TAME TOPOLOGY OF ARITHMETIC QUOTIENTS AND ALGEBRAICITY OF HODGE LOCI

B. BAKKER, B. KLINGLER, AND J. TSIMERMAN

Abstract. In this paper we prove the following results:

1) We show that any arithmetic quotient of a homogeneous space admits a natural real semi-algebraic structure for which its Hecke correspondences are semi-algebraic. A particularly important example is given by Hodge varieties, which parametrize pure polarized integral Hodge structures.

2) We prove that the period map associated to any pure polarized variation of integral Hodge structures $\mathbb{V}$ on a smooth complex quasi-projective variety S is definable with respect to an o-minimal structure on the relevant Hodge variety induced by the above semi-algebraic structure.

3) As a corollary of 2) and of Peterzil-Starchenko's o-minimal Chow theorem we recover that the Hodge locus of $(S, \mathbb{V})$ is a countable union of algebraic subvarieties of S, a result originally due to Cattani-Deligne-Kaplan. Our approach simplifies the proof of Cattani-Deligne-Kaplan, as it does not use the full power of the difficult multivariable SL_2-orbit theorem of Cattani-Kaplan-Schmid.

1. Introduction.

1.1. **Arithmetic quotients.** Arithmetic quotients are real analytic manifolds of the form $S_{\Gamma,G,M} := \Gamma\backslash G/M$, for $\mathbf{G}$ a connected semi-simple linear algebraic $\mathbb{Q}$-group, $G := \mathbf{G}(\mathbb{R})^+$ the real Lie group connected component of the identity of $\mathbf{G}(\mathbb{R})$, $M \subset G$ a connected compact subgroup and $\Gamma \subset \mathbf{G}(\mathbb{Q})^+ := \mathbf{G}(\mathbb{Q}) \cap G$ a neat arithmetic group. By a morphism of arithmetic quotients we mean a real analytic map $(\phi, g) : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$ of the form $\Gamma' h' M' \mapsto \Gamma\phi(h')gM$ for some morphism $\phi : \mathbf{G}' \longrightarrow \mathbf{G}$ of semi-simple linear algebraic $\mathbb{Q}$-groups and some element $g \in G$ such that $\phi(\Gamma') \subset \Gamma$ and $\phi(M') \subset gMg^{-1}$.

Such quotients are ubiquitous in various parts of mathematics. For $M = \{1\}$ the arithmetic quotients $S_{\Gamma,G,\{1\}} = \Gamma\backslash G$ are the main players in homogeneous dynamics, for example Ratner's theory [Rat91-0], [Rat91-1]. For $K \subset G$ a maximal compact subgroup the arithmetic quotients $S_{\Gamma,G,K}$ are the arithmetic riemannian locally symmetric spaces, for instance the arithmetic hyperbolic manifolds $\Gamma\backslash SO(n,1)^+/SO(n)$. They are intensively studied by differential geometers and group theorists. When G is moreover of Hermitian type then $S_{\Gamma,G,K}$ is a so-called arithmetic variety (also called a connected Shimura variety if Γ is a congruence subgroup): this is a smooth complex quasi-projective variety, naturally defined over $\overline{\mathbb{Q}}$ in the Shimura case. The simplest examples of connected Shimura varieties are the modular curves $\Gamma_0(N)\backslash SL(2,\mathbb{R})/SO(2)$. Connected

B.K. was partially supported by an Einstein Foundation's professorship.

B.B. was partially supported by NSF grant DMS-1702149.

Shimura varieties play a paramount role in arithmetic geometry and the Langlands program. Much more generally, the connected Hodge varieties are arithmetic quotients which play a crucial role in Hodge theory as target of period maps.

1.2. **Moderate geometry of arithmetic quotients.** For $S_{\Gamma,G,K}$ a connected Shimura variety, the study of the topological tameness properties of the uniformization map $\pi : G/K \longrightarrow S_{\Gamma,G,K}$ recently provided a crucial tool for solving longstanding algebraic and arithmetic questions (see [P11], [PT14], [KUY16], [Ts18], [KUY17], [MPT17]). Here tameness is understood in the sense proposed by Grothendieck [Gro, §5 and 6] and developed by model theory under the name "o-minimal structure" (see below). The first goal of this paper is to develop a similar study for a general arithmetic quotient $S_{\Gamma,G,M}$.

Among real analytic manifolds the ones with the tamest geometry are certainly the complex algebraic ones. However most arithmetic quotients have no complex algebraic structures, as they do not even admit a complex analytic one (for instance for obvious dimensional reasons). What about a real algebraic structure? In [Rag68] Raghunathan proved that any riemannian locally symmetric space is compactifiable, i.e. diffeomorphic to the interior of a compact smooth manifold with boundary; Akbulut and King [AK81] proved that any compactifiable manifold is diffeomorphic to a non-singular real algebraic set (generalizing a result of Tognoli [Tog73] in the compact case, conjectured by Nash [Na52]). Hence any riemannian locally symmetric space is diffeomorphic to a non-singular semi-algebraic set. On the other hand such abstract real algebraic models are useless if they don't satisfy some basic functorial properties. A crucial feature of the geometry of arithmetic quotients is the existence of infinitely many real-analytic finite self-correspondences: any element $g \in \mathbf{G}(\mathbb{Q})$ commensurates Γ (meaning that the intersection $g\Gamma g^{-1} \cap \Gamma$ is of finite index in both Γ and $g\Gamma g^{-1}$) hence defines a Hecke correspondence

(1.1)
$$c_g = (c_1, c_2) : \qquad \begin{tikzcd} S_{\Gamma,G,M} & S_{g^{-1}\Gamma g\cap\Gamma,G,M} \arrow[l, "c_1"'] \arrow[r, "g\cdot"] \arrow[rr, bend right, "c_2"'] & S_{\Gamma\cap g\Gamma g^{-1},G,M} \arrow[r] & S_{\Gamma,G,M} \end{tikzcd} .$$

Here the left and right morphisms of arithmetic quotients are the natural finite étale projections; the map in the middle is left-multiplication by g, i.e. the morphism of arithmetic quotients $(\mathrm{Int}(g), g)$, where $\mathrm{Int}(g) : \mathbf{G} \longrightarrow \mathbf{G}$ denotes the conjugation by g. We would like these Hecke correspondences to be real algebraic. Such functorial real algebraic models do exist in certain cases: see [Jaf75], [Jaf78], [Le79]; but we don't know of any general procedure for producing such a nice real algebraic structure on all arithmetic quotients. Hence our need to work with a more general notion of tame geometry.

Recall that a structure $\mathcal{S}$ on $\mathbb{R}$ expanding the real field is a collection $(S_n)_{n\in\mathbb{N}^*}$, where S_n is a set of subsets of $\mathbb{R}^n$ (called the $\mathcal{S}$-*definable sets*), such that: all algebraic subsets of $\mathbb{R}^n$ are in S_n; S_n is a boolean subalgebra of the power set of $\mathbb{R}^n$; if $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$; if $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is a linear projection and $A \in S_{n+1}$ then $p(A) \in S_n$. A function $f : A \longrightarrow B$ between $\mathcal{S}$-definable sets is said to be $\mathcal{S}$-definable if its graph is $\mathcal{S}$-definable. Such a structure $\mathcal{S}$ is said in addition to be *o-minimal* if the definable subsets of $\mathbb{R}$ are precisely the finite unions of points and intervals (i.e. the semi-algebraic subsets of $\mathbb{R}$). This o-minimal axiom guarantees the possibility of doing geometry using definable sets as basic blocks: it excludes infinite countable sets, like $\mathbb{Z} \subset \mathbb{R}$, as well as Cantor sets or space-filling curves, to be definable. Intuitively, subsets of $\mathbb{R}^n$ definable in an o-minimal structure are the ones having at the same time a reasonable local topology and a tame topology at infinity. Given an o-minimal structure $\mathcal{S}$, there is an obvious notion of $\mathcal{S}$-definable manifold: this is a manifold S admitting a *finite* atlas of charts $\varphi_i : U_i \longrightarrow \mathbb{R}^n$, $i \in I$, such that the intersections $\varphi_i(U_i \cap U_j)$, $i, j \in I$, are $\mathcal{S}$-definable subset of $\mathbb{R}^n$ and the change of coordinates $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ are $\mathcal{S}$-definable maps.

The simplest o-minimal structure is $\mathbb{R}_{\mathrm{alg}}$, the definable sets being the semi-algebraic subsets. There exist more general o-minimal structures. A result of Van den Dries based on Gabrielov's results [Ga68] shows that the structure

$$\mathbb{R}_{\mathrm{an}} := \langle \mathbb{R}, +, \times, <, \{f\} \text{ for } f \text{ restricted analytic function} \rangle$$

generated from $\mathbb{R}_{\mathrm{alg}}$ by adding the restricted analytic functions is o-minimal. Here a real function on $\mathbb{R}^n$ is restricted analytic if it is zero outside $[0,1]^n$ and coincides on $[0,1]^n$ with a real analytic function g defined on a neighbourhood of $[0,1]^n$. The $\mathbb{R}_{\mathrm{an}}$-definable sets of $\mathbb{R}^n$ are the globally subanalytic subsets of $\mathbb{R}^n$ (i.e. the ones which are subanalytic in the compactification $\mathbf{P}^n\mathbb{R}$ of $\mathbb{R}^n$). A deep result of Wilkie [Wil96] states that the structure $\mathbb{R}_{\exp} := \langle \mathbb{R}, +, \times, <, \exp : \mathbb{R} \longrightarrow \mathbb{R} \rangle$ generated from $\mathbb{R}_{\mathrm{alg}}$ by making the real exponential function definable is also o-minimal. Finally the structure $\mathbb{R}_{\mathrm{an,exp}} := \langle \mathbb{R}, +, \times, <, \exp, \{f\} \text{ for } f \text{ restricted analytic function} \rangle$ generated by $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\exp}$ is still o-minimal [VdM94].

The first result of this paper is the following:

Theorem 1.1. *Let* $\mathbf{G}$ *be a connected linear semi-simple algebraic* $\mathbb{Q}$*-group,* $\Gamma \subset \mathbf{G}(\mathbb{Q})^+$ *a torsion-free arithmetic lattice of* $G := \mathbf{G}(\mathbb{R})^+$*, and* $M \subset G$ *a connected compact subgroup.*

(1) *The arithmetic quotient* $S_{\Gamma,G,M} := \Gamma \backslash G/M$ *admits a natural structure of* $\mathbb{R}_{\mathrm{alg}}$*-definable manifold, characterized by the following property. Let* G/M *be endowed with its natural semi-algebraic structure (see Lemma 2.1) and* $\mathfrak{S} \subset G/M$ *be a semi-algebraic Siegel set (see Section 2.2 for the definition of Siegel sets). Then*

$$\pi_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow S_{\Gamma,G,M}$$

is $\mathbb{R}_{\mathrm{alg}}$*-definable.*

In particular, there exists a semi-algebraic fundamental set $\mathcal{F} \subset G/M$ *for the action of* Γ *on* G/M *such that*

$$\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S_{\Gamma,G,M}$$

is $\mathbb{R}_{\mathrm{alg}}$*-definable.*

The structure of $\mathbb{R}_{\mathrm{an}}$*-definable manifold on* $S_{\Gamma,G,M}$ *extending its* $\mathbb{R}_{\mathrm{alg}}$*-structure is the one induced by the real-analytic structure with corners of its Borel-Serre compactification* $\overline{S_{\Gamma,G,M}}^{BS}$.

(2) *Any morphism* $f : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$ *of arithmetic quotients is* $\mathbb{R}_{\mathrm{alg}}$*-definable. In particular the Hecke correspondences* c_g*,* $g \in \mathbf{G}(\mathbb{Q})^+$*, on* $S_{\Gamma,G,M}$ *are* $\mathbb{R}_{\mathrm{alg}}$*-definable.*

Theorem 1.1(1) can be thought as a strengthening and a generalization of the main result of [PS13] (for $S_{\Gamma,G,K} = \mathcal{A}_g$ the moduli space of principally polarized Abelian varieties of dimension g) and of [KUY16, Theor.1.9] (for a general arithmetic variety), which proved that for any arithmetic variety $S_{\Gamma,G,K}$, endowed with the $\mathbb{R}_{\mathrm{an}}$-definable manifold structure deduced from its complex algebraic Baily-Borel compactification, there exists a semi-algebraic fundamental set $\mathcal{F} \subset G/K$ for the action of Γ on G/K such that the map $\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S_{\Gamma,G,K}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. While these results claim only the definability of $\pi_{|\mathcal{F}}$ in $\mathbb{R}_{\mathrm{an,exp}}$ our Theorem 1.1(1) claim it in $\mathbb{R}_{\mathrm{alg}}$. This discrepancy comes from the fact that for $S_{\Gamma,G,K}$ an arithmetic variety, the $\mathbb{R}_{\mathrm{an}}$-definable structure on $S_{\Gamma,G,K}$ extending the natural $\mathbb{R}_{\mathrm{alg}}$-definable structure of Theorem 1.1(1) on $S_{\Gamma,G,K}$ is the one coming from the Borel-Serre compactification $\overline{S_{\Gamma,G,K}}^{BS}$ of $S_{\Gamma,G,K}$: it does not coincide with the one coming from the Baily-Borel compactification $\overline{S_{\Gamma,G,K}}^{BB}$ but the natural map $\overline{S_{\Gamma,G,K}}^{BS} \longrightarrow \overline{S_{\Gamma,G,K}}^{BB}$ is in fact $\mathbb{R}_{\mathrm{an,exp}}$-definable. As we won't need this result in this paper we just provide the simplest illustration:

Example 1.2. Let $\mathfrak{H}$ be the Poincaré upper half-plane and $Y_0(1)$ the modular curve $\mathbf{SL}(2,\mathbb{Z})\backslash\mathfrak{H}$. A semi-algebraic fundamental domain for the action of $\mathbf{SL}(2,\mathbb{Z})$ on $\mathfrak{H}$ is given by

$$\mathcal{F} := \{x + iy \in \mathfrak{H} \mid x^2 + y^2 \geq 1, -1/2 \leq x < 1/2\} \enspace .$$

The Borel-Serre compactification $\overline{Y_0(1)}^{BS}$ is obtained by adding a circle at infinity to $Y_0(1)$, corresponding to the compactification $\overline{\mathcal{F}}$ of $\mathcal{F}$ obtained by glueing the segment $\{y = \infty, -1/2 < x < 1/2\}$ to $\mathcal{F}$. The Baily-Borel compactification $X_0(1) := \overline{Y_0(1)}^{BB}$ is the one-point compactification of $Y_0(1)$ and is naturally identified with the complex projective line $\mathbf{P}^1\mathbb{C}$. The natural map $\overline{Y_0(1)}^{BS} \longrightarrow \overline{Y_0(1)}^{BB}$ contracting the circle at infinity to a point sends a point $(x, t = 1/y) \in [-1/2, 1/2] \times [0, 1)$ close to the circle at infinity $t = 0$ to the point $[1, z = \exp(2\pi i x)\exp(-2\pi/t)] \in \mathbf{P}^1\mathbb{C}$. This map is not globally subanalytic but it is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

It is worth noticing that the proof of the more general Theorem 1.1 is easier than the one in [PS13] (which uses explicit theta functions) or the one in [KUY16] (which uses the delicate toroidal compactifications of [AMRT75]): it relies exclusively on classical properties of Siegel sets (see Section 2.2 for the precise definition of Siegel sets), while the proofs of [PS13] and [KUY16], which apply only to arithmetic varieties, moreover insisted on using only complex analytic maps, thus obscuring to some extent the o-minimality issues.

1.3. **Moderate geometry of period maps.** Arithmetic quotients of interest to the algebraic geometers arise in Hodge theory as *connected Hodge varieties*, which are complex analytic quotients of period domains (or more generally Mumford-Tate domains). Let S be a smooth complex quasi-projective variety and let $\mathbb{V} \longrightarrow S$ be a polarized variation of $\mathbb{Z}$-Hodge structures (PVHS) of weight k on S. A typical example of such a PVHS is $\mathbb{V} = R^k f_* \mathbb{Z}$ for $f : \mathcal{X} \longrightarrow S$ a smooth proper morphism; in which case we say that $\mathbb{V}$ is geometric. We refer to [K17] and the references therein for the relevant background in Hodge theory, which we use thereafter. Let $\mathbf{MT}(\mathbb{V})$ be the generic Mumford-Tate group associated to $\mathbb{V}$ (this is a connected reductive $\mathbb{Q}$-group) and $\mathbf{G}$ its associated adjoint semi-simple $\mathbb{Q}$-group. The group $G := \mathbf{G}(\mathbb{R})^+$ acts by holomorphic transformations and transitively on the Mumford-Tate domain $D = G/M$ associated to $\mathbf{MT}(\mathbb{V})$, with compact isotropy denoted by M. If Γ is a torsion free arithmetic lattice of G the arithmetic quotient $S_{\Gamma,G,M}$ is a complex analytic manifold called a connected Hodge variety (which carries an algebraic structure in only very few cases). Replacing if necessary S by a finite étale cover, the PVHS $\mathbb{V}$ on S is completely described by its holomorphic period map $\Phi_S : S \longrightarrow \mathrm{Hod}^0(S, \mathbb{V}) := S_{\Gamma,G,M}$ for a suitable torsion-free arithmetic subgroup $\Gamma \subset G$.

We prove that the period map Φ_S has a moderate geometry. Let us endow S with the $\mathbb{R}_{\mathrm{an,exp}}$-definable manifold structure extending the $\mathbb{R}_{\mathrm{alg}}$-definable manifold structure on S coming from its complex algebraic structure; and the connected Hodge manifold $\mathrm{Hod}^0(S, \mathbb{V}) = S_{\Gamma,G,M}$ with the $\mathbb{R}_{\mathrm{an,exp}}$-definable manifold structure extending the $\mathbb{R}_{\mathrm{alg}}$-definable manifold structure defined in Theorem 1.1.

Theorem 1.3. *Let $\mathbb{V} \to S$ be a polarized variation of pure Hodge structures of weight k over a smooth complex quasi-projective variety S. Let $\Phi_S : S \longrightarrow \mathrm{Hod}^0(S, \mathbb{V}) = S_{\Gamma,G,M}$ be the holomorphic period map associated to $\mathbb{V}$. Then Φ_S is $\mathbb{R}_{\mathrm{an,exp}}$-definable.*

Remarks 1.4.
(1) Notice that Theorem 1.3 is easy in the rare case when the connected Hodge variety $S_{\Gamma,G,M}$ is compact. In that case, consider $\overline{S}$ a smooth projective compactification of S with normal crossing divisor at infinity. It follows from Borel's monodromy theorem [Sc73, Lemma (4.5)] and the fact that the cocompact lattice Γ does not contain any unipotent element [Rag72, Cor. 11.13] that the monodromy at infinity of $\mathbb{V}$ is finite. Thus, replacing if necessary S by a finite étale cover, the PVHS $\mathbb{V}$ extends to $\overline{S}$. Equivalently the period map $\Phi_S : S \longrightarrow \mathrm{Hod}^0(S, \mathbb{V}) := S_{\Gamma,G,M}$ extends to a period map $\Phi_{\overline{S}} : \overline{S} \longrightarrow S_{\Gamma,G,M}$. In particular the period map Φ is definable in $\mathbb{R}_{\mathrm{an}}$ in that case.
(2) When the connected Hodge variety $S_{\Gamma,G,M}$ is an arithmetic variety, Theorem 1.3 implies (see Section 4.6) that $\Phi_S : S \longrightarrow S_{\Gamma,G,M}$ is an algebraic map, thus recovering a classical result due to Borel [Bor72, Theor. 3.10]. Hence Theorem 1.3 can be thought as an extension of Borel's result to the general case where the connected Hodge variety $S_{\Gamma,G,M}$ has no algebraic structure. On the other hand, notice that Borel [Bor72, Theor.A] proves in the arithmetic variety case the stronger result that Φ_S extends to a holomorphic map $\Phi_{\overline{S}} : \overline{S} \longrightarrow \overline{S_{\Gamma,G,M}}^{BB}$, which does not directly follow from Theorem 1.3.

The main ingredient in the proof of Theorem 1.3 is the following finiteness result on the geometry of Siegel sets:

Theorem 1.5. *Let $\Phi : (\Delta^*)^n \longrightarrow S_{\Gamma,G,M}$ be a local period map with unipotent monodromy on a product of punctured disks. Let $\tilde{\Phi} : \mathfrak{H}^n \longrightarrow G/M$ be its lifting to the universal cover $\mathfrak{H}^n$ of $(\Delta^*)^n$. Given constants $R > 0$ and $\eta > 0$ let us define*

$$\mathfrak{H}^n_{R,\eta} := \{\mathbf{z} \in \mathfrak{H}^n \mid |\mathrm{Re}\,\mathbf{z}| \leq R \text{ and } \mathrm{Im}\,\mathbf{z} \geq \eta\}$$

where $|\mathrm{Re}\,\mathbf{z}| := \sup_{1\leq j\leq n} |\mathrm{Re}\, z_i|$ and $\mathrm{Im}\,\mathbf{z} := \inf_{1\leq j\leq n} \mathrm{Im}\, z_i$.

There exists finitely many Siegel sets $\mathfrak{S}_i \subset G/M$, $i \in I$, such that

$$\tilde{\Phi}(\mathfrak{H}^n_{R,\eta}) \subset \bigcup_{i\in I} \mathfrak{S}_i \quad .$$

In the one-variable case ($n = 1$) Theorem 1.5 is due to Schmid (see [Sc73, Cor. 5.29]), with $|I| = 1$. In the multivariable case, Green, Griffiths, Laza and Robles [GGLR17, Claims A.5.8 and A.5.9] show that the result with $|I| = 1$ does not hold.

The main point of the proof of Theorem 1.5 is to show there is a flat frame with respect to which the Hodge form remains Minkowski reduced, up to covering $\mathfrak{H}^n_{R,\eta}$ by finitely many sets. We note here that the proof does not use the higher dimensional SL_2^n-orbit theorem of [CKS86]. Rather, we deduce the higher-dimensional statement by restricting to curves and using the full power of Schmid's one-dimensional result, together with the work of [CKS86] and [Ka85] on the asymptotics of Hodge norms.

1.4. **Algebraicity of Hodge loci.** Recall that the Hodge locus $\mathrm{HL}(S, \mathbb{V}) \subset S$ associated to the PVHS $\mathbb{V}$ is the set of points s in S for which exceptional Hodge tensors for $\mathbb{V}_s$ do occur. The locus $\mathrm{HL}(S, \mathbb{V})$ is easily seen to be a countable union of irreducible complex analytic subvarieties of S, called special subvarieties of S associated to $\mathbb{V}$. If $\mathbb{V} = R^k f_* \mathbb{Q}$ for $f : \mathcal{X} \longrightarrow S$ a smooth proper morphism, it follows from the Hodge conjecture that the exceptional Hodge tensors in $\mathbb{V}_s$ come from exceptional algebraic cycles in some product $\mathcal{X}_s^N$. A Baire category type argument then implies that every special subvariety of S ought to be algebraic. As an immediate corollary of Theorem 1.1, Theorem 1.3, and Peterzil-Starchenko's o-minimal Chow Theorem 4.13 we obtain an alternative proof of the following result originally proven by Cattani, Deligne and Kaplan [CDK95]:

Theorem 1.6. *The special subvarieties of S associated to $\mathbb{V}$ are algebraic, i.e. the Hodge locus* $\mathrm{HL}(S, \mathbb{V})$ *is a countable union of closed irreducible algebraic subvarieties of S.*

The proof of Theorem 1.6 in [CDK95] works as follows. Let $\overline{S}$ be a smooth compactification of S with a simple normal crossing divisor D at infinity. Locally in the analytic topology S identifies with $(\Delta^*)^r \times \Delta^l$ inside $\overline{S} = \Delta^{r+l}$ (where Δ denotes the unit disk). The $SL_2{}^n$-orbit theorems of [Sc73] and [CKS86] describe extremely precisely the asymptotic behaviour of the period map Φ_S on $(\Delta^*)^r \times \Delta^l$. Using this description, Cattani, Deligne and Kaplan manage to write sufficiently explicitly the equation of the locus $S(v) \subset (\Delta^*)^r \times \Delta^l$ of the points at which some determination of a given multivalued flat section v of $\mathbb{V}$ is a Hodge class to prove that its closure $\overline{S(v)}$ in Δ^{r+l} is analytic in this polydisk. Our proof via Theorem 1.3 bypasses these delicate local computations, hence seems a worthwhile simplification.

In view of Theorem 1.3, its corollary Theorem 1.6, and the recent proof [BaT17] (using Theorem 1.6 and o-minimal techniques) of the Ax-Schanuel conjecture for pure Hodge varieties stated in [K17, Conj. 7.5], we hope to convey the idea that o-minimal geometry is an important tool in variational Hodge theory. We refer to [K17, section 1.5] for possible applications of these results to the structure of $\mathrm{HL}(S, \mathbb{V})$.

Theorem 1.6 has been extended to the case of (graded polarizable, admissible) variation of mixed Hodge structures in [BP09-1], [BP09-2], [BP13], [BPS10], [KNU11] using [CDK95] and the $SL_2{}^n$-orbit theorem of [KNU08] which extends [Sc73] and [CKS86] to the mixed case. Our o-minimal proof of Theorem 1.3 should certainly extend to this case, thus giving a simpler proof of the algebraicity of Hodge loci in full generality. We will come back to this problem in a sequel to this paper.

1.5. **Acknowledgments.** B.K would like to thank Patrick Brosnan, who asked him some time ago about a proof of Theorem 1.6 using o-minimal techniques; Mark Goresky, Lizhen Ji and Arvind Nair, who made him notice that the map from the Borel-Serre compactification to the Baily-Borel one is not subanalytic, and that Borel-Serre compactifications are not functorial; Colleen Robles, for the fruitful exchanges about Theorem 1.5; Wilfried Schmid, who confirmed to him that Theorem 1.5 should hold; and Kobi Peterzil and Sergei Starchenko, whose comments led to the upgrading of Theorem 1.1 from $\mathbb{R}_{\mathrm{an}}$ to $\mathbb{R}_{\mathrm{alg}}$. B.B. and J.T. would like to thank Wilfried Schmid and Yohan Brunebarbe for useful conversations.

2. Preliminaries

2.1. **Semi-algebraic structure on G/M.** The existence of a natural semi-algebraic structure on the model G/M of an arithmetic quotient, stated in the following lemma, is folkloric but we did not find a precise reference. For the convenience of the reader we provide two proofs: a "classical" algebraic one, and an o-minimal one announcing the proof of Theorem 1.1(1).

Lemma 2.1. *Let $\mathbf{G}$ be a connected semi-simple linear algebraic $\mathbb{Q}$-group, $G := \mathbf{G}(\mathbb{R})^+$ the real Lie group connected component of the identity of $\mathbf{G}(\mathbb{R})$, and $M \subset G$ a connected compact subgroup. Then G/M admits a natural structure of a semi-algebraic set, and the projection map $G \longrightarrow G/M$ is semi-algebraic. The action by left-multiplication of G on G/M is semi-algebraic.*

Remark 2.2. In general G/M does not admit a structure of real algebraic variety. This is already true for G: for instance the group $SO(p,q)$ is a real algebraic variety but its connected component $G := SO(p,q)^+$ is only semi-algebraic for $p \geq q > 0$. On the other hand any compact real Lie group M admits a natural structure $\mathbf{M}_\mathbb{R}$ of real algebraic group, see [OV90, Th. 5, p.133].

Proof. Let us start with the algebraic proof, inspired by [Sch75]. By a classical result of Chevalley [Che51], there exists a finite dimensional $\mathbf{G}_\mathbb{R}$-module W and a line $l \subset W$ such that the stabilizer in $\mathbf{G}_\mathbb{R}$ of l is precisely $\mathbf{M}_\mathbb{R}$ (the real algebraic subgroup of $\mathbf{G}_\mathbb{R}$ such that $\mathbf{M}_\mathbb{R}(\mathbb{R}) = M$). As the group M is compact connected, it not only stabilizes the line l but fixes any generator v of l. By another classical result, this time due to Hilbert (see [W46, Ch. VIII, §14]), the (graded) algebra $\mathbb{R}[W]^{\mathbf{M}_\mathbb{R}}$ of $\mathbf{M}_\mathbb{R}$-invariant polynomials on W is finitely generated, say by homogeneous elements $p_1, \cdots, p_d$. Consider the real algebraic map $p : \mathbf{G}(\mathbb{R}) \to W \to \mathbb{R}^d$ obtained by composing the orbit map of the vector $v \in W$ with $(p_1, \cdots, p_d) : W \longrightarrow \mathbb{R}^d$. It identifies $\mathbf{G}(\mathbb{R})/M$ with the image $p(\mathbf{G}(\mathbb{R}))$, hence G/M with a connected component of $p(\mathbf{G}(\mathbb{R}))$. As p is real algebraic the subset $p(\mathbf{G}(\mathbb{R}))$, hence its connected component G/M, is semi-algebraic. As p is real-algebraic, the projection $G \longrightarrow G/M$ is semi-algebraic.

Let us turn to the o-minimal proof. This is [VDD98, (2.18) p.167], which we summarize. The multiplication $G \times G \to G$ is the restriction of an algebraic map hence semi-algebraic. The equivalence relation

$$E = \{(g, gm) \mid g \in G \text{ and } m \in M\} \subset G \times G$$

is therefore semi-algebraic and definably proper in the sense of [VDD98, (2.13) p.166]. Hence the quotient $G \to G/M$ exists semi-algebraically [VDD98, (2.15) p.166]. □

2.2. **Siegel sets.** A crucial ingredient in this paper is the classical notion of Siegel sets for $\mathbf{G}$, which we recall now. We follow [BJ06a, §2] and refer to [Bor69, §12] for details.

Let $\mathbf{P}$ be a $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$. We denote by $\mathbf{N}_P$ its unipotent radical and by $\mathbf{L}_P$ the Levi quotient $\mathbf{N}_P \backslash \mathbf{P}$ of $\mathbf{P}$. Let N_P, P, and L_P be the Lie groups of real points of $\mathbf{N}_P$, $\mathbf{P}$ and $\mathbf{L}_P$ respectively. Let $\mathbf{S}_P$ be the split center of $\mathbf{L}_P$ and A_P the connected component of the identity in $\mathbf{S}_P(\mathbb{R})$. Let $\mathbf{M}_P := \cap_{\chi \in X^*(\mathbf{L}_P)} \ker \chi^2$ and $M_P = \mathbf{M}_P(\mathbb{R})$. Then L_P admits a decomposition $L_P = A_P M_P$.

Let X be the symmetric space of maximal compact subgroups of $G := \mathbf{G}(\mathbb{R})^+$. Choosing a point $x \in X$ corresponds to choosing a maximal compact subgroup K_x of G, or equivalently a Cartan involution θ_x of G. The choice of x defines a unique real Levi subgroup $\mathbf{L}_{P,x} \subset \mathbf{P}_{\mathbb{R}}$ lifting $(\mathbf{L}_P)_{\mathbb{R}}$ which is θ_x-invariant, see [BS73, 1.9]. We denote by $A_{P,x}$ and $M_{P,x}$ the subgroups of $\mathbf{L}_{P,x}(\mathbb{R})$ lifting A_P and M_P respectively. Although $\mathbf{L}_P$ is defined over $\mathbb{Q}$ this is not necessarily the case for $\mathbf{L}_{P,x}$. The parabolic group P decomposes as

$$P = N_P A_{P,x} M_{P,x} \quad , \tag{2.1}$$

inducing a horospherical decomposition of G:

$$G = N_P A_{P,x} M_{P,x} K_x \quad . \tag{2.2}$$

We recall (see [BJ06a, Lemma 2.3] that the right action of P on itself under the horospherical decomposition is given by

$$(n_0 a_0 m_0)(n, a, m) = (n_0 \cdot (a_0 m_0) n (a_0 m_0)^{-1}, a_0 a, m_0 m) \quad . \tag{2.3}$$

In the following the reference to the basepoint x in various subscripts is omitted. We let $\Phi(A_P, N_P)$ be the set of characters of A_P on the Lie algebra $\mathfrak{n}_P$ of N_P, "the roots of P with respect to A_P". The value of $\alpha \in \Phi(A_P, N_P)$ on $a \in A_P$ is denoted a^α. Notice that the map $a \mapsto a^\alpha$ from A_P to $\mathbb{R}^*$ is semi-algebraic.

There is a unique subset $\Delta(A_P, N_P)$ of $\Phi(A_P, N_P)$ consisting of $\dim A_P$ linearly independent roots, such that any element of $\Phi(A_P, N_P)$ is a linear combination with positive integral coefficients of elements of $\Delta(A_P, N_P)$ to be called the simple roots of P with respect to A_P.

Definition 2.3. *(Siegel set) For any $t > 0$, we define $A_{P,t} = \{a \in A_P \mid a^\alpha > t, \ \alpha \in \Delta(A_P, N_P)\}$. For any bounded sets $U \subset N_P$ and $W \subset M_P K$ the subset $\mathfrak{S} := U \times A_{P,t} \times W \subset G$ is called a Siegel set for G associated to $\mathbf{P}$ and x.*

When U and W are chosen to be relatively compact open semi-algebraic subsets of N_P and $M_P K$ respectively then the Siegel set $\mathfrak{S} = U \times A_{P,t} \times W$ is semi-algebraic in G. *We will only consider such semi-algebraic Siegel sets in the rest of the text.*

The following lemma follows immediately from Definition 2.3:

Lemma 2.4. *If $\mathfrak{S}$ is a Siegel set for G associated to $\mathbf{P}$ and x, then $g\mathfrak{S}g^{-1}$ is a Siegel set for G associated to $g\mathbf{P}g^{-1}$ and gx.*

Definition 2.5. *Let $M \subset G$ be a connected compact subgroup. A Siegel set $\mathfrak{S}$ for the homogeneous space G/M is a subset of G/M of the form $\pi_x(\mathfrak{S})$, where $\mathfrak{S} \subset G$ denotes a Siegel set for G associated to some parabolic $\mathbb{Q}$-subgroup $\mathbf{P} \subset \mathbf{G}$, K_x is a maximal compact subgroup containing a conjugate gMg^{-1} for some $g \in \mathbf{G}(\mathbb{Q})$ and $\pi_x : G \longrightarrow G/gMg^{-1} \simeq G/M$.*

Remark 2.6.
(1) It follows from the definition that for $\mathfrak{S} \subset G/M$ a Siegel set and $g \in \mathbf{G}(\mathbb{Q})$ the translate $g\mathfrak{S}$ is a Siegel set of G/M.
(2) As the projections π_x are semi-algebraic and we consider only semi-algebraic Siegel sets in G, Siegel sets in G/M are semi-algebraic.

Proposition 2.7. [BJ06a, Prop. 2.5] *Let $\Gamma \subset \mathbf{G}(\mathbb{Q})^+ := \mathbf{G}(\mathbb{Q}) \cap G$ be an arithmetic subgroup.*

(1) *There are only finitely many Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups. Let $\mathbf{P}_1, \dots, \mathbf{P}_k$ be a set of representatives of the Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups. There exists Siegel sets $\mathfrak{S}_i := U_i \times A_{P_i,t_i} \times W_i$ associated to $\mathbf{P}_i$ and x_i, $1 \leq i \leq k$, whose images in $\Gamma\backslash G/M$ cover the whole space.*
(2) *For any two parabolic subgroups $\mathbf{P}_i$ and Siegel sets $\mathfrak{S}_i$ associated to $\mathbf{P}_i$, $i = 1, 2$, the set*
$$\Gamma_{\mathfrak{S}_1,\mathfrak{S}_2} := \{\gamma \in \Gamma \mid \gamma\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq \emptyset\}$$
is finite.
(3) *Suppose that $\mathbf{P}_1$ is not Γ-conjugate to $\mathbf{P}_2$. Fix U_i, W_i, $i = 1, 2$. Then $\gamma\mathfrak{S}_1 \cap \mathfrak{S}_2 = \emptyset$ for all t_1, t_2 sufficiently large.*
(4) *For any fixed U, W, when $t >> 0$, $\gamma\mathfrak{S} \cap \mathfrak{S} = \emptyset$ for all $\gamma \in \Gamma - \Gamma_P$, where $\Gamma_P := \Gamma \cap P$.*
(5) *For any two different parabolic subgroups $\mathbf{P}_1$ and $\mathbf{P}_2$, when $t_1, t_2 \gg 0$ then $\mathfrak{S}_1 \cap \mathfrak{S}_2 = \emptyset$.*

2.3. **The Borel-Serre compactification $\overline{S_{\Gamma,G,M}}^{BS}$.** In [BS73] Borel and Serre construct a natural compactification $\overline{S_{\Gamma,G,K}}^{BS}$ of any arithmetic locally symmetric space $S_{\Gamma,G,K}$ in the category of real-analytic manifolds with corners, using the notion of geodesic actions and S-spaces. In [BJ06a, §3] Borel and Ji give a uniform construction of the so-called Borel-Serre compactification $\overline{S_{\Gamma,G,M}}^{BS}$ of any arithmetic quotient $S_{\Gamma,G,M}$ in the category of real-analytic manifolds with corners, simplifying the approach of [BS73] as they do not rely anymore on the notion of S-spaces and delicate inductions: they construct a partial compactification $\overline{G}^{BS}$ of G in the category of real-analytic manifolds with corners [BJ06a, Prop.6.3], such that the left $\mathbf{G}(\mathbb{Q})^+$-action on G (see [BJ06a, prop. 3.12]) and the commuting right K-action of a maximal compact subgroup K (see [BJ06a, Prop.3.17]) both extend to an action by weakly analytic maps to $\overline{G}^{BS}$ (see proof of [BJ06a, Prop. 6.4]). For any neat arithmetic subgroup Γ of G and compact subgroup M of G, the action of $\Gamma \times M$ on $\overline{G}^{BS}$ is free and proper. The quotient $\overline{S_{\Gamma,G,M}}^{BS} := \Gamma\backslash\overline{G}^{BS}/M$ provides a compactification of the arithmetic quotient $S_{\Gamma,G,M}$ in the category of real-analytic manifolds with corners.

Let us provide the details of this construction we will need. Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in $\Phi(A_P, N_P)$. Consider the semi-algebraic diffeomorphism $e_P : A_P \longrightarrow (\mathbb{R}_{>0})^r$ defined by

$$e_P(a) = (a^{-\alpha_1}, \dots, a^{-\alpha_r}) \in (\mathbb{R}_{>0})^r \subset \mathbb{R}^r \quad . \tag{2.4}$$

Let $\overline{A_P} = [0, \infty)^r \subset \mathbb{R}^r$ be the closure of $e_P(A_P)$ in $\mathbb{R}^r$. We denote by $\overline{A_{P,t}} \subset \overline{A_P}$ the closure of $e_P(A_{P,t})$.

Let

$$\overline{G}^{BS} = G \cup \coprod_{\mathbf{P} \subset \mathbf{G}} (N_P \times (M_P K))$$

be the Borel-Serre partial compactification of G constructed in [BJ06a, §3.2]. The topology on $\overline{G}^{BS}$ is such that an unbounded sequence $(y_j)_{j \in \mathbb{N}}$ in G converges to a point $(n, m) \in N_P \times (M_P K)$ if and only if, in terms of the horospherical decomposition $G = N_P \times A_P \times (M_P K)$, $y_j = (n_j, a_j, m_j)$ with $n_j \in N_P$, $a_j \in A_P$, $m_j \in M_P K$, and the components n_j, a_j and m_j satisfy the conditions:

1) For any $\alpha \in \Phi(A_P, N_P)$, $(a_j)^\alpha \longrightarrow +\infty$,
2) $n_j \longrightarrow n$ in N_P and $m_j \longrightarrow m$ in $M_P K$.

We refer to [BJ06a, p274-275] for the precise description of the similar glueing between $N_P \times (M_P K)$ and $N_Q \times (M_Q K)$ for two different parabolic subgroups $\mathbf{P} \subset \mathbf{Q}$.

Then:

Proposition 2.8. [BJ06a, Prop.3.3] *The embedding $N_P \times A_P \times (M_P K) = G \subset \overline{G}^{BS}$ extends naturally to an embedding $N_P \times \overline{A_P} \times (M_P K) \hookrightarrow \overline{G}^{BS}$.*

We denote by $G(P)$ the image of $N_P \times \overline{A_P} \times (M_P K)$ under this embedding. It is called the corner associated with $\mathbf{P}$. As explained in [BJ06a, Prop. 6.3] $\overline{G}^{BS}$ has the structure of a real-analytic manifold with corners, a system of real analytic neighbourhood of a point $(n, m) \in N_P \times (M_P K)$ being given by the $\overline{\mathfrak{S}_{U,t,W}} := U \times \overline{A_{P,t}} \times W$, for U a neighborhood of n in N_P, W a neighborhood of m in $M_P K$ and $t > 0$, see [BJ06a, Lemma 3.10, Prop. 6.1 and Prop. 6.3]. As the right action of any compact subgroup M of K on G extends to a proper real analytic action on $\overline{G}^{BS}$, the quotient $\overline{G}^{BS}/M$ is a partial compactification of G/M in the category of real-analytic manifolds with corners.

The left $\mathbf{G}(\mathbb{Q})$-multiplication on G extends to a real analytic action on $\overline{G}^{BS}/M$: see [BJ06a, Prop. 3.12] for the extension to a continous action and the proof of [BJ06a, Prop. 6.4] for the proof that the extended action is real analytic. The restriction of this extended action to a neat Γ is free and properly discontinuous (see [BJ06a, Prop. 3.13 and Prop. 6.4]). Then $\overline{S_{\Gamma,G,M}}^{BS} := \Gamma \backslash \overline{G}^{BS}/M$ is a compact real analytic manifold with corners compactifying $S_{\Gamma,G,M}$. We denote by $\overline{\pi} : \overline{G}^{BS}/M \longrightarrow \overline{S_{\Gamma,G,M}}^{BS}$ the extension of π.

3. Proof of Theorem 1.1

3.1. Proof of Theorem 1.1(1).

By Proposition 2.7(1) there exist finitely many $\mathbf{P}_1, \dots, \mathbf{P}_k$ parabolic subgroups of $\mathbf{G}$ and Siegel sets $\mathfrak{S}_i := U_i \times A_{P_i,t_i} \times W_i$, $1 \leq i \leq k$, with U_i, W_i compact semi-algebraic subsets of N_{P_i} and $M_{P_i}K/M$ respectively, whose images $V_i := \pi(\mathfrak{S}_i)$, $1 \leq i \leq k$ cover $S_{\Gamma,G,M}$.

The real analytic manifold $S_{\Gamma,G,M}$ can thus be obtained as the quotient of $\coprod_{i=1}^k \mathfrak{S}_i$ by the étale equivalence relation E defined by

$$x_1 \in \mathfrak{S}_{i_1} \sim_E x_2 \in \mathfrak{S}_{i_2} \iff \exists \gamma \in \Gamma \mid \gamma x_1 = x_2 \ .$$

The V_i's, $1 \leq i \leq k$, provide a cover of $S_{\Gamma,G,M}$ by open real-analytic charts.

For $1 \leq i \leq k$ let $\mathrm{cl}(\mathfrak{S}_i)$ denote the topological closure of $\mathfrak{S}_i$ in G/M. The quotient $S_{\Gamma,G,M}$ also identifies with the quotient of $\coprod_{i=1}^k \mathrm{cl}(\mathfrak{S}_i)$ by the same equivalence relation $\sim_E$. As each $\mathfrak{S}_i$, $1 \leq i \leq k$, is semi-algebraic, their closure $\mathrm{cl}(\mathfrak{S}_i)$ too, hence the set $\coprod_{i=1}^k \mathrm{cl}(\mathfrak{S}_i)$ is $\mathbb{R}_{\mathrm{alg}}$-definable. As the action of Γ is real-algebraic on G, the equivalence relation $\sim_E$ on $\coprod_{i=1}^k \mathrm{cl}(\mathfrak{S}_i)$ is $\mathbb{R}_{\mathrm{alg}}$-definably proper by Proposition 2.7(2) in the sense of [VDD98, (2.13) p.166]. By [VDD98, (2.15) p.166] the quotient $S_{\Gamma,G,M} = (\coprod_{i=1}^k \mathrm{cl}(\mathfrak{S}_i))/\sim_E$ is naturally an $\mathbb{R}_{\mathrm{alg}}$-definable manifold: each $\mathrm{cl}(V_i) := \pi(\mathrm{cl}(\mathcal{F}_i))$ is $\mathbb{R}_{\mathrm{alg}}$-definable and the restriction $\pi_i : \mathrm{cl}(\mathfrak{S}_i) \longrightarrow \mathrm{cl}(V_i)$ of π to $\mathfrak{S}_i$ is $\mathbb{R}_{\mathrm{alg}}$-definable. As each $\mathfrak{S}_i$ is semi-algebraic, it follows that $V_i := \pi_i(\mathfrak{S}_i)$, $1 \leq i \leq k$, is semi-algebraic and the V_i's, $1 \leq i \leq k$, form an explicit finite open atlas of the $\mathbb{R}_{\mathrm{alg}}$-definable manifold $S_{\Gamma,G,M}$.

Let $\mathfrak{S} \subset G/M$ be any Siegel set. By Proposition 2.7(1) there exists $\gamma \in \Gamma$ such that $\gamma\mathfrak{S}$ is associated to one of the parabolics $\mathbf{P}_i$ for some $1 \leq i \leq k$. Replacing $\mathfrak{S}_i$ by a bigger Siegel set for $\mathbf{P}_i$ if necessary in the previous construction, we can assume without loss of generality that $\gamma\mathfrak{S}$ is contained in $\mathfrak{S}_i$. Hence $\pi_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow S_{\Gamma,G,M}$ coincides with the composite

$$\mathfrak{S} \xrightarrow{\gamma\cdot} \gamma\mathfrak{S} \hookrightarrow \mathfrak{S}_i \xrightarrow{\pi_i} S_{\Gamma,G,M}$$

hence is $\mathbb{R}_{\mathrm{alg}}$-definable.

With the notation above, the set $\mathcal{F} := \cup_{i=1}^k \mathfrak{S}_i \subset G/M$ is a semi-algebraic fundamental set for the action of Γ on G/M. As each $\pi_i : \mathfrak{S}_i \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable, it follows that $\pi_{\mathcal{F}} : \mathcal{F} \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable.

By Proposition A.2 the compact real analytic manifold with corners $\overline{S_{\Gamma,G,M}}^{BS}$ admits a natural structure of $\mathbb{R}_{\mathrm{an}}$-definable manifold with corner. Explicitly: the images $\overline{V_i} := \overline{\pi}(\overline{\mathfrak{S}_i})$ of $\overline{\mathfrak{S}_i} := U_i \times \overline{A_{P_i,t_i}} \times W_i \subset \overline{\mathbf{G}}^{BS}$, $1 \leq i \leq k$, cover $\overline{S_{\Gamma,G,M}}^{BS}$ and form a finite atlas of the $\mathbb{R}_{\mathrm{an}}$-definable manifold with corners $\overline{S_{\Gamma,G,M}}^{BS}$. Let us show that the $\mathbb{R}_{\mathrm{an}}$-definable manifold structure on $S_{\Gamma,G,M}$ obtained by restriction of this structure of $\mathbb{R}_{\mathrm{an}}$-definable manifold with corner on $\overline{S_{\Gamma,G,M}}^{BS}$ coincide with the $\mathbb{R}_{\mathrm{an}}$-definable manifold structure extending the $\mathbb{R}_{\mathrm{alg}}$-definable manifold structure on $S_{\Gamma,G,M}$ we just constructed. We are reduced to showing that for each i, $1 \leq i \leq k$, the map $\mathfrak{S}_i \xrightarrow{\pi_i} V_i \hookrightarrow \overline{V_i}$ is $\mathbb{R}_{\mathrm{an}}$-definable. It factorises as

$$\mathfrak{S}_i \xrightarrow{1_{U_i} \times e_{P_i} \times 1_{W_i}} \overline{\mathfrak{S}_i} \xrightarrow{\overline{\pi_i}} \overline{V_i} \ .$$

On the one hand, it follows from the definition (2.4) of $e_{P_i} : A_{P_i,t_i} \longrightarrow \overline{A_{P_i,t_i}} \subset \mathbb{R}^{r_i}$ that e_{P_i}, hence also $1_{U_i} \times e_{P_i} \times 1_{W_i}$, is semi-algebraic. On the other hand, the map $\overline{\pi_i} : U_i \times \overline{A_{P_i,t_i}} \times W_i \longrightarrow \overline{V_i}$ is a real analytic map between compact sets, hence is $\mathbb{R}_{\mathrm{an}}$-definable. This concludes the proof that $\pi_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{an}}$-definable.

This concludes the proof of Theorem 1.1(1). $\square$

Remark 3.1. Although the $\overline{\mathfrak{S}_i}$, $1 \leq i \leq k$, are semi-algebraic, it is not true in general that $\overline{S_{\Gamma,G,M}}^{BS}$ admits a natural structure of $\mathbb{R}_{\mathrm{alg}}$-definable manifolds with corners: if $\mathbf{P} \subset \mathbf{Q}$ are two parabolics of $\mathbf{G}$ it follows from the proof of [BJ06a, Prop. 6.2] that the inclusion of the corner $G(Q) \subset G(P)$ is real-analytic but not semi-algebraic in general.

3.2. Morphisms of arithmetic quotients are definable: proof of Theorem 1.1(2).

Notice that the statement of Theorem 1.1(2) is non-trivial even in $\mathbb{R}_{\mathrm{an}}$. For instance in the case where $f : \mathbf{G}' \longrightarrow \mathbf{G}$ is a strict inclusion the morphism f does not usually extend to a real analytic morphism (or even a continuous one) $\overline{f}$ between the Borel-Serre compactifications (in other words the Borel-Serre compactification is not functorial). The problem is that two parabolic subgroups $\mathbf{P}_i \subset \mathbf{G}$, $i = 1, 2$, can be non conjugate under Γ while their intersections $\mathbf{P}_i \cap \mathbf{G}'$ are Γ'-conjugate parabolic subgroups of $\mathbf{G}'$. However Theorem 1.1(2) will follow from a finiteness result for Siegel sets due to Orr (see Theorem 3.2).

First, we claim that for $g \in \mathbf{G}(\mathbb{Q})$ the morphism of arithmetic quotients $(\mathrm{Int}(g), g) : S_{g^{-1}\Gamma g,G,M} \longrightarrow S_{\Gamma,G,M}$ (the left multiplication by g) is $\mathbb{R}_{\mathrm{alg}}$-definable: this follows immediately from Remark 2.6(1).

Let $(f, g) : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$ be a general morphism of arithmetic quotients. As $(f, g) = (\mathrm{Int}(g^{-1}) \circ f, 1) \circ (\mathrm{Int}(g), g)$ we are reduced to considering morphism of arithmetic quotients $f := (f, 1) : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$ deduced from a morphism $f : \mathbf{G}' \longrightarrow \mathbf{G}$ of semi-simple linear algebraic $\mathbb{Q}$-group such that $f(M') \subset M$ and $f(\Gamma') \subset \Gamma$.

Let $(V_i')_{1 \leq i \leq k}$ be an $\mathbb{R}_{\mathrm{alg}}$-atlas for $S_{\Gamma',G',M'}$ as in Section 3.1. Showing that $f : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable is equivalent to showing that for each i, $1 \leq i \leq k$, the restriction $f : V_i' \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable. As the diagram

$$\begin{tikzcd}
\mathfrak{S}_i' := U_i' \times A'_{P_i',t_i'} \times W_i' \arrow[r, "f"] \arrow[d, "\pi_i'"'] & G \arrow[d, "\pi"] \\
V_i' \arrow[r, "f"'] & S_{\Gamma,G,M}
\end{tikzcd}$$

is commutative, it is enough to show that the composite

$$\mathfrak{S}_i' \xrightarrow{f} G \xrightarrow{\pi} S_{\Gamma,G,M} \tag{3.1}$$

is $\mathbb{R}_{\mathrm{alg}}$-definable.

The case where $\mathbf{G}' = \mathbf{G}$ is clear from Theorem 1.1(1) (notice that in that case Goresky and MacPherson show in [GM03, Lemma 6.3] (and its proof) that f extends uniquely to a real analytic morphism $\overline{f} : \overline{S_{\Gamma',G',M'}}^{BS} \longrightarrow \overline{S_{\Gamma,G,M}}^{BS}$).

Suppose that $f : \mathbf{G}' \longrightarrow \mathbf{G}$ is surjective. Without loss of generality we can assume that $\mathbf{G}$, and then $\mathbf{G}'$, are adjoint. Then $\mathbf{G}' = \mathbf{G} \times \mathbf{H}$, $S_{\Gamma',G',M'} = S_{\Gamma' \cap G,G,M' \cap G} \times S_{\Gamma' \cap H,H,M' \cap H}$, the map f coincides with the projection onto the first factor (this projection is obviously $\mathbb{R}_{\mathrm{alg}}$-definable) composed with the morphism of arithmetic quotients $i : S_{\Gamma' \cap G,G,M' \cap G} \longrightarrow S_{\Gamma,G,M}$, which is $\mathbb{R}_{\mathrm{alg}}$-definable from the case $\mathbf{G}' = \mathbf{G}$. This proves Theorem 1.1(2) when $f : \mathbf{G}' \longrightarrow \mathbf{G}$ is surjective.

We are thus reduced to proving Theorem 1.1(2) in the case where $f : \mathbf{G}' \longrightarrow \mathbf{G}$ is a strict inclusion. We use the following:

Theorem 3.2. *([O17, Theor.1.2]) Let $\mathbf{G}$ and $\mathbf{H}$ be reductive linear $\mathbb{Q}$-algebraic groups, with $\mathbf{H} \subset \mathbf{G}$. Let $\mathfrak{S}_H := U_H \times A_{P_H,t} \times W_H \subset \mathbf{H}(\mathbb{R})$ be a Siegel set for $\mathbf{H}$.*

Then there exists a finite set $C \subset \mathbf{G}(\mathbb{Q})$ and a Siegel set $\mathfrak{S} := U \times A_{P,t} \times W \subset \mathbf{G}(\mathbb{R})$ such that $\mathfrak{S}_H \subset C \cdot \mathfrak{S}$.

Applying this result to $\mathbf{G}' \subset \mathbf{G}$ and the Siegel set $\mathfrak{S}'_i$ of $\mathbf{G}'$, there exists a finite set $C_i \subset \mathbf{G}(\mathbb{Q})$ and a Siegel set $\mathfrak{S}_i := U_i \times A_{P_i,t_i} \times W_i$ such that the composition (3.1) factorizes as

$$\mathfrak{S}'_i \longrightarrow C_i \cdot \mathfrak{S}_i \xrightarrow{\pi} S_{\Gamma,G,M} \quad . \tag{3.2}$$

The inclusion $\mathfrak{S}'_i \longrightarrow C_i \cdot \mathfrak{S}_i$ is semi-algebraic. The map $C_i \cdot \mathfrak{S}_i \xrightarrow{\pi} S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{alg}}$-definable by Theorem 1.1(1).

This finishes the proof of Theorem 1.1(2). □

4. Definability of the period map

4.1. **Reduction of Theorem 1.3 to a local statement.** In the situation of Theorem 1.3, let $S \subset \overline{S}$ be a smooth compactification such that $\overline{S} - S$ is a normal crossing divisor. Let $(\overline{S_i})_{1 \leq i \leq m}$ be a finite open cover of $\overline{S}$ such that the pair $(\overline{S_i}, S_i := S \cap \overline{S_i})$ is biholomorphic to $(\Delta^n, (\Delta^*)^{r_i} \times \Delta^{l_i := n - r_i})$. To show that the period map $\Phi_S : S \longrightarrow \mathrm{Hod}^0(S, \mathbb{V}) = S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable, it is enough to show that for each i, $1 \leq i \leq m$, the restricted period map

$$\Phi_{S|S_i} : S_i = (\Delta^*)^{r_i} \times \Delta^{l_i} \longrightarrow S_{\Gamma,G,M} \tag{4.1}$$

is $\mathbb{R}_{\mathrm{an,exp}}$-definable. Without loss of generality we can assume that $r_i = n$ and $l_i = 0$ by allowing some factors with trivial monodromies. Finally we are reduced to proving:

Theorem 4.1. *Let $\mathbb{V} \to (\Delta^*)^n$ be a polarized variation of pure Hodge structures of weight k over the punctured polydisk $(\Delta^*)^n$, with period map $\Phi : (\Delta^*)^n \longrightarrow S_{\Gamma,G,M}$. Then Φ is $\mathbb{R}_{\mathrm{an,exp}}$-definable.*

4.2. **Proof of Theorem 4.1 assuming Theorem 1.5.**

Let us fix x_0 a basepoint in $(\Delta^*)^n$. We denote by $V_{\mathbb{Z}}$ the fiber $\mathbb{V}_{x_0}$ of $\mathbb{V}$ at x_0 (modulo torsion) and $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. We further denote by $Q_{\mathbb{Z}}$ the polarization form on $V_{\mathbb{Z}}$, and $Q_{\mathbb{C}}$ its $\mathbb{C}$-linear extension to $V_{\mathbb{C}}$.

It follows from Borel's monodromy theorem [Sc73, Lemma (4.5)] that the monodromy transformation $T_i \in \mathbf{G}(\mathbb{Z}) \subset \mathbf{GL}(V_{\mathbb{Z}})$, $1 \leq i \leq n$, of the local system $\mathbb{V}$, corresponding to counterclockwise simple circuits around the various punctures, are quasi-unipotent. Replacing $(\Delta^*)^n$ by a finite étale cover if necessary we can assume without loss of generality that all the T_i's are unipotent. Let $N_i \in \mathfrak{g}_{\mathbb{Q}}$, $1 \leq i \leq n$, be the logarithm of T_i; this is a nilpotent element in the (rational) Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ of $\mathbf{G}$.

Let $\mathfrak{H}$ denote the Poincaré upper half-plane and $\exp(2\pi i \cdot\ ) : \mathfrak{H} \longrightarrow \Delta^*$ the uniformizing map of Δ^*. Let $\mathfrak{S}_{\mathfrak{H}} \subset \mathfrak{H}$ be the usual Siegel fundamental set

$$\{(x, y) \in \mathfrak{H} \mid 0 < x < 1,\ 1 < y\} \quad .$$

Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{S}_{\mathfrak{H}}^n \subset \mathfrak{H}^n \arrow[r, "\tilde{\Phi}"] \arrow[d, "p=e(\cdot\,)^n"'] & D = G/M \arrow[d, "\pi"] \\ (\Delta^*)^n \arrow[r, "\Phi"'] & S_{\Gamma,G,M} \end{tikzcd} ,$$

where $e(\cdot) := \exp(2\pi i \cdot)$ and $\tilde{\Phi}$ is the lifting of Φ to the universal cover. As the restriction $\exp(2\pi i\ \cdot\ )_{|\mathfrak{S}_{\mathfrak{H}}}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable, the map $p_{|\mathfrak{S}_{\mathfrak{H}}^n} : \mathfrak{S}_{\mathfrak{H}}^n \longrightarrow (\Delta^*)^n$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. We are reduced to proving that the composition $\mathfrak{S}_{\mathfrak{H}}^n \xrightarrow{\tilde{\Phi}} G/M \xrightarrow{\pi} S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Lemma 4.2. *The map $\tilde{\Phi} : \mathfrak{S}_{\mathfrak{H}}^n \longrightarrow G/M$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable.*

Proof. The nilpotent orbit theorem [Sc73, (4.12)] states that (after maybe shrinking the polydisk) the map $\tilde{\Phi} : {\mathfrak{S}_{\mathfrak{H}}}^n \longrightarrow G/M$ is of the form

$$\tilde{\Phi}(z) = \exp\left(\sum_{j=1}^{n} z_j N_j\right) \cdot \Psi(p(z))$$

for $\Psi : \Delta^n \longrightarrow \check{D}$ a holomorphic map and $\check{D} \supset D$ the compact dual of D. The map Ψ is the restriction to a relatively compact set of a real analytic map. As $p_{|{\mathfrak{S}_{\mathfrak{H}}}^n} : \mathfrak{S}_{\mathfrak{H}}^n \longrightarrow \Delta^n$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable, it follows that $(z \mapsto \Psi(p(z))$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

The action of $\mathbf{G}(\mathbb{C})$ on the compact dual $\check{D}$ is algebraic, hence $\mathbb{R}_{\mathrm{an,exp}}$-definable; D is a semi-algebraic subset of $\check{D}$ and $\exp(\sum_{j=1}^n z_j N_j)$ is a polynomial in the z_j's as the N_j's are nilpotent.

Hence the result. □

Remark 4.3. Notice that Lemma 4.2 appears also in [BaT17, Lemma 3.1].

It moreover follows from Theorem 1.5, proven below, that there exist finitely many Siegel sets $\mathfrak{S}_i$ $(i \in I)$ for G/M such that $\tilde{\Phi}(\mathfrak{S}_{\mathfrak{H}}^n) \subset \bigcup_{i\in I} \mathfrak{S}_i$. As $\pi_{|\mathfrak{S}_i} : \mathfrak{S}_i \longrightarrow S_{\Gamma,G,M}$, $i \in I$, is $\mathbb{R}_{\mathrm{alg}}$-definable by Theorem 1.1(1), and the set I is finite, we deduce from Lemma 4.2 that $\pi \circ \tilde{\Phi} : \mathfrak{S}_{\mathfrak{H}}^n \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. This concludes the proof of Theorem 4.1, hence of Theorem 1.3, assuming Theorem 1.5. □

4.3. **Roughly polynomial functions.** Before the proof of Theorem 1.5 we discuss a class of functions with the same asymptotics as the Hodge form.

Definition 4.4. *Let $\Sigma_n \subset \mathfrak{S}_{\mathfrak{H}}^n$ be the region $0 < x_i < 1$ and $y_1 \geq \cdots \geq y_n > 1$ for $x_i = \mathrm{Re}\, z_i$ and $y_i = \mathrm{Im}\, z_i$. Let $\mathcal{O}$ be the ring of real restricted analytic functions on Δ^n pulled back to Σ_n via $p : \mathfrak{H}^n \to (\Delta^*)^n$, $\mathcal{O}[x, y, y^{-1}]$ the ring of polynomials in $x_1, \ldots, x_n, y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}$ with coefficients in $\mathcal{O}$, and $\mathcal{O}(x, y)$ its fraction field. By a* monomial *we mean an element of $\mathcal{O}[x, y, y^{-1}]$ of the form $y_1^{s_1} \cdots y_n^{s_n}$ for integers $s_i \in$* $\mathbf{Z}$*. We say a function $f \in \mathcal{O}(x, y)$ is* roughly monomial *if it is within a multiplicatively bounded constant of a monomial on Σ_n. We say that $f \in \mathcal{O}(x, y)$ is* roughly polynomial *if it is of the form $\frac{g}{h}$ where $g \in \mathcal{O}[x, y, y^{-1}]$ and h is roughly monomial. Note that roughly polynomial functions form a ring which we denote $\mathcal{T}_n$.*

In the following we write $f \ll g$ to mean $f < Cg$ for a constant $C > 0$ and $f \sim g$ to mean $f \ll g$ and $g \ll f$. Thus, $f \in \mathcal{O}(x,y)$ is roughly monomial if $f \sim y_1^{s_1} \cdots y_n^{s_n}$. The next lemma will allow us to understand the asymptotics of roughly polynomial functions by restricting to curves.

Lemma 4.5. *Let $f, g \in \mathcal{O}(x,y)$ with f roughly polynomial and g roughly monomial. Assume that $|f| \ll |g|$ when restricted to any set of the form*

$$\Sigma_n \cap \{\alpha_1 z_1 + \beta_1 = \alpha_2 z_2 + \beta_2 = \cdots = \alpha_{n_0} z_{n_0} + \beta_{n_0}, z_{n_0+1} = \zeta_{n_0+1}, \ldots, z_n = \zeta_n\}$$

for some $1 \leq n_0 \leq n$, $\zeta_{n_0+1}, \ldots, \zeta_n \in \mathfrak{H}$, $\alpha_1, \ldots, \alpha_{n_0} \in \mathbb{Q}_+^{n_0}$, and $\beta_1, \ldots, \beta_{n_0} \in \mathbb{R}$. Then $|f| \ll |g|$ on all of Σ_n.

Proof. By clearing denominators it is clearly sufficient to handle the case where g is a monomial, and in fact where $g = 1$ since y_i is a unit in $\mathcal{O}[x, y, y^{-1}]$. We proceed by induction on n, with the case $n = 1$ being immediate from the assumptions. Separating out the powers of x_1 and y_1 we may write $f = \sum_j a_j x_1^{j_1} y_1^{j_2}$, where the sum is over finitely many pairs $j = (j_1, j_2)$ in $\mathbb{Z}^2$. Now, the a_j are real analytic functions in $t_1 := e(z_1), x_2, y_2, y_2^{-1}, t_2 := e(z_2), \ldots, x_n, y_n, y_n^{-1}, t_n := e(z_n)$ up to an error on Σ_n of $O(y_1^A e^{-2\pi y_1})$ for some positive $A > 0$. Since this decays faster than any monomial, we may restrict to the case where the a_j are independant of t_1.

The lemma will follow immediately once we prove the following claim:

Claim: For each j we have that $|a_j| y_1^{j_2} \ll 1$ on Σ_n.

First, we claim that all the powers j_2 of y_1 in f are non-positive. If this is not the case, we can fix the other variables at a point where the coefficient of a positive power of y_1 is non-zero, and get a contradiction as $y_1 \to \infty$.

Now, since the powers of y_1 are non-positive and $y_1 \geq y_2$, it is sufficient to prove the claim when we restrict to $y_1 = Cy_2$ for any $C > 0$. Consider for each positive integer m and real number c the function $f_{m,c}$ which we obtain from f by setting $z_1 = mz_2 + c$. Note that the assumptions of the lemma still apply to $f_{m,c}$, and it follows by induction on n that $f_{m,c} \ll 1$ on all of Σ_{n-1} (taken with respect to the ordered coordinates $z_2, \ldots, z_n$). Thus, we have that $\sum_j a_j (mx_2 + c)^{j_1} (my_2)^{j_2} \ll 1$ on all of Σ_{n-1}.

Let $j_{\max} = (r_1, r_2)$ be the lexicographically maximal j that occurs among all j with $a_j \neq 0$. Let $F_m := \frac{1}{r_1!} \sum_{i=0}^{r_1} \binom{r_1}{i} (-1)^i f_{m,i}$. Note that $F_m \ll 1$ on Σ_{n-1} and also that $F_m = \sum_{j=(r_1,j_2)} a_J m^{j_2} y_2^{j_2}$. By taking a finite (constant) linear combination of the F_m with distinct values of m we may isolate the $a_{j_{\max}} y_2^{r_2}$ term, from which it follows that $a_{j_{\max}} y_2^{r_2} \ll 1$, thus proving the claim for this monomial. Subtracting it off and proceeding inductively, the claim follows.

$\square$

4.4. **The Hodge form is roughly polynomial.**

For $u, v \in V_{\mathbb{C}}$ we denote by $h(u,v) : \Sigma_n \to \mathbb{C}$ the function mapping $z \in \Sigma_n$ to the Hodge form $h_z(u,v) := Q_{\mathbb{C}}(C_z u, \overline{v})$ of u and v at $\tilde{\Phi}(z)$, where C_z is the Weil operator at $\tilde{\Phi}(z)$. Likewise we denote $h_z(u) = h_z(u,u)$. The main result of this section is the following:

Proposition 4.6. *For any $u, v \in V_{\mathbb{C}}$, the function $h(u,v)$ is roughly polynomial.*

Let us introduce the notation needed for proving Proposition 4.6.

Let $C \subset \mathfrak{g}_{\mathbb{R}}$ be the open convex cone generated by the monodromy logarithms N_i. Recall from [CKS86, p.468] that for each $M \in \overline{C}$ we have a weight filtration $W(M)$ on $V_{\mathbb{Q}}$ (centered at 0). Let $M_j = \sum_{i=1}^{j} N_i$ and $F = \Psi(0) \in \check{D}$. We write $\tilde{\Phi}(z) = \gamma(z)F$ where $\gamma : \mathfrak{H}^n \to \mathbf{G}(\mathbb{C})$ is a lift of the form $\gamma(z) = e^{z \cdot N} g(p(z))$ for $g : \Delta^n \to \mathbf{G}(\mathbf{C})$ a holomorphic function taken as follows. Writing $\mathfrak{g}^{p,q}$ for the Deligne splitting of the mixed Hodge structure on $\mathfrak{g}_{\mathbb{R}}$ induced by $(F, W(M_n))$, there is a unique holomorphic lift $v(t) \in \bigoplus_{p<0} \mathfrak{g}^{p,q}$ with $\Psi(t) = e^{v(t)}F$.

Recall that there is a splitting (see [Ka85, Lemma 2.4.1]) $V_{\mathbb{C}} = \bigoplus_{p,q_1,\ldots,q_r} I^{p,q_1,\ldots,q_n}$ such that $F^s = \bigoplus_{p\geq s} I^{p,q_1,\ldots,q_n}$ and $W(M_j)_s = \bigoplus_{p+q_j\leq s} I^{p,q_1,\ldots,q_n}$. For simplicity we denote $\alpha = (p, q_1, \ldots, q_n)$. There is also a rational splitting $V_{\mathbb{Q}} = \bigoplus_{s_1,\ldots,s_n} J^{s_1,\ldots,s_n}$ such that $W(M_j)_s = \bigoplus_{s_j\leq s} J^{s_1,\ldots,s_n}$, and again for simplicity we denote $\sigma = (s_1, \ldots, s_n)$.

Proposition 4.6 follows immediately from:

Lemma 4.7. *Let $u \in I^{\alpha}$ and $v \in I^{\beta}$.*

(1) $h(u)$ is roughly monomial.
(2) $h(\gamma(z)u)$ is roughly monomial.
(3) $h(u, v)$ is roughly polynomial.

Proof of Lemma 4.7. The asymptotics of Hodge forms are well-studied, and we state the precise result we will need:

Theorem 4.8 (Theorem 5.21 of [CKS86] or Theorems 3.4.1 and 3.4.2 of [Ka85])**.** *Let $u \in J_{\mathbb{C}}^{s_1,\ldots,s_n}$. Then on Σ_n we have*

(1) $h(u) \sim (y_1/y_2)^{s_1} \cdots (y_{n-1}/y_n)^{s_{n-1}} y_n^{s_n}$;
(2) $h\left(e^{z\cdot N}u\right) \sim (y_1/y_2)^{s_1} \cdots (y_{n-1}/y_n)^{s_{n-1}} y_n^{s_n}$.

Remark 4.9. We remark that Kashiwara's proof of Theorem 4.8 does not require the use of the SL_2^n-orbit theorem of [CKS86].

Given Theorem 4.8(1), proving part (1) of Lemma 4.7 reduces to showing that $h(u)$ is in $\mathcal{O}(x, y)$. Choose a basis w_i of $V_{\mathbb{C}}$ such that each $w_i \in I^{\alpha_i}$ for some $\alpha_i = (p_i, q_1^i, \ldots, q_n^i)$ and the sequence p_i is non-increasing. Define an increasing filtration $K^\bullet$ as follows: K^j is the span of $w_1, \ldots, w_j$. Note that $K^\bullet$ is a full flag refining the filtration $F^{-\bullet}$. Define $B(u, v) := Q_{\mathbb{C}}(u, \bar{v})$ and for simplicity call $\gamma = \gamma(z)$. An h-orthogonal basis of the Hodge filtration at γF is obtained from the γw_i by the Gram-Schmidt procedure (with respect to B); call this basis $\tilde{w}_i$. Let $u = \sum \tilde{u}_i$ with $\tilde{u}_i$ a multiple of $\tilde{w}_i$ and likewise for v. We then have, denoting $w_{\det K^i} = w_1 \wedge \cdots \wedge w_i$ and extending the use of the symbols B and h to the corresponding hermitian forms on any wedge power (or other tensor operation) of $V_{\mathbb{C}}$:

$$B(\tilde{w}_i) = \frac{B\left(\gamma w_{\det K^i}\right)}{B(\gamma w_{\det K^{i-1}})} \tag{4.2}$$

We also have

$$B(u, \tilde{w}_i) = \frac{B\left((\gamma w_{\det K^{i-1}}) \wedge u, \gamma w_{\det K^i}\right)}{B(\gamma w_{\det K^{i-1}})}\ , \tag{4.3}$$

$$h(\tilde{u}_i,\tilde{v}_i)=i^{2p_i-k}B(\tilde{u}_i,\tilde{v}_i)=i^{2p_i-k}\frac{B(u,\tilde{w}_i)B(\tilde{w}_i,v)}{B(\tilde{w}_i)}\ . \tag{4.4}$$

Now, (4.2) and (4.3) are both in $\mathcal{O}(x,y)$, so $h(u,v)=\sum_i h(\tilde{u}_i,\tilde{v}_i)$ is in $\mathcal{O}(x,y)$ as well.

That $h(\gamma u)$ is in $\mathcal{O}(x,y)$ similarly follows from the fact that the B-norm of the projection of γu to $\tilde{w}_i$ is likewise computed via

$$B(\gamma u,\tilde{w}_i)=\frac{B(\gamma(w_{\det K^{i-1}}\wedge u),\gamma w_{\det K^i})}{B(\gamma w_{\det K^{i-1}})}. \tag{4.5}$$

To finish the proof of part (2), we need the following lemma, which is also proven in [CKS86, Theorem 5.21].

Lemma 4.10. *On Σ_n we have $h(\gamma u)\sim h\left(e^{z\cdot N}u\right)$.*

Proof. Recall that $\gamma(z)=e^{z\cdot N}g(p(z))$ and $g(t)=e^{v(t)}$. Griffiths transversality requires

$$e^{-\mathrm{ad}(v(t))}N_i+t_i\frac{1-e^{-\mathrm{ad}(v(t))}}{\mathrm{ad}(v(t))}\frac{\partial v}{\partial t_i}\in F^{-1}\mathfrak{g}.$$

Since $N_i\in\mathfrak{g}^{-1,-1}$ and $v(t)\in\bigoplus_{p<0}\mathfrak{g}^{p,q}$, this implies that

$$v(t)=\sum_{i=1}^n t_iv_i(t)$$

where $v_i(t)$ is a holomorphic function of $t_i,\dots,t_n$ and $\mathrm{ad}(N_j)v_i(t)=0$ for $j<i$. Thus, $v_i(t)$ preserves each $W(M_j)$ for $j<i$. It follows likewise that $g(t)=1+\sum_i t_ig_i(t)$ with $g_i(t)\in\mathrm{End}\,(V_{\mathbf{C}})$ preserving $W(M_j)$ for $j<i$. Thus by Theorem 4.8, we have on Σ_n

$$h(e^{z\cdot N}g_i(t)u)\ll(y_1/y_2)^{s_1}\cdots(y_{i-1}/y_i)^{s_{i-1}}(y_i/y_{i+1})^{s_i'}\cdots y_n^{s_n'}$$

for some $s_i',\dots,s_n'$. For any monomial y^I only in $y_i,\dots,y_n$ and any $\epsilon>0$, we have that $|t_i|^2y^I<\epsilon$ for sufficiently large y_n and so the same is true of $h(\gamma u-e^{z\cdot N}u)/h(e^{z\cdot N}u)$, whence the claim. ☐

Finally, $h(\gamma w_{\det K^i})=|B(\gamma w_{\det K^i})|$ is roughly monomial by part (2). It then follows that the same is true for (4.2), and thus that $h(\tilde{u}_i,\tilde{v}_i)$ is roughly polynomial, as the numerator in (4.4) is clearly in $\mathcal{O}[x,y,y^{-1}]$. ☐

4.5. Proof of Theorem 1.5.

Proof. First, observe that the set $\mathfrak{S}^n_{\mathfrak{H}}$ is covered by finitely many sets of the form Σ_n (corresponding to the finitely many possible orderings of the coordinates). Hence it is enough to show that $\tilde{\Phi}(\Sigma_n)$ is contained in finitely many Siegel sets of $D=G/M$.

The natural embedding $\mathbf{G}(\mathbb{Q})\subset\mathbf{SL}(V_{\mathbb{Q}})$ defines a natural map $\iota:D\to X$, given by $\iota(x):=h_x$, where X denotes the symmetric space of positive definite symmetric forms on $V_{\mathbb{R}}$. By [B-HC62, 7.5] the preimage of any Siegel set $\mathfrak{S}\subset X$ is contained in the union of finitely many Siegel sets of D. It is thus enough to show that $\iota\circ\tilde{\Phi}(\Sigma_n)$ is contained in finitely many Siegel sets of X.

Siegel sets in X can be understood in terms of reducedness of positive definite forms with respect to a basis:

Definition 4.11. *Given an integral (ordered) basis $e = \{e_i\}$ of $V_{\mathbb{Z}}$, a constant $C > 0$, and a positive definite symmetric form b, we say b is (e, C)-reduced if:*

(1) $|b(e_i, e_j)| < Cb(e_i)$ for all i, j;
(2) $b(e_i) < Cb(e_j)$ for $i < j$;
(3) $\prod_i b(e_i) < C \det(b)$.

Given an integral (ordered) basis $e = \{e_i\}$ of $V_{\mathbb{Z}}$ and $C > 0$ we define the subset $\mathfrak{T}_{e,C} = \{b \in X \mid b \text{ is } (e, C)\text{-reduced}\}$. By classical reduction theory, any $\mathfrak{T}_{e,C}$ is contained in a Siegel set of X, and any Siegel set of X is contained in $\mathfrak{T}_{e,C}$ for some choice of e and $C > 0$ (see for example [K90, Prop. 2 p.18]). Moreover, if b is (e', C')-reduced and e is a basis for which condition (3) of Definition 4.11 holds for some $C > 0$, then b will also be (e, C'')-reduced for some $C'' = C''(e, C, e', C') > 0$. We are thus reduced to proving the following:

Claim: There is a basis $e = \{e_i\}$ of $V_{\mathbb{Q}}$ and $C > 0$ such that h_z is (e, C)-reduced for all $z \in \Sigma_n$.

Choosing a basis e for which each $e_i \in J^{\sigma_i}$ for some σ_i, we have condition (3) in Definition 4.11 by Theorem 4.8 since each weight filtration is centered around 0, while we may assume (2) as there are only finitely many orderings of the basis. By a result of Schmid [Sc73, Corollary 5.29] we know Theorem 1.5 is true in the $n = 1$ case. Since any Siegel set of D is contained in finitely many Siegel sets of X by Theorem 3.2, it follows that h_z is (e, C_τ)-reduced for $p(z)$ restricted to any curve τ in Σ_n. Taking Proposition 4.6 into account, Lemma 4.5 implies condition (1) for some fixed $C > 0$ on all of Σ_n and this completes the proof. □

4.6. Theorem 1.1 implies Borel's algebraicity theorem.

Theorem 4.12. [Bor72, Theor. 3.10] *Let S be a complex algebraic variety and $f : S \longrightarrow S_{\Gamma,G,K}$ a complex analytic map to an arithmetic variety $S_{\Gamma,G,K}$. Then f is algebraic.*

Proof. The map f is a period map, hence is $\mathbb{R}_{\text{an,exp}}$-definable by Theorem 1.3. The graph of f is thus a complex analytic, $\mathbb{R}_{\text{an,exp}}$-definable, subset of the smooth complex algebraic manifold $S \times S_{\Gamma,G,K}$. Recall the following o-minimal Chow theorem of Peterzil-Starchenko [PS09, Theor. 4.4 and Corollary 4.5] (see also [MPT17, Theor. 2.2] and [Sc18, Theor. 2.11] for more precise versions), generalizing a result of Fortuna-Łojasiewicz [FL81] in the semi-algebraic case:

Theorem 4.13. *(Peterzil-Starchenko) Let S be a smooth complex algebraic manifold (hence endowing the $\mathbb{C}$-analytic manifold S with a canonical $\mathbb{R}_{\text{alg}}$-definable manifold structure). Let $W \subset S$ be a complex analytic subset which is also an $\mathcal{S}$-definable subset for some o-minimal structure $\mathcal{S}$ expanding $\mathbb{R}_{\text{an}}$. Then W is an algebraic subset of S.*

It follows that the graph of f is an algebraic subvariety of $S \times S_{\Gamma,G,K}$, hence that f is algebraic (see [Se56, Prop. 8]). □

5. Algebraicity of Hodge loci: proof of Theorem 1.6

We refer to [K17, Section 3.1] for the notions of (connected) Hodge datum and morphism of (connected) Hodge data, connected Hodge varieties and Hodge morphisms of connected Hodge varieties. Notice that any connected Hodge variety is in particular an arithmetic quotient and that any Hodge morphism of connected Hodge varieties is in particular a morphism of arithmetic quotients.

A special subvariety Y of the connected Hodge variety $S_{\Gamma,G,M}$ is by definition the image $Y := f(S_{\Gamma',G',M'})$ of some Hodge morphism $f : S_{\Gamma',G',M'} \longrightarrow S_{\Gamma,G,M}$. It follows from Theorem 1.1(2) and the remark above that any special subvariety of $S_{\Gamma,G,M}$ is an $\mathbb{R}_{\mathrm{an,exp}}$-definable subset of $S_{\Gamma,G,M}$ (endowed with its $\mathbb{R}_{\mathrm{an}}$-structure of Theorem 1.1(1). The Hodge locus $\mathrm{HL}(S_{\Gamma,G,M})$ is defined as the (countable) union of special subvarieties of $S_{\Gamma,G,M}$.

The Hodge locus $\mathrm{HL}(S,\mathbb{V})$ coincides with the preimage $\Phi_S^{-1}(\mathrm{HL}(S_{\Gamma,G,M}))$. Hence to prove Theorem 1.6 we are reduced to proving that the preimage $W := \Phi^{-1}(Y)$ of any special subvariety $Y \subset S_{\Gamma,G,M}$ is an algebraic subvariety of S. By Theorem 1.3 the period map $\Phi_S : S \longrightarrow S_{\Gamma,G,M}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. As $Y \subset S_{\Gamma,G,M}$ is an $\mathbb{R}_{\mathrm{an,exp}}$-definable subset of $S_{\Gamma,G,M}$ it follows that $W = \Phi_S^{-1}(Y)$ is an $\mathbb{R}_{\mathrm{an,exp}}$-definable subset of S (in particular has finitely many connected components). As W is also a complex analytic subvariety, the o-minimal Chow Theorem 4.13 of Peterzil-Starchenko implies that W is an algebraic subvariety of S, which finishes the proof of Theorem 1.6.

□

Appendix A. Real analytic manifolds with corners and definability

A.1. **Real analytic manifolds with corners.** From the analytic point of view, the class of real analytic manifolds with corners is natural: a compact real-analytic manifold with corners is the real version of the compactification of a complex analytic manifold by a normal crossing divisor. However this class of manifolds has been poorly studied and even their definition is not universally agreed. We use the one given by [Dou61], which has been clarified and developed in [Joy12]. For the convenience of the reader we recall the basic definitions but we refer to [Joy12] for more details. Notice that Joyce works in the C^∞ context, but all the definitions we need translate literally to the real-analytic setting by replacing "smooth" with "real-analytic".

Let X be a paracompact Hausdorff topological space X and $n \geq 1$ an integer. An n-dimensional chart with corners on X is a pair (U, φ) where U is an open subset in $\mathbb{R}^n_k := \mathbb{R}^k_{\geq 0} \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$ and $\varphi : U \longrightarrow X$ is a homeomorphism with a non-empty open set $\varphi(U)$.

Given $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ and $\alpha : A \longrightarrow B$ continuous, we say that α is real-analytic if it extends to a real-analytic map between open neighborhoods of A, B.

Two n-dimensional charts with corners (U, φ), (V, ψ) on X are said real-analytically compatible if $\psi^{-1} \circ \varphi : \varphi^{-1}(\varphi(U) \cap \psi(V)) \longrightarrow \psi^{-1}(\varphi(U) \cap \psi(V))$ is a homeomorphism and $\psi^{-1} \circ \varphi$ (resp. its inverse) are real-analytic in the sense above.

An n-dimensional real analytic atlas with corners for X is a system $\{(U_i, \varphi_i) : i \in I\}$ of pairwise real-analytically compatible charts with corners on X with $X = \bigcup_{i \in I} \varphi_i(U_i)$. We call such an atlas maximal if is not a proper subset of any other atlas. Any atlas is contained in a unique maximal atlas: the set of all charts with corners (U, φ) on X compatible with (U_i, φ_i) for all $i \in I$.

A real-analytic manifold with corners of dimension n is a paracompact Hausdorff topological X equipped with a maximal n-dimensional real-analytic atlas with corners.

Weakly real-analytic maps between real-analytic manifolds with corners are the continuous maps which are real-analytic in charts (cf. [Joy12, def. 3.1], where a stronger notion of real-analytic map is also defined; we won't need this strengthened notion).

Given X a real-analytic n-manifold with corners, one defines its boundary ∂X (cf. [Joy12, def. 2.6]. This is a real-analytic n-manifold with corners for $n > 0$, endowed with an immersion (not necessarily injective) $i_X : \partial X \longrightarrow X$ (cf. [Joy12, prop.2.7]) which is real-analytic ([Joy12, Theor. 3.4.(iv)]) in particular weakly real-analytic.

A.2. **$\mathcal{R}$-definable manifolds with corners.** Let $\mathcal{R}$ be any fixed o-minimal expansion of $\mathbb{R}$. The notion of $\mathcal{R}$-definable manifold is given in [VDD98, chap.10] and in [VdM96, p.507]. We will need the extended notion of $\mathcal{R}$-definable manifold with corners.

Let X be a paracompact Hausdorff topological space X. An n-dimensional chart with corners (U, φ) on X is said to be $\mathcal{R}$-definable if U is an $\mathcal{R}$-definable subset of $\mathbb{R}^n$ (equivalently: of $\mathbb{R}^n_k$).

Two n-dimensional $\mathcal{R}$-definable charts with corners (U, φ), (V, ψ) on X are said $\mathcal{R}$-compatible if $\psi^{-1} \circ \varphi : \varphi^{-1}(\varphi(U) \cap \psi(V)) \longrightarrow \psi^{-1}(\varphi(U) \cap \psi(V))$ is an $\mathcal{R}$-definable homeomorphism between $\mathcal{R}$-definable subsets $\varphi^{-1}(\varphi(U) \cap \psi(V))$ and $\psi^{-1}(\varphi(U) \cap \psi(V))$ of $\mathbb{R}^n$.

An n-dimensional $\mathcal{R}$-definable atlas with corners for X is a system $\{(U_i, \varphi_i) : i \in I\}$, I *finite*, of pairwise $\mathcal{R}$-compatible $\mathcal{R}$-definable charts with corners on X with $X = \cup_{i \in I} \varphi_i(U_i)$. Two such atlases $\{(U_i, \varphi_i) : i \in I\}$ and$\{(V_j, \psi_j) : j \in J\}$ are said $\mathcal{R}$-equivalent if all the "mixed" transition maps $\psi_j \circ \varphi_i^{-1}$ are $\mathcal{R}$-definable.

An $\mathcal{R}$-definable manifold with corners of dimension n is a paracompact Hausdorff topological X equipped with an $\mathcal{R}$-equivalence class of n-dimensional $\mathcal{R}$-definable atlas with corners.

Remark A.1. Notice that the definitions of real-analytic manifold with corners and $\mathcal{R}$-definable manifold with corners are parallel, except the crucial fact that we work in a strictly finite setting for $\mathcal{R}$-definable manifolds: the set I of charts has to be finite. This finiteness condition, in addition to the definability condition, ensures the tameness at infinity of the $\mathcal{R}$-definable manifolds with corners.

We say that a subset $Z \subset X$ is $\mathcal{R}$-definable (resp. open or closed) if $\varphi_i^{-1}(Z \cap \varphi_i(U_i))$ is an $\mathcal{R}$-definable (resp. open or closed) subset of U_i for all $i \in I$. An $\mathcal{R}$-definable map between $\mathcal{R}$-definable manifolds (with corners) is a map whose graph is an $\mathcal{R}$-definable subset of the $\mathcal{R}$-definable product manifold (with corners).

A.3. **Compact real-analytic manifolds with corners are $\mathbb{R}_{\mathrm{an}}$-definable.**

Proposition A.2. *Let X be a compact real-analytic n-manifold with corners. Then X has a natural structure of $\mathbb{R}_{\mathrm{an}}$-definable manifold with corners. Moreover the map $i_X : \partial X \longrightarrow X$ is $\mathbb{R}_{an}$-definable. In particular the interior $X \setminus i_X(\partial X)$ is an $\mathbb{R}_{\mathrm{an}}$-definable manifold.*

Proof. For each point x of X choose $\varphi_x : U_x \longrightarrow (X, x)$ a real-analytic chart with corners whose image $\varphi(U_x)$ is a neighborhood of x. Without loss of generality we can assume that $U_x \subset \mathbb{R}^n_k$ is relatively compact and semi-analytic, hence $\mathbb{R}_{\mathrm{an}}$-definable. Hence (U_x, φ_x) is a real-analytic chart with corners for X which is also an $\mathbb{R}_{\mathrm{an}}$-definable chart with corners for X.

Fix x, y two points in X. The fact that the two real-analytic charts (U_x, φ_x) and (U_y, φ_y) are real-analytically compatible implies immediately that they are $\mathbb{R}_{\mathrm{an}}$-compatible.

The space X is compact hence one can extract from the covering family $\{(U_x, \varphi_x),\ x \in X\}$ a finite subfamily $\{(U_i, \varphi_i),\ i \in I\}$, such that $X = \bigcup_{i \in I} \varphi_i(U_i)$: this is an n-dimensional $\mathbb{R}_{\mathrm{an}}$-definable atlas with corners for X, which defines a structure of $\mathbb{R}_{\mathrm{an}}$-definable manifold with corners on X.

One easily checks that this structure is independent of the choice of the finite extraction $\{(U_i, \varphi_i),\ i \in I\}$ of $\{(U_x, \varphi_x),\ x \in X\}$, and also of the choice of the relatively compact and semi-analytic subsets U_x.

Hence X has a natural structure of $\mathbb{R}_{\mathrm{an}}$-definable manifold with corners. The same procedure endows the compact real-analytic $(n-1)$-manifold with corners ∂X with a natural $\mathbb{R}_{\mathrm{an}}$-definable structure. The fact that $i_X : \partial X \longrightarrow X$ is weakly real-analytic implies immediately that i_X is $\mathbb{R}_{\mathrm{an}}$-definable and that the manifold $X \setminus i_X(\partial X)$ is $\mathbb{R}_{\mathrm{an}}$-definable. $\square$

References

[AK81] S. Akbulut, H.C. King, *The topology of real algebraic sets with isolated singularities*, Ann. of Math. **113** (1981), no. 3, 425-446

[AMRT75] A. Ash, D. Mumford, M. Rapoport, Y. Tai, *Smooth Compactification of Locally Symmetric varieties*, Lie Groups: History Frontiers and applications vol. **4**. Math Sci Press. (1975).

[BB66] W.L. Baily, A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Annals of Math., **84** (1966), 442-528

[BaT17] B. Bakker, J. Tsimerman, *The Ax-Schanuel conjecture for variations of Hodge structures*, arxiv.org/abs/1712.05088v1

[Bor69] A. Borel, *Introduction aux groupes arithmétiques*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341 Hermann, Paris (1969)

[Bor72] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry **6** (1972), 543-560

[B-HC62] A. Borel, Harish-Chandra, *Arithmetic subgroups of algebraic groups*, *Ann. of Math.* **75** (1962), 485-535

[BJ06a] , A. Borel, L.Ji, *Compactifications of locally symmetric spaces*, J. Differential Geometry **73**, (2006), 263-317

[BJ06b] A. Borel, L.Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkäuser Boston (2006)

[BS73] A. Borel, J.P. Serre, *Corners and arithmetic groups*, Comm. Math. Helv. **4** (1973), 436-491

[BT65] A. Borel, J.Tits, *Groupes réductifs*, Publ.Math. IHES **27** (1965), 55-150

[BP09-1] P. Brosnan, G. Pearlstein, *Zero loci of admissible normal functions with torsion singularities*, Duke Math. J. **150** (2009) no.1, 77-100

[BP09-2] P. Brosnan, G. Pearlstein, *The zero locus of an admissible normal function*, Ann. of Math. (2) **170** (2009), no. 2, 883-897

[BP13] P. Brosnan, G. Pearlstein, *On the algebraicity of the zero locus of an admissible normal function*, Compos. Math. **149** (2013), no. 11, 1913-1962.

[BPS10] P.Brosnan, G. Pearlstein, C. Schnell, *The locus of Hodge classes in an admissible variation of mixed Hodge structure*, C.R. Math. Acad. Sci. Paris **348** (2010), no. 11-12, 657-660

[CK82] E. Cattani, A. Kaplan, *Polarized Mixed Hodge structures and the monodromy of a variation of Hodge structure*, Invent. Math. **67**, 101-115 (1982)

[CDK95] E. Cattani, P. Deligne, A. Kaplan, *On the locus of Hodge classes.* J. of AMS, **8** (1995), 483-506

[CKS86] E. Cattani, A. Kaplan, W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), no. 3, 457-535.

[CKS87] E. Cattani, A. Kaplan, W. Schmid, *Variations of polarized Hodge structure: asymptotics and monodromy*, Hodge theory (Sant Cugat 1985), 16-31, Lecture Notes in Math. **1246** Springer (1987)

[Che51] C. Chevalley, *Théorie des groupes de Lie*, Tome 2, Hermann (1951)

[De74] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES **44** (1974), 5-77

[Dou61] A. Douady, *Variétés à bord anguleux et voisinages tubulaires*, Séminaire Henri Cartan 1961/1962, Exp. 1

[VDD98] L. van den Dries, *Tame Topology and o-minimal structures.* LMS lecture note series, 248. Cambridge University Press, 1998.

[VdM94] L. van den Dries, C. Miller *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), 19–56.

[VdM96] L. van den Dries, C. Miller *Geometric categories and o-minimal structures*, Duke Math. J. **84** no.2, (1996), 497-540.

[FL81] E. Fortuna, S. Łojasiewicz, *Sur l'algébricité des ensembles analytiques complexes*, J. Reine Angew. Math. **329** (1981) 215-220

[Gao15] Z. Gao, *Towards the André-Oort conjecture for mixed Shimura varieties: the Ax-Lindemann theorem and lower bounds for Galois orbits of special points*, J. Reine Angew. Math. **732** (2017) 85-146

[Ga68] A.M Gabrielov, *Projections of semi-analytic sets*, Funkt. Ana. i Prilozen **2** (1968), 18-30

[GM03] M. Goresky, R. Macpherson, *The topological trace formula*, J. Reine Angew. Math. 560 (2003), 77-150

[GGLR17] M. Green, P. Griffiths, R. Laza, C. Robles, *Completion of period mappings and ampleness of the Hodge bundle*, arXiv:1708.09523

[Gro] A. Grothendieck, *Esquisse d'un programme*, published in London Math. Soc. Lecture Note Ser. **242**, Geometric Galois actions, 1, 5-48, Cambridge Univ. Press, Cambridge

[Jaf75] H. A. Jaffee, *Real forms of hermitian symmetric spaces*, Bull. Amer. Math. Soc. **81** (1975), 456-458

[Jaf78] H.A. Jaffee, *Anti-holomorphic automorphisms of the exceptional symmetric domains*, J. Differential Geom. **13** (1978), no.1, 79-86

[Joy12] D. Joyce, *On manifolds with corners*, Advances in geometric analysis, Adv. Lect. Math. **21** 225-258, Int. Press, Somerville, MA, 201

[Ka85] M. Kashiwara, *The asymptotic behavior of a variation of polarized Hodge structure*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 4, 853-875

[KNU08] K. Kato, C. Nakayama, S. Usui, *SL(2)-orbit theorem for degeneration of mixed Hodge structure*, J. of Algebraic Geom. **17** (2008), no. 3, 401-479

[KNU11] K. Kato, C. Nakayama, S. Usui, *Analyticity of the closures of some Hodge theoretic subspaces*, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), no. 9, 167-172

[KU00] K. Kato, S. Usui, *Borel-Serre spaces and spaces of SL(2)-orbits*, in *Algebraic geometry 2000*, Adv. Stud. Pure Math., **36**, Math. Soc. Japan, Tokyo (2002)

[K90] H. Klingen, *Introductory lectures on Siegel modular forms*, Cambridge Studies in Advanced Mathematics, 20. Cambridge University Press, Cambridge, 1990.

[KUY16] B. Klingler, E. Ullmo, A. Yafaev, *The hyperbolic Ax-Lindemann-Weierstraß conjecture*, Publ.Math. IHES **123** (2016), 333-360

[KUY17] B. Klingler, E. Ullmo, A. Yafaev, *bi-algebraic geometry and the André-Oort conjecture*, to appear in *Proceedings of 2015 AMS Summer Institute in Algebraic Geometry*

[K17] B. Klingler, *Hodge loci and atypical intersections: conjectures*, http://arxiv.org/abs/1711.09387, accepted for publication in the book *Motives and complex multiplication*, Birkhaüser

[Le79] D. S. P. Leung, *Reflective submanifolds. IV. Classification of real forms of Hermitian symmetric spaces*, J. Differential Geom. **14** (1979), no.2., 179-185.

[MPT17] N. Mok, J. Pila, J. Tsimerman, *Ax-Schanuel for Shimura varieties*, arXiv:1711.02189

[Na52] J. Nash, *Real algebraic manifolds*, Ann. of Math. **56** (1952), 405-421

[OV90] A.L Onishchick, E.B. Vinberg, *Lie groups and algebraic groups*, Springer Series in Soviet Mathematics, Springer-Verlag (1990)

[O17] M. Orr, *Height bounds and the Siegel property*, Algebra Number Theory **12** (2018), no. 2, 455-478.

[PS13] Y. Peterzil, S. Starchenko, *Definability of restricted theta functions and families of abelian varieties*, Duke Math. J. **162**, (2013), 731-765

[PS09] Y. Peterzil, S. Starchenko, *Complex analytic geometry and analytic-geometric categories*, J. reine angew. Math. 626 (2009), 39-74

[P11] J. Pila, *O-minimality and the Andre-Oort conjecture for* $\mathbb{C}^n$, Annals Math. **173** (2011), 1779-1840.

[PT14] J. Pila, J. Tsimerman, *Ax-Lindemann for* $\mathcal{A}_g$. Ann. of Math **179** (2014), 659-681

[Rag68] M.S. Raghunathan, *A note on quotients of real algebraic groups by arithmetic subgroups*, Invent. Math. **4**, (1967/1968), 318-335

[Rag72] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Math. und Ihrer Grenzgebiete **68**, Springer-Verlag (1972)

[Rat91-0] M. Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. **134** (1991), no.3, 545-607

[Rat91-1] M. Ratner, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J., **63** (1991), no.1, 235-280

[Sc18] T. Scanlon, *Algebraic differential equations from covering maps*, Adv. Math. **330** (2018), 1071-1100

[Sc73] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211-319

[Sch75] G.W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63-68

[Se56] J.P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1955-56), 1-42

[So78] A.J.Sommese, *On the rationality of the period mapping*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 683-717

[Ts18] J. Tsimerman, *The André-Oort conjecture for* $\mathcal{A}_g$, Annals of Math. (2) **187** (2018), no. 2, 379-390

[UY14] E. Ullmo, A. Yafaev, *Hyperbolic Ax-Lindemann theorem in the cocompact case,* Duke Math. J. **163** (2014) 433-463

[Tog73] A. Tognoli, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Pisa **27** (1973), 167-185

[W46] H. Weyl, *Classical Groups*, Princeton University Press, Princeton, N.J., 1946.

[Wil96] J. Wilkie, *Model completeness results for expansions of the ordered field or real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), no.4, 1051-1094

Benjamin Bakker : University of Georgia
email : `bakker.uga@gmail.com`.

Bruno Klingler : Humboldt Universität zu Berlin
email : `bruno.klingler@hu-berlin.de`.

Jacob Tsimerman : University of Toronto
email : `jacobt@math.toronto.edu`.