using the geometry of $\mathcal{E}$. When E is a mere bundle, the only natural geometric structure on $\mathcal{E}$ is the foliated structure given by the projection $\pi : \mathcal{E} \longrightarrow X$. If in addition we assume that the bundle E is endowed with a flat connection ∇, the total space $\mathcal{E}$ has a natural *local product structure* in the sense of Definition 2.1, the additional foliation being given by the flat leaves of ∇.

For any manifold M endowed with a local product structure, the De Rham complex of sheaves of real differential forms $(\Omega^\bullet_M, d)$ is enriched with a natural bigrading $(\Omega^{\bullet,\bullet}_{\mathcal{E}}, d', d'')$, d' being the differential in the "horizontal" direction and d'' the one in the "vertical" direction. This bigrading defines two filtrations ${}_{d'}F^\bullet$ and ${}_{d''}F^\bullet$, on $H^\bullet_c(\mathcal{E}, \mathbb{R})$ and also on $H^\bullet(\mathcal{E}, \mathbb{R})$. As usual the graded pieces of these filtrations are computed by spectral sequences ${}_{d'}E^{\bullet,\bullet}_\bullet$ and ${}_{d''}E^{\bullet,\bullet}_\bullet$ (both in the compact support case and the usual one). I don't know how to compute these filtrations for a general local product structure.

On the other hand when M is the total space $\mathcal{E}$ of a flat bundle E on X, one can compute these filtrations with the exception of ${}_{d''}F^\bullet$ on $H^\bullet_c(\mathcal{E}, \mathbb{R})$, see Proposition 3.2 and Proposition 3.4.

The morphism $H^\bullet_c(\mathcal{E}, \mathbb{R}) \longrightarrow H^\bullet(\mathcal{E}, \mathbb{R})$ we want to study is induced by a morphism of spectral sequences $\varphi^{\bullet,\bullet}_\bullet : {}_{d''}E^{\bullet,\bullet}_{c,\bullet} \longrightarrow {}_{d''}E^{\bullet,\bullet}_\bullet$ and the relation between the local product structure on $\mathcal{E}$ and the vanishing of $e_{\mathbb{R}}(E)$ is given by the following refinement of Proposition 1.7:

Proposition 1.9. *Let X be a connected oriented closed n-manifold. Let E be an oriented flat real vector bundle on X of rank $r > 0$ with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. The Euler class $e_{\mathbb{R}}(E) \in H^r(X, \mathbb{R})$ vanishes if and only if the map*

$$\varphi^{0,r}_\infty : \mathrm{Gr}^0_{{}_{d''}F^\bullet} H^r_c(\mathcal{E}, \mathbb{R}) = {}_{d''}E^{0,r}_{c,\infty} \longrightarrow {}_{d''}E^{0,r}_\infty = \mathrm{Gr}^0_{{}_{d''}F^\bullet} H^r(\mathcal{E}, \mathbb{R}) = \mathbb{R}$$

vanishes.

We will be mainly interested in the case $n = r$. In this case the local product structure on $\mathcal{E}$ is called a *para-complex structure* on $\mathcal{E}$. The bigrading $(\Omega_{\mathcal{E}}^{\bullet,\bullet}, d', d'')$ is formally similar to the bigrading of the complex analytic De Rham complex on a complex manifold, *except that there is no involution of* $(\Omega_{\mathcal{E}}^{\bullet}, d)$ *exchanging* d' *and* d'' *(like the conjugation in the complex setting).*

Suppose now that the vector bundle E is the tangent bundle TX. Any linear connection ∇ on TX defines a natural almost complex structure I on $\mathcal{E}$. Moreover Dombrowski [Dom62] proved that I is a complex structure if and only if ∇ is flat *and torsion-free*, i.e. X is an affine manifold. This complex structure was further studied by Cheng and Yau [ChengYau82].

The interplay of this complex structure on $\mathcal{E}$ and the natural para-complex structure on $\mathcal{E}$ is our main tool for studying the vanishing of $e_{\mathbb{R}}(TX)$: the total space $\mathcal{E}$ of the tangent bundle of an affine manifold acquires a very rich *para-hypercomplex structure* (see Definition 4.1), a notion analogous to an hypercomplex structure in complex geometry. In particular, and this will be crucial for us, *the standard para-complex structure on* $\mathcal{E}$ *is the value at* $\theta = 0 \in [0, 2\pi[$ *of an* S^1*-family of para-complex structures, induced by a canonical* $\mathrm{SO}(2)$*-action on* $T\mathcal{E}$. Such an S^1-family simply does not exist if ∇ is flat but has non-trivial torsion. Notice moreover that for $\theta \neq 0 \mod \pi/2$, the para-complex structure on $\mathcal{E}$ corresponding to θ does not come from a flat bundle structure on TX.

For each $\theta \in S^1$ the corresponding para-complex structure defines, as above, a filtration ${}_{d''_\theta}F^\bullet$, on $H_c^\bullet(\mathcal{E}, \mathbb{R})$ and on $H^\bullet(\mathcal{E}, \mathbb{R})$. It satisfies ${}_{d''_{\theta=0}}F^\bullet = {}_{d''}F^\bullet$ and ${}_{d''_{\theta=\pi/2}}F^\bullet = {}_{d'}F^\bullet$. The main idea in the proof of Theorem 1.5 is that while the filtrations ${}_{d'}F^\bullet$ and ${}_{d''}F^\bullet$ are unrelated when $\mathcal{E}$ is the total space of a mere flat bundle, the S^1-family of para-complex structures on the total space $\mathcal{E}$ of the tangent bundle of an affine manifold induces an S^1-family of filtrations interpolating between them. Technically speaking, we construct a spectral sequence in the category of sheaves over S^1, obtaining a morphism

$$\varphi_{\infty,S^1}^{0,n} : {}_{d''}\mathcal{E}_{c,\infty}^{0,n} \longrightarrow {}_{d''}\mathcal{E}_{\infty}^{0,n}$$

of sheaves over S^1. The subtle relation between this spectral sequence of sheaves, and the spectral sequence we are interested in at the point $\theta = 0$, lies in the existence of a canonical factorisation (see Lemma 4.10) of the morphism $\varphi_\infty^{0,n}$ as

$$(1) \qquad \varphi_\infty^{0,n} : {}_{d''}E_{c,\infty}^{0,n} \xrightarrow{\sim} \left({}_{d''}\mathcal{E}_{c,\infty}^{0,n}\right)_{\theta=0} \xrightarrow{\left(\varphi_{\infty,S^1}^{0,n}\right)_{\theta=0}} \left({}_{d''}\mathcal{E}_{\infty}^{0,n}\right)_{\theta=0} \longrightarrow {}_{d''}E_{\infty}^{0,n} .$$

Let us warn the reader that the canonical morphism $\left({}_{d''}\mathcal{E}_{\infty}^{0,n}\right)_{\theta=0} \longrightarrow {}_{d''}E_{\infty}^{0,n}$ relating the stalk of the sheaf ${}_{d''}\mathcal{E}_{\infty}^{0,n}$ at the point $\theta = 0$ to ${}_{d''}E_{\infty}^{0,n}$ in the factorization (1) is a priori neither injective nor surjective.

A crucial feature of the sheaves ${}_{d''}\mathcal{E}_{c,\infty}^{0,n}$ and ${}_{d''}\mathcal{E}_{\infty}^{0,n}$ on S^1 is their constructibility, as they are quotients of the constant sheaf $\mathbb{R}_{S^1}$. Suppose now that X is *special* affine. We use this constructibility, the fact that ${}_{d''_{\theta=\pi/2}}F^\bullet = {}_{d'}F^\bullet$ and the existence of an affine volume form on X to show that the sheaf ${}_{d''}\mathcal{E}_\infty^{0,n}$ is identically zero (see Proposition 4.13). It follows from (1) that the morphism $\varphi_\infty^{0,n}$ vanishes. By Proposition 1.9, this finishes the proof of Theorem 1.5.

1.5. **Notations.** From now on manifolds are connected and oriented. If Y is a manifold one denotes by Sh_Y the Abelian category of sheaves of real vector spaces on Y, by $C_Y^\infty \in \mathrm{Sh}_Y$ its sheaf of infinitely differentiable real functions and by $(\Omega_Y^\bullet, d)$ its De Rham complex of sheaves of real differential forms. One denotes by $\mathbb{R}_Y \in \mathrm{Sh}_Y$ the constant sheaf of rank one. More generally if Λ denotes a $\pi_1(Y)$-module we denote by Λ_Y the locally constant sheaf on Y defined by Λ. The notation T^*Y denotes either the cotangent bundle of Y or the associated sheaf on Y.

We refer to [KS90, chap. II] for a survey on sheaves and their properties. In particular, given $f : X \longrightarrow Y$ a morphism of manifolds one denotes as usual by $f^{-1} : \mathrm{Sh}_Y \longrightarrow \mathrm{Sh}_X$, $f_* : \mathrm{Sh}_X \longrightarrow \mathrm{Sh}_Y$ and $f_! : \mathrm{Sh}_X \longrightarrow \mathrm{Sh}_Y$ the pull-back functor, direct image functor and direct image with proper support functor. If $\mathcal{F}$ is a C_Y^∞-module the notation $f^*\mathcal{F}$ denotes the C_X^∞-module $C_X^\infty \otimes_{f^{-1}C_Y^\infty} \mathcal{F}$.

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2. Differential forms on spaces with a local product structure

2.1. **Spectral sequence associated to a foliation.** Recall that a foliation $\mathcal{F}$ of dimension r on a manifold M of dimension $n+r$ is the datum of an atlas of charts $(V_\alpha, \varphi_\alpha : V_\alpha \xrightarrow{\sim} \varphi_\alpha(V_\alpha) \subset \mathbb{R}^n \times \mathbb{R}^r)$ such that the change of coordinates $(\varphi_\alpha \circ \varphi_\beta^{-1})_{|\varphi_\beta(V_\alpha \cap V_\beta)} : \varphi_\beta(V_\alpha \cap V_\beta) \xrightarrow{\sim} \varphi_\alpha(V_\alpha \cap V_\beta)$ is a diffeomorphism of open subsets of $\mathbb{R}^n \times \mathbb{R}^r$ of the form $\varphi_{\alpha\beta}(x, y) = (\varphi^1_{\alpha\beta}(x), \varphi^2_{\alpha\beta}(x, y))$. The connected components of the sets $x =$ constant in a chart V_α are called *plaques*. Let $F \subset TM$ be the (integrable) subbundle of vectors tangent to the plaques. The exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow TM \longrightarrow Q := TM/F \longrightarrow 0$$

defines the dual sequence

$$0 \longrightarrow Q^* \longrightarrow T^*M \longrightarrow F^* \longrightarrow 0 \ .$$

This 1-step filtration of T^*M induces a filtration $F^\bullet$ on the De Rham complex of sheaves $(\Omega_M^\bullet, d)$ hence spectral sequences ${}_{\mathcal{F}}E_1^{pq} \Rightarrow H^{p+q}(M, \mathbb{R})$ and ${}_{\mathcal{F}}E_{c,1}^{pq} \Rightarrow H_c^{p+q}(M, \mathbb{R})$. We won't need a precise description of these spectral sequences for general foliations, see [To97, chap.4] as we consider only spaces with a local product structure (two foliations).

2.2. Local product structures and para-complex structures.

Definition 2.1. *Let n and r be two positive integers. A local product structure of type (n,r) on a manifold M of dimension $n+r$ is the datum of an atlas of charts $(V_\alpha, \varphi_\alpha : V_\alpha \xrightarrow{\sim} \varphi_\alpha(V_\alpha) \subset \mathbb{R}^n \times \mathbb{R}^r)$ such that the change of coordinates $(\varphi_\alpha \circ \varphi_\beta^{-1})_{|\varphi_\beta(V_\alpha \cap V_\beta)} : \varphi_\beta(V_\alpha \cap V_\beta) \xrightarrow{\sim} \varphi_\alpha(V_\alpha \cap V_\beta)$ is a diffeomorphism of open subsets of $\mathbb{R}^n \times \mathbb{R}^r$ of the form*

$$\varphi_{\alpha\beta}(x,y) = (\varphi^1_{\alpha\beta}(x), \varphi^2_{\alpha\beta}(y)) \quad . \tag{2}$$

Remark 2.2. Recall that an *almost product structure* on a manifold M is an endomorphism $J \in \operatorname{End} TM$ such that $J^2 = 1$. Let $TM^+ \subset TM$ (resp. $TM^- \subset TM$) be the subbundle eigenspace of I associated to the eigenvalue $+1$ (resp. -1) of I. Hence $TM = TM^+ \oplus TM^-$. The almost product structure J is said of type (n,r) if TM^+ is of rank n and TM^- of rank r. Equivalently an almost product structure of type (n,r) is a $(\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(r,\mathbb{R}))$-structure on M. A local product structure of type (n,r) is the same thing as an integrable almost product structure of type (n,r). We refer to [Wa61] for details.

In this paper we will essentially be concerned with the special case $n = r$:

Definition 2.3. *A para-complex structure on a manifold M of dimension $2n$ is a local product structure of type (n,n).*

We refer to [CFG96] for a survey on para-complex geometry.

2.3. The bigraded De Rham complex of a local product structure.

A pair of supplementary foliations $\mathcal{F}'$ and $\mathcal{F}''$ on a manifold M gives rise to two pairs of spectral sequences as in Section 2.1. In the case of a local product structure, these spectral sequences are the two spectral sequences associated to a bigraduation of the De Rham complex of M, as we now show.

Let M be a manifold with a local product structure of type (n,r), and $(V_\alpha, \varphi_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^r)$ an atlas of charts for the local product structure. Let $p' : \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}^n$ and $p'' : \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$ be the two natural projections. The map φ_α induces a decomposition of the sheaves of differential forms on V_α:

$$\Omega^l_{V_\alpha} \simeq \bigoplus_{p+q=l} \Omega^{p,q}_{V_\alpha} \quad , \tag{3}$$

where

$$\Omega^{p,q}_{V_\alpha} := \varphi^*_\alpha(p'^*\Omega^p_{\mathbb{R}^n} \otimes_{C^\infty_{\mathbb{R}^n \times \mathbb{R}^r}} p''^*\Omega^q_{\mathbb{R}^r}) \quad . \tag{4}$$

The shape (2) of the change of coordinates guarantees that these local decompositions glue together and one obtains a canonical decomposition of sheaves

$$\Omega^l_M = \bigoplus_{p+q=l} \Omega^{p,q}_M \quad . \tag{5}$$

Moreover the decomposition $d_{\mathbb{R}^{n+r}} = d'_{\mathbb{R}^n} + d''_{\mathbb{R}^r}$ of the differential on $\mathbb{R}^{n+r}$ induces a canonical decomposition of $d : \Omega^\bullet_M \longrightarrow \Omega^{\bullet+1}_M$ into the sum of two differential operators $d = d' + d''$ where

$$d' : \Omega^{\bullet,\bullet}_M \longrightarrow \Omega^{\bullet+1,\bullet}_M \qquad \text{and} \qquad d'' : \Omega^{\bullet,\bullet}_M \longrightarrow \Omega^{\bullet,\bullet+1}_M$$

satisfy

$$d'^2 = d''^2 = d'd'' + d''d' = 0 \ .$$

Hence the bigraded C^∞_M -algebra $\Omega^{\bullet,\bullet}_M$ carries two natural derivations d' and d'' of type $(1,0)$ and $(0,1)$ respectively. Considering the double complex $\Omega^{\bullet,\bullet}_M$ of sheaves on M

(6)
$$\begin{tikzcd}
\Omega^{0,0}_M \arrow[r, "d'"] \arrow[d, "d''"'] & \Omega^{1,0} \arrow[r, "d'"] \arrow[d, "d''"'] & \cdots \arrow[r, "d'"] \arrow[d, "d''"'] & \Omega^{n,0} \arrow[d, "d''"'] \\
\Omega^{0,1}_M \arrow[r, "d'"] \arrow[d, "d''"'] & \Omega^{1,1} \arrow[r, "d'"] \arrow[d, "d''"'] & \cdots \arrow[r, "d'"] \arrow[d, "d''"'] & \Omega^{n,1} \arrow[d, "d''"'] \\
\vdots \arrow[r, "d'"] \arrow[d, "d''"'] & \vdots \arrow[r, "d'"] \arrow[d, "d''"'] & \vdots \arrow[r, "d'"] \arrow[d, "d''"'] & \vdots \arrow[d, "d''"'] \\
\Omega^{0,r}_M \arrow[r, "d'"] & \Omega^{1,r} \arrow[r, "d'"] & \cdots \arrow[r, "d'"] & \Omega^{n,r}
\end{tikzcd}$$

one obtains an identification:

(7)
$$(\Omega^\bullet_M, d) = \mathrm{Tot}(\Omega^{\bullet,\bullet}_M, d', d'') \ .$$

Remark 2.4. The situation is formally similar to the decomposition of the sheaves of complex differential forms on a complex manifold, *except there is no real involution exchanging* $\Omega^{p,q}_M$ *with* $\Omega^{q,p}_M$.

Definition 2.5. *Let M be a manifold with a local product structure. We define the sheaves $\mathcal{L}'^p$ and $\mathcal{L}''^p$, $p \in \mathbb{Z}^{>0}$, on M as $\mathcal{L}'^p := \mathrm{Ker}(d'' : \Omega^{p,0}_M \longrightarrow \Omega^{p,1}_M)$ and $\mathcal{L}''^p := \mathrm{Ker}(d' : \Omega^{0,p}_M \longrightarrow \Omega^{1,p}_M)$.*

2.4. **Cohomological properties of the bicomplex $\Omega^{\bullet,\bullet}_M$.**

Lemma 2.6. *Let M be a manifold with a local product structure. Then $\mathbb{R}_M \longrightarrow (\mathcal{L}'^\bullet, d')$ and $\mathbb{R}_M \longrightarrow (\mathcal{L}''^\bullet, d'')$ are resolutions of the constant sheaf $\mathbb{R}_M$.*

Proof. Let $(V_\alpha, \varphi_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^r)$ be a chart of the local product structure M. Then $(\mathcal{L}'^\bullet, d')_{|V_\alpha} = (p' \circ \varphi_\alpha)^{-1}((\Omega^\bullet_{\mathbb{R}^n}, d))$. As $\mathbb{R}_{\mathbb{R}^n} \longrightarrow (\Omega^\bullet_{\mathbb{R}^n}, d)$ is a resolution and the functor $(p' \circ \varphi_\alpha)^{-1}$ is exact, one obtains a quasi-isomorphism $\mathbb{R}_{V_\alpha} = (p' \circ \varphi_\alpha)^{-1}\mathbb{R}_{\mathbb{R}^n} \simeq (\mathcal{L}'^\bullet, d')_{|V_\alpha}$. This proves that $\mathbb{R}_M \longrightarrow (\mathcal{L}'^\bullet, d')$ is a resolution. Similarly, replacing $\mathcal{L}'$, p' and d' by $\mathcal{L}''$, p'' and d'' respectively, one obtains that $\mathbb{R}_M \longrightarrow (\mathcal{L}''^\bullet, d'')$ is a resolution. □

Recall the following classical definitions:

Definition 2.7. *Let $\mathcal{A}$ be an Abelian category and K a complex of objects of $\mathcal{A}$. The (decreasing)* filtration bête $F^\bullet K$ *of K is defined as*

$$(F^pK)^n = \begin{cases} 0 & \text{if } n < p \\ K^n & \text{if } n \geq p. \end{cases} \tag{8}$$

Definition 2.8. *We denote by ${}_{d'}E^{p,q}$ the classical hypercohomology spectral sequence associated to the filtration bête $F^\bullet$ on the complex of sheaves $((\mathcal{L}')^\bullet, d')$ on M:*

$${}_{d'}E_1^{p,q} = H^q(M, (\mathcal{L}')^p) \Rightarrow H^{p+q}(M, \mathbb{R}) \ . \tag{9}$$

We denote by ${}_{d'}F^\bullet$ the associated filtration on $H^\bullet(M, \mathbb{R})$.

Similarly, we denote by ${}_{d''}E^{p,q}$ the spectral sequence obtained by replacing $\mathcal{L}'$ and d' by $\mathcal{L}''$ and d'' respectively:

$${}_{d''}E_1^{p,q} = H^q(M, (\mathcal{L}'')^p) \Rightarrow H^{p+q}(M, \mathbb{R}) \ . \tag{10}$$

We denote by ${}_{d''}F^\bullet$ the associated filtration on $H^\bullet(M, \mathbb{R})$.

Replacing cohomology with compactly supported cohomology we obtain the two spectral sequences

$${}_{d'}E_{c,1}^{p,q} = H_c^q(M, (\mathcal{L}')^p) \Rightarrow H_c^{p+q}(M, \mathbb{R}) \ . \tag{11}$$

$${}_{d''}E_{c,1}^{p,q} = H_c^q(M, (\mathcal{L}'')^p) \Rightarrow H_c^{p+q}(M, \mathbb{R}) \ . \tag{12}$$

We still denote by ${}_{d'}F^\bullet$ and ${}_{d''}F^\bullet$ the two associated filtrations on $H_c^\bullet(M, \mathbb{R})$.

Remark 2.9. These spectral sequences are first quadrant spectral sequences and hence converge. However I don't know if they always degenerate in E_2.

Remark 2.10. Notice that the sheaves $\Omega_M^{p,q}$ are fine sheaves on M. Hence both $\mathcal{L}'^p \longrightarrow (\Omega_M^{p,\bullet}, d'')$ and $\mathcal{L}''^p \longrightarrow (\Omega_M^{\bullet,p}, d')$ are acyclic resolutions for the functors $\Gamma(M, \cdot)$ and $\Gamma_c(M, \cdot)$. Thus the spectral sequences we defined above are nothing else than the two spectral sequences of the double complexes of real vector spaces obtained by applying $\Gamma(M, \cdot)$ and $\Gamma_c(M, \cdot)$ to the double complex of sheaves (6). Once more the situation is formally similar to the Hodge to De Rham spectral sequence for complex manifolds.

Remark 2.11. In view of Remark 2.10, the language of sheaves in not really needed at this step. However the use of sheaves will be unavoidable in the heart of the proof (see Section 4.3).

3. The case of flat bundles

Let X be a manifold of dimension n. Let E be a real oriented vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. The bundle structure on $\mathcal{E}$ defines a "trivial" foliation of dimension r on $\mathcal{E}$, with closed linear leaves. Suppose from now on that the bundle E is endowed with a flat connection ∇ associated to a linear representation $\rho : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(\vec{V})$. Then the manifold $\mathcal{E}$ has a natural local product structure given by the fibers of π and the leaf of the flat connection ∇. In the description of Definition 2.1 the maps $\varphi^2_{\alpha\beta}$ are moreover linear. In this situation the complex $(\mathcal{L}'^\bullet, d')$ and $(\mathcal{L}''^\bullet, d'')$ are easier to describe.

3.1. **The complex $(\mathcal{L}'^\bullet, d')$.** Notice that the sheaf $\mathcal{L}'^p$ is nothing else, by definition, than $\pi^{-1}\Omega^p_X$ and the complex $(\mathcal{L}'^\bullet, d')$ coincides with the complex $(\pi^{-1}\Omega^\bullet_X, \pi^{-1}d_X)$, which is well defined for any (not necessarily flat) vector bundle E on X.

Proposition 3.1. *There is a canonical isomorphism*

$$H^i(\mathcal{E}, \pi^{-1}\Omega^p_X) = H^i(\Omega^{p,\bullet}(\mathcal{E}), d'') = \begin{cases} 0 & \textit{if } i \neq 0; \\ \Omega^p(X) & \textit{if } i = 0. \end{cases}$$

Similarly, considering cohomology with compact support:

$$H^i_c(\mathcal{E}, \pi^{-1}\Omega^p_X) = H^i(\Omega^{p,\bullet}_c(\mathcal{E}), d'') = \begin{cases} 0 & \textit{if } i \neq r; \\ \Omega^p_c(X) & \textit{if } i = r. \end{cases}$$

Proof. Let us give the proof for the compactly supported cohomology, the proof for ordinary cohomology is similar. Consider the Leray spectral sequence for π

$$E_2^{i,j} = H^i_c(X, R^j\pi_!(\pi^{-1}\Omega^p_X)) \Rightarrow H^{i+j}_c(\mathcal{E}, \pi^{-1}\Omega^p_X) \ .$$

By the projection formula: $R\pi_!(\pi^{-1}\Omega^p_X) = R\pi_!(\mathbb{R}) \otimes \Omega^p_X = \Omega^p_X[-r]$.

As Ω^p_X is a fine sheaf (hence Γ_c-acyclic):

$$E_2^{i,j} = \begin{cases} 0 & \text{if } i \neq r; \\ \Omega^j_c(X) & \text{if } i = r; \end{cases}$$

Hence the E_2-page has only one non-zero column at $i = r$, the Leray spectral sequence degenerates in E_2 and we get the result. $\square$

Hence the first page ${}_{d'}E_1^{p,q}$ is given by

$${}_{d'}E_1^{p,q} = H^q(\mathcal{E}, \pi^{-1}\Omega^p_X) = \begin{cases} \Omega^p(X) & \text{if q=0 ;} \\ 0 & \text{if } q \neq 0. \end{cases}$$

It coincides with the usual de Rham complex of X on the line $q = 0$, thus the spectral sequence ${}_{d'}E$ degenerates in E_2 (not in E_1 as in the compact Kähler case!) and we recover (in a complicated way...) the isomorphism

$${}_{d'}E_\infty^{p,0} = \mathrm{Gr}^p_{{}_{d'}F} H^p(\mathcal{E}, \mathbb{R}) = H^p(\mathcal{E}, \mathbb{R}) \simeq H^p(X, \mathbb{R}) \ .$$

For the compactly supported cohomology we get:

$${}_{d'}E_{c,1}^{p,q} = H^q_c(\mathcal{E}, \pi^{-1}\Omega^p_X) = \begin{cases} \Omega^p_c(X) & \text{if q=r ;} \\ 0 & \text{if } q \neq r. \end{cases}$$

Hence the E_1-page of the spectral sequence ${}_{d'}E_c$ is the compactly supported global de Rham complex for X on the line $q = r$, the spectral sequence${}_{d'}E_c$ degenerates in E_2 and we recover (a version of) the Thom isomorphism (see Appendix A):

$$ {}_{d'}E^{p,r}_{c,\infty} = \mathrm{Gr}^p_{{}_{d'}F} H^{p+r}_c(\mathcal{E}, \mathbb{R}) = H^{p+r}_c(\mathcal{E}, \mathbb{R}) \simeq H^p_c(X, \mathbb{R}) \ . $$

Hence we have proven:

Proposition 3.2. *Let X be a connected oriented n-manifold. Let E be an oriented flat real vector bundle on X of rank $r > 0$ with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. Then*

$$ {}_{d'}E^{p,0}_{\infty} = \mathrm{Gr}^p_{{}_{d'}F} H^p(\mathcal{E}, \mathbb{R}) = H^p(\mathcal{E}, \mathbb{R}) \simeq H^p(X, \mathbb{R}) \,, \textit{ and} $$
$$ {}_{d'}E^{p,r}_{c,\infty} = \mathrm{Gr}^p_{{}_{d'}F} H^{p+r}_c(\mathcal{E}, \mathbb{R}) = H^{p+r}_c(\mathcal{E}, \mathbb{R}) \simeq H^p_c(X, \mathbb{R}) \ . $$

3.2. **The complex $(\mathcal{L}''^{\bullet}, d'')$ for flat bundles.** The cohomology of the sheaves $(\mathcal{L}'')^q$, $0 \leq q \leq r$, is not as simple as the one of the $\pi^{-1}\Omega^p_X$, $0 \leq p \leq n$. We will interpret it directly in terms of the monodromy representation $\rho : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(\vec{V})$ defining the flat structure on E. This $\pi_1(X, x_0)$-module structure on $\vec{V}$ induces a natural structure of $\pi_1(X, x_0)$-module on the infinite dimensional real vector spaces $\Omega^q(\vec{V})$ and $\Omega^q_c(\vec{V})$, $0 \leq q \leq r$, which makes $(\Omega^{\bullet}(\vec{V}), d)$ and $(\Omega^{\bullet}_c(\vec{V}), d)$ complexes of $\pi_1(X, x_0)$-modules.

Recall (see Section 1.5) that if Λ is a $\pi_1(X)$-module we denote by Λ_X the corresponding local system on X. Hence we obtain the following resolutions *in the category of infinite dimensional local systems on* X:

$$ \mathbb{R}_X \simeq ((\Omega^{\bullet}(\vec{V}))_X, d'') \qquad \text{and} \qquad \mathbb{R}_X[-r] \simeq ((\Omega^{\bullet}_c(\vec{V}))_X, d'') \ . \tag{13} $$

Proposition 3.3.

$$ H^i(\mathcal{E}, \mathcal{L}''^q) = H^i(\Omega^{q,\bullet}(\mathcal{E}), d'') = H^i(X, (\Omega^q(\vec{V}))_X) \quad \text{and} $$
$$ H^i_c(\mathcal{E}, \mathcal{L}''^q) = H^i_c(\Omega^{q,\bullet}(\mathcal{E}), d'') = H^i_c(X, (\Omega^q_c(\vec{V}))_X) \ . $$

Proof. We give the proof for the compactly supported cohomology, the other case is similar. Consider the Leray spectral sequence:

$$ E_2^{i,j} = H^i_c(X, R^j\pi_! \mathcal{L}''^q) \Rightarrow H^{i+j}_c(\mathcal{E}, \mathcal{L}''^q) \ . $$

As π is a locally trivial fibration the sheaf $R^j\pi_!\mathcal{L}''^q$ is nothing else than the locally constant sheaf $(H^j_c(\vec{V}, \Omega^q_{\vec{V}}))_X$. The sheaf $\Omega^q_{\vec{V}}$ is fine and hence is Γ_c-acyclic on $\vec{V}$. Thus $H^j_c(\vec{V}, \Omega^q_{\vec{V}})$ is zero for $j \neq 0$ and

$$ E_2^{i,j} = \begin{cases} 0 & \text{if} \quad j \neq 0; \\ H^i_c(X, (\Omega^q_c(\vec{V}))_X) & \text{if} \quad j = 0 \ . \end{cases} $$

The spectral sequence degenerates trivially in E_2 and the result follows. □

Although the cohomology of the sheaves $(\mathcal{L}'')^q$, $0 \leq q \leq r$, is not as simple as the one of the $\pi^{-1}\Omega^p_X$, $0 \leq p \leq n$, we can still show that the filtration ${}_{d''}F$ is trivial on $H^{\bullet}(\mathcal{E}, \mathbb{R})$:

Proposition 3.4. *Let X be a connected oriented n-manifold. Let E be an oriented flat real vector bundle on X of rank $r > 0$ with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. For any integer p, the natural map*

$$H^p(\mathcal{E}, \mathbb{R}) \simeq H^p(X, \mathbb{R}) \longrightarrow {}_{d''}E_\infty^{0,p} := \mathrm{Gr}^0_{{}_{d''}F} H^p(\mathcal{E}, \mathbb{R})$$

is an isomorphism.

Proof. It follows from Proposition 3.3 that the spectral sequence ${}_{d''}E_1^{p,q} = H^q(\mathcal{E}, \mathcal{L}''^p) \Rightarrow H^{p+q}(\mathcal{E}, \mathbb{R})$ is canonically isomorphic, via the isomorphism $\pi^* : H^\bullet(X, \cdot) \longrightarrow H^\bullet(\mathcal{E}, \cdot)$, to the spectral sequence $E_{\rho,1}^{p,q} = H^q(X, (\Omega^p(\vec{V}))_X) \Rightarrow H^{p+q}(X, \mathbb{R}_X)$ associated to the filtration bête of the resolution $\mathbb{R}_X \simeq (\Omega^\bullet(\vec{V}))_X$. Notice that the natural morphism of $\pi_1(X, x_0)$-modules $\mathbb{R} \longrightarrow \Omega^0(V)$ is split injective: the splitting is defined by associating to $f \in \Omega^0(V)$ its value $f(0)$. Hence the edge map $H^q(X, \mathbb{R}_X) \longrightarrow H^q(X, (\Omega^0(\vec{V}))_X)$ of our spectral sequence is injective for all q, hence the map $H^p(\mathcal{E}, \mathbb{R}) \longrightarrow \mathrm{Gr}^0_{{}_{d''}F} H^p(\mathcal{E}, \mathbb{R})$ is an isomorphism. □

Notice that similarly the spectral sequence ${}_{d''}E_{c,1}^{p,q} = H_c^q(\mathcal{E}, \mathcal{L}''^p) \Rightarrow H^{p+q}(\mathcal{E}, \mathbb{R})$ is canonically isomorphic, via the Thom isomorphism $\Phi : H_c^\bullet(X, \mathbb{R}) \longrightarrow H_c^{\bullet+r}(\mathcal{E}, \mathbb{R})$, to the spectral sequence $E_{c,\rho,1}^{p,q} = H_c^q(X, (\Omega^p(\vec{V}))_X) \Rightarrow H_c^{p+q}(X, \mathbb{R}_X[-r])$ associated to the filtration bête of the resolution $\mathbb{R}_X[-r] \simeq (\Omega_c^\bullet(\vec{V}))_X$. This time the morphism $\Omega_c^r(\vec{V}) \longrightarrow \mathbb{R}$ of $\pi_1(X, x_0)$-modules given by integrating over $\vec{V}$ does not split anymore and I don't know how to compute the filtration ${}_{d''}F$ on $H_c^\bullet(\mathcal{E}, \mathbb{R})$.

Definition 3.5. *We denote by $\varphi : {}_{d''}E_c \longrightarrow {}_{d''}E$ the canonical morphism of spectral sequences defined by the morphism of functors $H_c^\bullet(\mathcal{E}, \cdot) \longrightarrow H^\bullet(\mathcal{E}, \cdot)$.*

3.3. **A criterion for the vanishing of the Euler class of a flat bundle : proof of Proposition 1.9.** *From now on we assume that X is closed.*

Proof. Let E be a flat oriented real vector bundle of rank $r > 0$ with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. Let $\omega \in \Omega_c^r(\mathcal{E})$ be a Thom form for E, see Definition A.6. In particular the n-form ω is d-closed. By Lemma A.4 the Euler class of E can be computed as:

$$e(X) = i^*[\omega] \in H^n(X, \mathbb{R}) \ ,$$

where $i : X \longrightarrow \mathcal{E}$ denotes the zero section.

Let us now relate the form ω to the spectral sequence ${}_{d''}E_c$. Decompose the form ω into types: $\omega = \sum_{i=0}^r \omega^{i,r-i}$, with $\omega^{i,r-i} \in \Omega_c^{i,r-i}(\mathcal{E})$. As ω is d-closed we obtain:

$$\forall\, i,\ 0 \leq i \leq r, \quad d'\omega^{i,r-i} = -d''\omega^{i+1,r-i-1} \ . \tag{14}$$

In particular: $d'\omega^{r,0} = 0$. As $\mathcal{L}''^p \longrightarrow (\Omega_{\mathcal{E}}^{\bullet,p}, d')$ is a fine resolution, the spectral sequence ${}_{d''}E_{c,1}^{p,q} = H_c^q(\mathcal{E}, \mathcal{L}^p) \Rightarrow H_c^{p+q}(\mathcal{E}, \mathbb{R})$ coincides with the double complex spectral sequence:

$$E_{c,1}^{p,q} = H^q(\Omega_c^{\bullet,p}(\mathcal{E}), d') \Rightarrow H_c^{p+q}(\mathcal{E}, \mathbb{R}) \ .$$

Hence the d'-closed form $\omega^{r,0}$ defines a class $[\omega^{r,0}] \in {}_{d''}E_{c,1}^{0,r}$. Notice that the sub-quotient ${}_{d''}E_\infty^{0,r}$ of ${}_{d''}E_1^{0,r}$ is in fact a subspace of ${}_{d''}E_1^{0,r}$ as $E_1^{p,q} = 0$ for $p < 0$.

Lemma 3.6. *The class* $[\omega^{r,0}]$ *belongs to* ${}_{d''}E_{c,\infty}^{0,r}$.

Proof. By definition $d_1 : {}_{d''}E_{c,1}^{0,r} \longrightarrow {}_{d''}E_{c,1}^{1,r}$ maps $[\omega^{r,0}]$ to the class of $d''\omega^{r,0}$ in ${}_{d''}E_{c,1}^{1,r} = H^r(\Omega_c^{\bullet,1}(\mathcal{E}), d')$. From Equation (14) we get $d''\omega^{r,0} = d'(-\omega^{r-1,1})$ hence $d_1[\omega^{r,0}] = 0$ and $[\omega^{r,0}]$ belongs to ${}_{d''}E_{c,2}^{r,0}$.

More generally it follows by induction on i that $[\omega^{r,0}]$ belongs to ${}_{d''}E_{c,i}^{0,r}$ and that $d_i[\omega^{0,r}]$ coincides with the class of $d''\omega^{r-i+1,i-1}$ in ${}_{d''}E_{c,i}^{i,r-i+1}$, which vanishes as $d''\omega^{r-i+1,i-1} = d'(-\omega^{r-i,i})$.

The lemma follows. □

We now prove the proposition. Suppose first that $\varphi_\infty^{0,r} : {}_{d''}E_{c,\infty}^{0,r} \longrightarrow {}_{d''}E_{\infty}^{0,r}$ vanishes and let us show that $e_{\mathbb{R}}(e) = 0$. As the diagram

$$\begin{tikzcd}
{}_{d''}E_{c,\infty}^{0,r} \arrow[r, hook] \arrow[d, "\varphi_\infty^{0,r}"'] & {}_{d''}E_{c,1}^{0,r} \arrow[d, "\varphi_1^{0,r}"] \\
{}_{d''}E_{\infty}^{0,r} \arrow[r, hook] & {}_{d''}E_{c,1}^{0,r}
\end{tikzcd}$$

commutes, it follows from Lemma 3.6 that $\varphi_1^{0,r}([\omega^{r,0}]) = 0$. This means that $\omega^{r,0}$ is d'-exact in $(\Omega^{\bullet,0}(\mathcal{E}), d')$: there exists $\alpha \in \Omega^{r-1,0}(\mathcal{E})$ such that $\omega^{r,0} = d'\alpha$. Hence

$$e(X) = [i^*\omega] = [i^*\omega^{r,0}] = [i^*(d'\alpha)] = [d(i^*\alpha)] = 0 \in H^n(X, \mathbb{R}) \ .$$

Conversely suppose that $\varphi_\infty^{0,r} : {}_{d''}E_{c,\infty}^{0,r} \longrightarrow {}_{d''}E_{\infty}^{0,r}$ does not vanish. Consider the commutative diagram:

$$\begin{tikzcd}
H_c^r(\mathcal{E}, \mathbb{R}) \arrow[r, two heads] \arrow[d] & {}_{d''}E_{c,\infty}^{0,r} = \mathrm{Gr}_F^0 H_c^r(\mathcal{E}, \mathbb{R}) \arrow[d, "\varphi_\infty^{0,r}"'] \\
H^r(\mathcal{E}, \mathbb{R}) \arrow[r, two heads] & {}_{d''}E_{\infty}^{0,r} = \mathrm{Gr}_F^0 H^r(\mathcal{E}, \mathbb{R}).
\end{tikzcd}$$

As $H_c^r(\mathcal{E}, \mathbb{R}) \simeq H^0(X, \mathbb{R}) \simeq \mathbb{R}$ (Thom isomorphism) the non-vanishing of $\varphi_\infty^{0,r}$ implies that the quotient map $\mathbb{R} = H_c^r(\mathcal{E}, \mathbb{R}) \twoheadrightarrow {}_{d''}E_{c,\infty}^{0,r}$ is an isomorphism and that the map $H_c^r(\mathcal{E}, \mathbb{R}) \longrightarrow H^r(\mathcal{E}, \mathbb{R})$ is injective. As the Thom class $[\omega] \in H_c^r(\mathcal{E}, \mathbb{R})$ generates $H_c^r(\mathcal{E}, \mathbb{R})$, this implies that its image $e_{\mathbb{R}}(e)$ is non-zero in $H^r(\mathcal{E}, \mathbb{R})$. □

4. The case of affine manifolds

Let X be a closed n-manifold whose tangent bundle $E := TX$ admits a flat connection. As explained in Section 3 the flat structure on TX endows the total space $\mathcal{E}$ of TX with a para-complex structure, which we call its *standard para-complex structure* and Proposition 1.9 provides a criterion for the vanishing of $\chi(X)$.

Suppose now that X is affine i.e. the flat connection on TX is torsion-free. In that case and in that case only, the standard para-complex structure on TX can be upgraded to a much richer structure: a *para-hypercomplex structure* (see Section 4.1).

Any para-hypercomplex manifold M admits a canonical $\mathrm{GL}(2,\mathbb{R})$-action on its tangent bundle, whose restriction to $\mathrm{SO}(2)$ defines an S^1-family of para-complex structures on M. Hence when X is affine the space $\mathcal{E}$ is canonically endowed with an S^1-family of para-complex structures, the para-complex structure at $\theta = 0$ being the standard one.

The main idea in the proof of Theorem 1.5 is that *this S^1-family of para-complex structures on $\mathcal{E}$ induces an S^1-family of spectral sequences, interpolating in a subtle way between the spectral sequence ${}_{d''}E_c$ and the much better understood spectral sequence ${}_{d'}E_c$ (resp. between ${}_{d''}E$ and ${}_{d'}E$).* This will enable us to show that the Proposition 1.9 is satisfied for special affine manifolds.

4.1. **Para-hypercomplex structures.**

4.1.1. *Definition.* Let n be a positive integer. The canonical isomorphism $\mathbb{R}^n \otimes \mathbb{R}^2 \simeq \mathbb{R}^{2n}$ induces an embedding

$$\mathrm{GL}(n,\mathbb{R}) \times_{\mathbb{R}^*} \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R}). \tag{15}$$

Definition 4.1. *Let M be a differential manifold of dimension $2n$. A para-hypercomplex structure on M is an integrable $\mathrm{GL}(n,\mathbb{R})$-structure on M (for the embedding of $\mathrm{GL}(n,\mathbb{R})$ in $\mathrm{GL}(2n,\mathbb{R})$ defined by (15)).*

Equivalently [And05] *a para-hypercomplex structure on M is the data of a complex structure $I \in \mathrm{End}\,(TM)$ and a para-complex structure $J \in \mathrm{End}\,(TM)$ satisfying $IJ = -JI$.*

Para-hypercomplex manifolds are for para-complex manifolds what hypercomplex manifolds are for complex manifolds. This analogy is explained in detail in Appendix B, hoping it might clarify the well-known many analogies between complex geometry and affine geometry.

Para-hypercomplex structures are also called "complex-product" structure. In addition to Appendix B we refer to [IvZam05], [And05] and references mentioned there for more details on such structures.

4.1.2. *S^1-family of para-complex structures on a para-hypercomplex manifold.* Let M be a para-hypercomplex manifold. The 4-dimensional $\mathbb{R}$-algebra generated by the two elements J and I of Definition 4.1 satisfying $J^2 = 1$, $I^2 = -1$ and $JI = -IJ$ identifies with $\mathfrak{gl}_2(\mathbb{R})$, which thus acts on TM. As the $\mathrm{GL}(n,\mathbb{R})$-structure on M is integrable, this $\mathfrak{gl}_2(\mathbb{R})$-action integrates to an action of the centralizer $\mathrm{GL}(2,\mathbb{R})$ of $\mathrm{GL}(n,\mathbb{R})$ in $\mathrm{GL}(2n,\mathbb{R})$ on TM. The $\mathrm{GL}(2,\mathbb{R})$-orbit of J in $\mathrm{End}\,(TM)$ identifies with the $\mathrm{GL}(2,\mathbb{R})$-adjoint orbit of $J \in \mathfrak{gl}_2(\mathbb{R})$, whose Killing norm is positive. Hence the $\mathrm{GL}(2,\mathbb{R})$-orbit of J is a hyperboloid of one sheet $\mathcal{H} := \mathrm{GL}(2,\mathbb{R})/(\mathbb{R}^* \times_{\mathbb{Z}/2} \mathbb{R}^*) \simeq \mathbb{R} \times S^1$, the section S^1 being given by the $\mathrm{SO}(2)$-orbit of J. Each point of this orbit $\mathcal{H}$ defines a para-complex on M.

We thus obtain an $\mathcal{H}$-family of para-complex structures on M, with an S^1-subfamily for which the complex structure I is preserved.

4.1.3. *Para-hypercomplex structure on the tangent bundle of an affine manifold.* Let X be a closed n-dimensional manifold. Let $\mathcal{E}$ the total space of its tangent bundle TX and $\pi : \mathcal{E} \longrightarrow X$ the corresponding projection. Let ∇ be any linear connection on TX. It induces:

- a direct sum decomposition $T\mathcal{E} = T_v\mathcal{E} \oplus T_h\mathcal{E}$ into a vertical and horizontal part. Moreover $T_v\mathcal{E} \simeq T_h\mathcal{E} \simeq \pi^*TX$.

- an almost complex structure I on $\mathcal{E}$ given by $I = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the previous decomposition.

- an almost product structure J on $\mathcal{E}$ defined by $J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. They obviously satisfy the relation $IJ = -JI$.

One easily shows that J is integrable if and only if ∇ is flat. Moreover Dombrowski [Dom62] proved that I is a complex structure if and only if ∇ is flat *and torsion-free*. It follows from these remarks and Definition 4.1 that if X is an affine manifold then the total space $\mathcal{E}$ of TX is canonically endowed with a para-hypercomplex structure, which we call *the* para-hypercomplex structure on $\mathcal{E}$.

4.2. S^1-family of para-complex structures on the total space of the tangent bundle of an affine manifold. It follows from Section 4.1.3 and Section 4.1.2 that the para-hypercomplex space $\mathcal{E}$, total space of the tangent bundle $E := TX$ of an affine manifold X, is endowed with an S^1-family of para-complex structures. We now describe in more details these structures in term of the developing map of the affine structure on X.

4.2.1. *The standard para-complex structure on $\mathcal{E}$.* We define the *standard action* of $\mathrm{Aff}(V)$ on $\mathcal{E}_V := V \times \vec{V}$ as the one obtained on the total space $\mathcal{E}_V$ of the tangent bundle $TV = V \times \vec{V}$ from the standard action of $\mathrm{Aff}(V)$ on V. If $l : \mathrm{Aff}(V) \twoheadrightarrow \mathrm{GL}(\vec{V})$ denotes the *linear part* this standard action is given by

$$\forall\, g \in \mathrm{Aff}(V),\ \forall (u, v) \in \mathcal{E}_V = V \times \vec{V}, \quad g \cdot (u, v) = (g \cdot u,\ l(g) \cdot v) \ .$$

Let X be an oriented $(\mathrm{Aff}(V), V)$-manifolds. Fix a base-point x_0 in X, $\tilde{x_0}$ the corresponding base point of $\tilde{X}$ and let $(h : \pi_1(X, x_0) \longrightarrow \mathrm{Aff}(V), D : \tilde{X} \longrightarrow V)$ be the corresponding holonomy and developing map.

Let $\mathcal{E}$ denote the total space of the tangent bundle TX, with projection $\pi : \mathcal{E} \longrightarrow X$ and base point $e_0 := (x_0, 0)$. Then the $(\mathrm{Aff}(V), V)$-structure (h, D) on X defines an $(\mathrm{Aff}(V), V \times \vec{V})$-structure on $\mathcal{E}$ (for the standard action of $\mathrm{Aff}(V)$ on $V \times \vec{V}$) with holonomy $h : \pi_1(\mathcal{E}, e_0) \simeq \pi_1(X, x_0) \longrightarrow \mathrm{Aff}(V)$ and developing map

$$D_{\mathcal{E}} := dD : \tilde{\mathcal{E}} \longrightarrow \mathcal{E}_V = V \times \vec{V} \ , \tag{16}$$

called the *standard* $(\mathrm{Aff}(V), V \times \vec{V})$*-structure on* $\mathcal{E}$. It induces the standard para-complex structure on $\mathcal{E}$ associated as in Section 3 to the flat structure on the bundle $E = TX$ defined by the monodromy representation $l \circ h : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(\vec{V})$.

4.2.2. *The S^1-family of para-complex structures on the tangent bundle of an* $(\mathrm{Aff}(V), V)$*-manifold.* Suppose given an origin $O \in V$. It defines a splitting $\mathrm{Aff}(V) = \mathrm{GL}(\vec{V}) \ltimes \vec{V}$. The decomposition of an element $g \in \mathrm{Aff}(V)$ for this splitting will be denoted by $(l(g), t(g))$ (where $t(g)$ denotes the *translational part* of g).

Definition 4.2. *Let $\theta \in [0, 2\pi]$. The θ-deformed action of* $\mathrm{Aff}(V)$ *on $\mathcal{E}_V$ is defined by*

$$\forall\, g \in \mathrm{Aff}(V),\ \forall (u, v) \in \mathcal{E}_V = V \times \vec{V}, \quad g \cdot_\theta (u, v) = (l(g) \cdot u + \cos\theta \cdot t(g),\ l(g) \cdot v - \sin\theta \cdot t(g)) \ .$$

We denote by $\mathcal{E}_{V,\theta}$ the space $\mathcal{E}_V$ with this θ-deformed affine action of $\mathrm{Aff}(V)$.

Remark 4.3. The 0-deformed action is the standard action of Section 4.2.1.

For $\theta \in [0, 2\pi]$ let us define $R_\theta := \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{GL}(\vec{V} \oplus \vec{V})$ as the block rotation matrix with angle θ. Fix $\tilde{e_0} \in \tilde{\mathcal{E}}$ a point over $e_0 \in \mathcal{E}$ and choose $O := D_{\mathcal{E}}(\tilde{e_0})$ as an origin for V. The local diffeomorphism $D_{\mathcal{E},\theta} := R_\theta \circ D_{\mathcal{E}} : \tilde{\mathcal{E}} \longrightarrow \mathcal{E}_V$ satisfies

$$\forall\, \gamma \in \pi_1(\mathcal{E}, e_0), \quad D_{\mathcal{E},\theta} \circ \gamma = h(\gamma) \cdot_\theta D_{\mathcal{E},\theta} \ .$$

In other words the pair $(h, D_{\mathcal{E},\theta})$ defines an $(\mathrm{Aff}(V), \mathcal{E}_{V,\theta})$-structure on $\mathcal{E}$. In particular it defines a para-complex structure on $\mathcal{E}$, called the θ-para-complex structure, which, in local affine coordinates, is obtained by applying the rotation R_θ to the standard one. This S^1-family of para-complex structures on $\mathcal{E}$ coincides with the one defined in Section 4.1.2.

Remark 4.4. Notice that these $(\mathrm{Aff}(V), \mathcal{E}_{V,\theta})$-structures on $\mathcal{E}$ are not deduced by differentiation from an affine structure on X as in Equation (16), except if $\theta = 0 \mod \pi$.

Remark 4.5. All these $(\mathrm{Aff}(V), \mathcal{E}_{V,\theta})$-structures on $\mathcal{E}$ are equivalent as $(\mathrm{Aff}(V \times V), V \times V)$-structure. On the other hand the para-complex structures associated to θ_1 and θ_2 coincide if and only if $\theta_1 = \theta_2 \mod \pi$.

4.3. **The one-parameter family of spectral sequences.**

4.3.1. *Definitions.*

Definition 4.6. *For each $\theta \in [0, 2\pi]$, we define:*

- the complex of sheaves $(\mathcal{L}''^{\bullet}_{\theta}, d''_\theta)$ on $\mathcal{E}$ associated to the θ-para-complex structure on $\mathcal{E}$ as in Definition 2.5,

- the spectral sequences ${}_{d''_\theta}E^{\bullet;\bullet}_{\bullet}$ and ${}_{d''_\theta}E^{\bullet;\bullet}_{c,\bullet}$ associated to the θ-para-complex structure on $\mathcal{E}$ as in Definition 2.8,

- the bigraded complex of sheaves $(\Omega^{\bullet;\bullet}_{\mathcal{E},\theta}, d'_\theta, d''_\theta)$ on $\mathcal{E}$ associated to the θ-para-complex structure on $\mathcal{E}$ as defined in Section 2.3.

We now sheafify the situation over S^1:

Definition 4.7. *Consider the submersion $\Psi : \tilde{\mathcal{E}} \times S^1 \longrightarrow \mathcal{E}_V$ defined by $\Psi(e, \theta) = D_{\mathcal{E},\theta}(e)$. We define the complex of sheaves $(\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1}, d'')$ on $\mathcal{E} \times S^1$ as the descent to $\mathcal{E} \times S^1$ of the $\pi_1(\mathcal{E})$-equivariant complex of sheaves $\Psi^{-1}((\mathcal{L}'')^{\bullet}_{\mathcal{E}_V}, d'')$ on $\tilde{\mathcal{E}} \times S^1$; and $(\Omega^{\bullet;\bullet}_{\mathcal{E}\times S^1/S^1}, d', d'')$ as the descent to $\mathcal{E} \times S^1$ of the $\pi_1(\mathcal{E})$-equivariant complex of sheaves $\Psi^{-1}(\Omega^{\bullet;\bullet}_{\mathcal{E}_V}, d', d'')$.*

The complex $((\mathcal{L}'')^{\bullet}_{\mathcal{E}_V}, d'')$ resolves the constant sheaf $\mathbb{R}_{\mathcal{E}_V}$ and is quasi-isomorphic to the total complex of $(\Omega^{\bullet;\bullet}_{\mathcal{E}_V}, d', d'')$. As the functor Ψ^{-1} is exact, it follows once more that the complex $(\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1}, d'')$ resolves the constant sheaf $\mathbb{R}_{\mathcal{E}\times S^1}$ and is quasi-isomorphic to the total complex of $(\Omega^{\bullet;\bullet}_{\mathcal{E}\times S^1/S^1}, d', d'')$. For $\theta \in S^1$, the restriction of the complex of sheaves $(\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1}, d'')$ to the fiber $\mathcal{E}_\theta$ coincides with the complex of sheaves $(\mathcal{L}''^{\bullet}_{\theta}, d''_\theta)$. Similarly the restriction of the bigraded complex of sheaves $(\Omega^{\bullet;\bullet}_{\mathcal{E}\times S^1/S^1}, d', d'')$ to the fiber $\mathcal{E}_\theta$ coincide with $(\Omega^{\bullet;\bullet}_{\mathcal{E},\theta}, d'_\theta, d''_\theta)$.

Remark 4.8. The sheaves $\Omega^{r;s}_{\mathcal{E},\theta}$ on $\mathcal{E}$ are fine sheaves and hence provide an *acyclic* resolution $(\Omega^{\bullet;s}_{\mathcal{E},\theta}, d'_\theta)$ of $\mathcal{L}''^{s}_{\theta}$. In contrast the sheaves $\Omega^{r;s}_{\mathcal{E}\times S^1/S^1}$ are not fine, hence the resolution $(\Omega^{\bullet;s}_{\mathcal{E}\times S^1/S^1}, d')$ of $\mathcal{L}''^{s}_{\mathcal{E}\times S^1/S^1}$ has no reason to be acyclic.

Definition 4.9. *Let $p_2 : \mathcal{E}\times S^1 \longrightarrow S^1$ denote the second projection.*

a) We denote by ${}_{d''}\mathcal{E}^{\bullet;\bullet}_{c,\bullet}$ the spectral sequence computing $Rp_{2!}\mathbb{R}_{\mathcal{E}\times S^1} = Rp_{2!}(\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1})$ associated to the filtration bête on $\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1}$:

$$ {}_{d''}\mathcal{E}^{p,q}_{c,1} = R^q p_{2!}\mathcal{L}''^{p}_{\mathcal{E}\times S^1/S^1} \Rightarrow R^{p+q}p_{2!}\mathbb{R}_{\mathcal{E}\times S^1} \quad , $$

and by ${}_{d''}F^\bullet$ the associated filtration on the sheaves $R^\bullet p_{2!}\mathbb{R}_{\mathcal{E}\times S^1}$.

b) Similarly we denote by ${}_{d''}\mathcal{E}^{\bullet;\bullet}_{\bullet}$ the spectral sequence computing $Rp_{2*}\mathbb{R}_{\mathcal{E}\times S^1} = Rp_{2*}(\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1})$ associated to the filtration bête on $\mathcal{L}''^{\bullet}_{\mathcal{E}\times S^1/S^1}$:

$$ {}_{d''}\mathcal{E}^{p,q}_{1} = R^q p_{2*}\mathcal{L}''^{p}_{\mathcal{E}\times S^1/S^1} \Rightarrow R^{p+q}p_{2*}\mathbb{R}_{\mathcal{E}\times S^1} \quad , $$

and by ${}_{d''}F^\bullet$ the associated filtration on the sheaves $R^\bullet p_{2*}\mathbb{R}_{\mathcal{E}\times S^1}$.

c) We denote by $\varphi^{\bullet;\bullet}_{\bullet,S^1} : {}_{d''}\mathcal{E}^{\bullet;\bullet}_{c,\bullet} \longrightarrow {}_{d''}\mathcal{E}^{\bullet;\bullet}_{\bullet}$ the morphism of spectral sequences of sheaves over S^1 induced by the morphism of functors $Rp_{2!} \longrightarrow Rp_{2*}$.

4.3.2. *The sheaf spectral sequence versus the pointwise spectral sequences.* Let us now describe the relation between the morphisms of sheaves

$$ \varphi^{r,s}_{\infty,S^1} : {}_{d''}\mathcal{E}^{r,s}_{c,\infty} \longrightarrow {}_{d''}\mathcal{E}^{r,s}_{\infty} $$

and the morphisms of vector spaces $\varphi^{r,s}_{\infty} : {}_{d''}E^{r,s}_{c,\infty} \longrightarrow {}_{d''}E^{r,s}_{\infty}$ of Definition 3.5.

As usual for a sheaf $\mathcal{F}$ over a locally compact topological space Y we denote by $\mathcal{F}_y$ its stalk at the point y. Recall that if $f : Y \longrightarrow Z$ is a continuous morphism of topological spaces then for any point $z \in Z$ there are natural morphisms $(R^p f_* F)_z \longrightarrow H^p(Y_z, F_{|Y_z})$ and $(R^p f_! F)_z \longrightarrow H^p_c(Y_z, F_{|Y_z})$. Moreover this last morphism is an isomorphism by the proper base change theorem ([KS90, Prop.2.5.2]), while the first one is not surjective in general. Let us apply this to $f = p_2$ and $z = \theta$.

For all non negative integers r, s and all $\theta \in S^1$ we obtain a natural isomorphism

$$ \left(R^r p_{2!}\mathcal{L}''^{s}_{\mathcal{E}\times S^1/S^1}\right)_\theta \xrightarrow{\sim} H^r_c(\mathcal{E}, \mathcal{L}''^{s}_{\theta}) \quad , $$

i.e. a natural isomorphism $\left({}_{d''}\mathcal{E}^{r,s}_{c,1}\right)_\theta \xrightarrow{\sim} {}_{d''_\theta}E^{r,s}_{c,1}$. This isomorphism is compatible with the differential d_1, hence for every $\theta \in S^1$ we obtain an isomorphism of spectral sequences

$\left({}_{d''}\mathcal{E}_{c,\bullet}^{\bullet,\bullet}\right)_\theta \xrightarrow{\sim} {}_{d''_\theta}E_{c,\bullet}^{\bullet,\bullet}$. For all $\theta \in S^1$ we denote by $\alpha_\theta^{r,s}$ the inverse isomorphism

(17) $$\alpha_\theta^{r,s} : \left({}_{d''}\mathcal{E}_{c,\infty}^{r,s}\right)_\theta \xrightarrow{\sim} {}_{d''_\theta}E_{c,\infty}^{r,s} \quad .$$

Arguing with p_{2*} rather than $p_{2!}$, one obtains for every $\theta \in S^1$ a morphism

(18) $$\beta_\theta^{r,s} : \left({}_{d''}\mathcal{E}_{\infty}^{r,s}\right)_\theta \longrightarrow {}_{d''_\theta}E_{\infty}^{r,s} \quad ,$$

which is a priori neither injective nor surjective as the fibers of p_2 are non-compact.

As these morphisms are natural we obtain in particular:

Lemma 4.10. *The morphism of vector spaces $\varphi_\infty^{0,n} : {}_{d''}E_{c,\infty}^{0,n} \longrightarrow {}_{d''}E_\infty^{0,n}$ factorises as*

$${}_{d''}E_{c,\infty}^{0,n} \xrightarrow[\sim]{\left(\alpha_0^{0,n}\right)^{-1}} \left({}_{d''}\mathcal{E}_{c,\infty}^{0,n}\right)_{\theta=0} \xrightarrow{\left(\varphi_{\infty,S^1}^{0,n}\right)_{\theta=0}} \left({}_{d''}\mathcal{E}_{\infty}^{0,n}\right)_{\theta=0} \xrightarrow{\beta_0^{0,n}} {}_{d''}E_\infty^{0,n} \quad .$$

The following criterion is thus an immediate corollary of Lemma 4.10 and Proposition 1.9:

Corollary 4.11. *Let X be an affine n-dimensional manifold. If the morphism of sheaves $\varphi_{\infty,S^1}^{0,n} : {}_{d''}\mathcal{E}_{c,\infty}^{0,n} \longrightarrow {}_{d''}\mathcal{E}_\infty^{0,n}$ vanishes then $e_{\mathbb{R}}(X) = 0$.*

4.3.3. *Structure of the sheaves ${}_{d''}\mathcal{E}_\infty^{r,n-r}$.* By definition the sheaf ${}_{d''}\mathcal{E}_\infty^{r,n-r}$ is the graded sheaf $\mathrm{Gr}^r_{{}_{d''}F^\bullet} R^n p_* \mathbb{R}_{\mathcal{E}\times S^1}$ on S^1. Now $R^n p_{2*}\mathbb{R}_{\mathcal{E}\times S^1}$ is nothing else than the constant sheaf $\mathbb{R}_{S^1}$ as the fibration $p_2 : \mathcal{E} \times S^1 \longrightarrow S^1$ is trivial. Hence all the sheaves ${}_{d''}\mathcal{E}_\infty^{r,n-r}$ are constructible on S^1 as constructibility is preserved by taking subquotients. The following lemma will in particular be useful for us:

Lemma 4.12. *There exists a unique open subset $j : U \hookrightarrow S^1$ such that the sheaf ${}_{d''}\mathcal{E}_\infty^{n,0}$ is isomorphic to the subsheaf $j_!\mathbb{R}_U \hookrightarrow \mathbb{R}_{S^1}$.*

Proof. The sheaf ${}_{d''}\mathcal{E}_\infty^{n,0} = \mathrm{Gr}^n_{{}_{d''}F^\bullet} R^n p_{2*}\mathbb{R}_{\mathcal{E}\times S^1} = {}_{d''}F^n R^n p_{2*}\mathbb{R}_{\mathcal{E}\times S^1} = {}_{d''}F^n \mathbb{R}_{S_1}$ is not only a subquotient, but a subsheaf of $\mathbb{R}_{S_1}$. Hence its support U is open in S^1, and ${}_{d''}\mathcal{E}_\infty^{n,0}$ is of the form $j_!\mathbb{R}_U$, where $j : U \hookrightarrow S^1$ denotes the natural inclusion. □

4.4. **End of the proof of the Theorem 1.5.** Our main Theorem 1.5 follows immediately from Corollary 4.11 and the following:

Proposition 4.13. *Suppose that X is special affine. Then the inclusion morphism*

$${}_{d''}\mathcal{E}_\infty^{n,0} \hookrightarrow \mathbb{R}_{S^1}$$

is an isomorphism. In particular ${}_{d''}\mathcal{E}_\infty^{0,n}$ is zero.

Proof. The real vector spaces $\{\Omega^{0,n}_{\mathcal{E}_{V,\theta}}(\mathcal{E}_{V,\theta})\}^{\mathrm{SAff}(V)} \subset \Omega^n(\mathcal{E}_V)$ of $\mathrm{SAff}(V)$-invariant $(0,n)$-forms on $\mathcal{E}_V$ for the θ-deformed action coincide for all $\theta \in S^1$ and are 1-dimensional. Let $\omega^{0,n}_{\mathcal{E}_V}$ be a common generator of these spaces. Hence $\Psi^{-1}\omega_V^{0,n}$ defines a $\pi_1(\mathcal{E})$-invariant global section of $\Psi^{-1}\Omega^{0,n}_{\mathcal{E}_V}$, hence a global section $\omega^{0,n}_{\mathcal{E}\times S^1/S^1}$ of the sheaf $\Omega^{0,n}_{\mathcal{E}\times S^1/S^1}$ on $\mathcal{E}\times S^1$. As the section $\omega^{0,n}_{\mathcal{E}_V}$ is d-closed, the section $\omega^{0,n}_{\mathcal{E}\times S^1/S^1}$ is also d-closed.

As ${}_{d''}\mathcal{E}_1^{n,0} = p_{2*}\mathcal{L}''^n_{\mathcal{E}\times S^1/S^1}$ and $\mathcal{L}''^n_{\mathcal{E}\times S^1/S^1} = \ker(d' : \Omega^{0,n}_{\mathcal{E}\times S^1/S^1} \longrightarrow \Omega^{1,n}_{\mathcal{E}\times S^1/S^1})$, the d'-closed global section $\omega^{0,n}_{\mathcal{E}\times S^1/S^1}$ of $\Omega^{0,n}_{\mathcal{E}\times S^1/S^1}$ over $\mathcal{E}\times S^1$ defines a global section $[\omega^{0,n}_{\mathcal{E}\times S^1/S^1}]$ of the sheaf ${}_{d''}\mathcal{E}_1^{n,0}$ over S^1, hence of its quotient ${}_{d''}\mathcal{E}_\infty^{n,0}$.

Let us show that this global section $[\omega^{0,n}_{\mathcal{E}\times S^1/S^1}] \in {}_{d''}\mathcal{E}_\infty^{n,0}(S^1)$ is non-zero. For $\theta \in S^1$, consider the morphism

$$ {}_{d''}\mathcal{E}_\infty^{n,0}(S^1) \longrightarrow \left({}_{d''}\mathcal{E}_\infty^{n,0}\right)_\theta \xrightarrow{\beta_\theta^{n,0}} {}_{d''_\theta}E^{n,0}_{\infty,\theta} \ , $$

where the first morphism associates to a global section its germ at $\theta \in S^1$. By definition it maps the class $[\omega^{0,n}_{\mathcal{E}\times S^1/S^1}]$ to the class $[\omega_\theta^{0,n}] \in {}_{d''_\theta}E_\infty^{n,0}$, where $\omega_\theta^{0,n} := D^*_{\mathcal{E},\theta}(\omega_{\mathcal{E}_V}) \in {}_{d''_\theta}E_1^{n,0}$.

Consider $\theta = \pi/2$. Then by definition ${}_{d''_{\frac{\pi}{2}}}E_\infty^{n,0}$ is nothing else than ${}_{d'}E_\infty^{n,0}$ and $[\omega_{\frac{\pi}{2}}^{0,n}]$ is a generator of ${}_{d'}E_\infty^{n,0} \simeq \mathbb{R}$ (see Proposition 3.2). This shows that the germ of $[\omega^{0,n}_{\mathcal{E}\times S^1/S^1}] \in {}_{d''}\mathcal{E}_\infty^{n,0}(S^1)$ is non-zero, hence $[\omega^{0,n}_{\mathcal{E}\times S^1/S^1}] \in {}_{d''}\mathcal{E}_\infty^{n,0}(S^1)$ is non-zero.

By Lemma 4.12, the sheaf ${}_{d''}\mathcal{E}_\infty^{n,0}$ is of the form $j_!\mathbb{R}_U$ for some open subset $j : U \hookrightarrow S^1$. Such a sheaf admits a non-trivial global section over S^1 if and only if $U = S^1$. Hence the morphism ${}_{d''}\mathcal{E}_\infty^{n,0} \hookrightarrow \mathbb{R}_{S^1}$ is an isomorphism.

This finishes the proof of Proposition 4.13, and of Theorem 1.5.

□

Remark 4.14. It is worth noticing that if one argues similarly replacing $\omega^{0,n}_{\mathcal{E}_V}$ by $\omega^{n,0}_{\mathcal{E}_V}$ (with its obvious meaning) one obtains a global section of $\Omega^{n,0}_{\mathcal{E}\times S^1/S^1}$ which is d-closed. However it does not define a section of ${}_{d''}\mathcal{E}_1^{0,n}(S^1) = R^n p_{2*}\mathcal{L}''^0(S^1)$.

Appendix A. Thom class and Euler class: proof of Proposition 1.7

Let X be a connected oriented n-manifold. Let E be an oriented real vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. In this section we recall the definition of the Euler class $e(X) \in \tilde{H}^r(X, \mathbb{Z})$ from the Thom class of E (where $\tilde{H}^\bullet$ denotes the reduced cohomology).

Recall that the Thom space $\mathrm{Th}(E)$ of the bundle E is the space uniquely defined in the homotopy category by one of the following equivalent constructions:

(a) apply one-point compactification to each fiber of E to obtain a new bundle $\mathrm{Sph}(E)$ over X whose fibers are spheres S^r with basepoints, namely the points at ∞. These basepoints specify a section $s_\infty : X \longrightarrow \mathrm{Sph}(E)$. Define the Thom space as the quotient $\mathrm{Th}(E) = \mathrm{Sph}(E)/s_\infty(X)$.

(b) introduce a auxiliary Riemannian metric on E and denote by $D(E)$ and $S(E)$ the associated unit disk bundle and unit sphere bundle in E. Define $\mathrm{Th}(E)$ as the quotient $D(E)/S(E)$.

(c) define $\mathrm{Th}(E) = \mathbf{P}(E \oplus 1)/\mathbf{P}(E)$.

Notice that the constructions (a) and (c) are clearly functorial for morphisms of vector bundles. For a point $x \in X$ let $\iota_x : E_x \longrightarrow E$ denotes the inclusion of the fiber E_x in E, seen as a morphism of vector bundles over $\{x\}$ and X respectively. It induces a map still denoted $\iota_x : \mathrm{Th}(E_x) \simeq S^r \longrightarrow \mathrm{Th}(E)$ between Thom spaces.

Theorem A.1. *(Thom isomorphism) Let X be a connected oriented n-manifold. Let E be an oriented real vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$.*

There exists a unique class $u \in \tilde{H}^r(\mathrm{Th}(E), \mathbb{Z})$, called the Thom class of E, such that for any $x \in X$ the pull-back $\iota_x^* u$ is the preferred generator of $\tilde{H}^r(\mathrm{Th}(E_x), \mathbb{Z}) \simeq \tilde{H}^r(S^r, \mathbb{Z})$ given by the orientation.

Moreover the map $\Phi = \pi^*(\cdot) \cup u : H^\bullet(X, \mathbb{Z}) \longrightarrow \tilde{H}^{\bullet+r}(\mathrm{Th}(E), \mathbb{Z})$ is an isomorphism of $\mathbb{Z}$-modules.

Let $\mathcal{E}^0$ the complement of the zero-section in $\mathcal{E}$. Then $\mathrm{Th}(E)$ is the cofiber of the inclusion $\mathcal{E}^0 \hookrightarrow \mathcal{E}$, in particular the reduced cohomology $\tilde{H}^\bullet(\mathrm{Th}(E), \mathbb{Z})$ is nothing else than the relative cohomology $H^\bullet(\mathcal{E}, \mathcal{E}^0; \mathbb{Z})$. Hence we obtain the following version of the Thom class of E and of the Thom isomorphism in terms of relative cohomology:

Theorem A.2. *Let X be a connected oriented n-manifold. Let E be an oriented real vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$.*

There exists a unique class $u \in H^r(\mathcal{E}, \mathcal{E}^0; \mathbb{Z})$, called the Thom class of E, such that for any $x \in X$ the pull-back $\iota_x^* u$ is the preferred generator of $H^r(E_x, E_x \setminus \{0\}; \mathbb{Z}) \simeq \tilde{H}^r(S^r, \mathbb{Z})$ given by the orientation.

Moreover the map $\Phi = \pi^*(\cdot) \cup u : H^\bullet(X, \mathbb{Z}) \longrightarrow H^{\bullet+r}(\mathcal{E}, \mathcal{E}^0; \mathbb{Z})$ is an isomorphism of $\mathbb{Z}$-modules.

Definition A.3. *Let X be a connected oriented n-manifold. Let E be an oriented real vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$.*

The Euler class of E is the class

$$e(E) = \Phi^{-1} u^2 \in H^r(X, \mathbb{Z}) \quad .$$

The following lemma follows easily from the definition (see [BottTu82, prop.12.4] for a proof with real coefficients):

Lemma A.4. *Let X be a connected oriented n-manifold. Let E be an oriented real vector bundle on X of rank $r > 0$, with total space $\mathcal{E}$ and projection $\pi : \mathcal{E} \longrightarrow X$. The Euler class $e(E) \in H^r(X, \mathbb{Z})$ is the image of the Thom class $u \in H^r(\mathcal{E}, \mathcal{E}^0; \mathbb{Z})$ under the composite*

$$H^r(\mathcal{E}, \mathcal{E}^0; \mathbb{Z}) \xrightarrow{\cdot|_{\mathcal{E}}} H^r(\mathcal{E}, \mathbb{Z}) \overset{i^*}{\simeq} H^r(X, \mathbb{Z}) \quad ,$$

where $i : X \longrightarrow \mathcal{E}$ denotes the zero-section.

We now give a differential geometric interpretation of the Thom class. Denote by $(\Omega^\bullet_{cv}(\mathcal{E}), d)$ the De Rham complex of differential forms on $\mathcal{E}$ with *vertical compact support*, see [BottTu82]. Notice that when X is closed this complex coincides with the complex of differential forms on $\mathcal{E}$ with compact support $(\Omega_c^\bullet(\mathcal{E}), d)$. By integrating along the fibers one defines a push-forward map of complexes

$$\pi_* : (\Omega_{cv}^\bullet(\mathcal{E}), d) \longrightarrow (\Omega^{\bullet - n}(X), d) \ .$$

One easily shows:

Proposition A.5. *There exists a canonical isomorphism $\varphi : H^*(\Omega_{cv}^\bullet(\mathcal{E}), d) \longrightarrow H^*(\mathcal{E}, \mathcal{E}^0; \mathbb{R})$ making the following diagram of isomorphisms commutative:*

$$\begin{tikzcd} H^*(\Omega_{cv}^\bullet(\mathcal{E}), d) \arrow[r, "\varphi"] \arrow[d, "\pi_*"'] & H^*(\mathcal{E}, \mathcal{E}^0; \mathbb{R}) \arrow[d, "\Phi^{-1}"] \\ H^{*-n}(\Omega^\bullet(X), \mathbb{R}) \arrow[r, "\sim"] & H^{*-n}(X, \mathbb{R}) \end{tikzcd} \ .$$

Definition A.6. *A form $\omega \in \Omega_{cv}^r(\mathcal{E})$ is called a Thom form if its class $[\omega] \in H_{cv}^r(\mathcal{E}, \mathbb{R}) \simeq H^0(X, \mathbb{R}) \simeq \mathbb{R}$ is a generator.*

Proposition 1.7 follows immediately from Lemma A.4 and from Proposition A.5.

Appendix B. Quaternionic and para-quaternionic geometry

In this appendix we illustrate the analogy between hypercomplex structures (or even the more general quaternionic structures) among complex structures and para-hypercomplex structures (and more generally para-quaternionic structures) among para-complex ones.

Let us start with the classical quaternionic geometry, for which we refer to the original paper of Salamon [Sal86] and more recently the work of Verbitsky (see for example [Ver99]). Let n be a positive integer. Let $\mathbb{H}$ denotes the Hamilton's quaternion algebra over $\mathbb{R}$ (which identifies with the Clifford algebra $\mathrm{Cl}_{0,2}(\mathbb{R})$). The natural left-action of the group $\mathrm{GL}(n, \mathbb{H})$ on the left quaternionic vector space $\mathbb{H}^n$, which identifies with $\mathbb{R}^{4n}$ as a real vector space, defines an embedding $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(4n, \mathbb{R})$, with centralizer $\mathrm{GL}(1, \mathbb{H})$ (acting by right scalar multiplication on $\mathbb{H}^n$). The intersection of $\mathrm{GL}(n, \mathbb{H})$ with $\mathrm{GL}(1, \mathbb{H})$ is $\mathbb{R}^*$, thus defining a maximal subgroup

$$\mathrm{GL}(n, \mathbb{H}) \times_{\mathbb{R}^*} \mathrm{GL}(1, \mathbb{H}) = \mathrm{GL}(n, \mathbb{H}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Sp}(1) \subset \mathrm{GL}(4n, \mathbb{R}) \ ,$$

where $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ is the 3-sphere of quaternionic units.

Definition B.1. *Let M be a differential manifold of dimension $4n$.*

An almost quaternionic structure on M is a $\mathrm{GL}(n, \mathbb{H}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Sp}(1)$-structure on M. A quaternionic structure on M is an almost quaternionic structure admitting a torsion-free connection.

An almost hypercomplex structure on M is a $\mathrm{GL}(n, \mathbb{H})$-structure on M. A hypercomplex structure on M is an almost hypercomplex structure admitting a torsion-free connection.

Equivalently, an almost hypercomplex structure on M is the data of two endomorphisms $I_1, I_2 \in \mathrm{End}\,(TM)$ satisfying $I_1^2 = I_2^2 = -1$ (i.e. I_1 and I_2 are almost complex structures on

M) and $I_1 I_2 = -I_2 I_1$, hence generating an action of the algebra $\mathbb{H}$ on TM. This action integrates to an action of the Lie group $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ on TM, generated by the one-parameter subgroups with tangent vectors I_1, I_2 and $I_1 I_2$. This almost hypercomplex structure is a hypercomplex structure if and only if the almost complex structures I_1 and I_2 are integrable, or equivalently if there exists a (unique) torsion-free connection ∇ on TM such that $\nabla I_1 = \nabla I_2 = 0$ (the Obata connection).

Para-quaternionic geometry is defined similarly, replacing the quaternion algebra by the para-quaternion algebra $\mathrm{Cl}_{1,1}(\mathbb{R}) \simeq \mathfrak{gl}_2(\mathbb{R})$, i.e. the 4-dimensional $\mathbb{R}$-algebra generated by two elements j and i satisfying $j^2 = 1$, $i^2 = -1$ and $ji = -ij$. Consider the inclusion of groups

$$\mathrm{GL}(n,\mathbb{R}) = \mathrm{GL}(n,\mathbb{C}) \cap (\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})) \subset \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R}) \quad , \tag{19}$$

given by $A \in \mathrm{GL}(n,\mathbb{R}) \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right) \in \mathrm{GL}(2n,\mathbb{R})$. The centralizer of $\mathrm{GL}(n,\mathbb{R})$ in $\mathrm{GL}(2n,\mathbb{R})$ is $\mathrm{GL}(2,\mathbb{R})$, with intersection $\mathbb{R}^*$, thus defining an embedding

$$\mathrm{GL}(n,\mathbb{R}) \times_{\mathbb{R}^*} \mathrm{GL}(2,\mathbb{R}) = \mathrm{GL}(n,\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R}) \quad .$$

Definition B.2. *Let M be a differential manifold of dimension $2n$.*

An almost para-quaternionic structure on M is a $\mathrm{GL}(n,\mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{SL}(2,\mathbb{R})$-structure on M. A quaternionic structure on M is an almost quaternionic structure admitting a torsion-free connection.

An almost para-hypercomplex structure on M is a $\mathrm{GL}(n,\mathbb{R})$-structure on M. A para-hypercomplex structure on M is an almost para-hypercomplex structure admitting a torsion-free connection.

Remark B.3. Para-hypercomplex structures are also called "complex-product" structure. We refer to [IvZam05], [And05] and references mentioned there for a survey on such structures.

Equivalently [And05] an almost para-hypercomplex structure on M is the data of an almost-complex structure $I \in \mathrm{End}\,(TM)$ and an almost product structure $J \in \mathrm{End}\,(TM)$ satisfying $IJ = -JI$ (it follows immediately that J and IJ are necessarily para-complex structures). The endomorphisms I and J generate an action of the algebra $\mathrm{Cl}_{1,1}(\mathbb{R})$ of para-quaternions on TM. This action integrates to an action of the Lie group $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$ on TM, generated by the one-parameter subgroups with tangent vectors I, J and IJ. This is a para-hypercomplex structure if and only if the almost complex structure I and the almost product structure J are integrable, or equivalently if there exists a (unique) torsion-free connection ∇ on TM such that $\nabla I = \nabla J = 0$.

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