ON THE FIELDS OF DEFINITION OF HODGE LOCI

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ABSTRACT. For $\mathbb{V} \to S$ a polarizable variation of Hodge structure defined over $\overline{\mathbb{Q}}$, the special subvarieties of S on which $\mathbb{V}$ admits exceptional Hodge tensors are conjectured to be defined over $\overline{\mathbb{Q}}$. We prove this conjecture for special subvarieties satisfying a simple monodromy condition, and illustrate this result for the universal family of smooth hypersurfaces of fixed degree in projective space. Using the same ideas, we moreover reduce the conjecture for special subvarieties of arbitrary dimension to the conjecture for special points.

RÉSUMÉ. Étant donnée une variation polarisable de structures de Hodge $\mathbb{V} \to S$ définie sur $\overline{\mathbb{Q}}$, il est conjecturé que les sous-variétés spéciales de S le long desquelles $\mathbb{V}$ admet des tenseurs de Hodge exceptionnels sont définies sur $\overline{\mathbb{Q}}$. Nous démontrons cette conjecture pour les sous-variétés spéciales satisfaisant une condition simple de monodromie, et illustrons ce résultat dans le cas de la famille universelle des hypersurfaces lisses de degré fixé dans l'espace projectif. En utilisant les mêmes méthodes, nous réduisons la conjecture au cas particulier des points spéciaux.

1. INTRODUCTION

1.1. **Hodge loci.** The main objects of study in this article are Hodge loci. Let us start by recalling their definition in the geometric case, where their behaviour is predicted by the Hodge conjecture.

Let $f : X \to S$ be a smooth projective morphism of smooth irreducible complex quasi-projective varieties and let i a positive integer. The Betti and De Rham incarnations of the $2i$-th cohomology of the fibers of f give rise to a weight zero polarizable variation of Hodge structure $(\mathbb{V} := R^{2i} f^{\mathrm{an}}_* \mathbb{Z}(i), \mathcal{V} := R^{2i} f_* \Omega^\bullet_{X/S}, F^\bullet, \nabla)$ on S. Here $\mathbb{V}$ is the local system on the complex manifold S^{an} associated to S parametrising the $2i$-th Betti cohomology of the fibers of f; $\mathcal{V}$ is the corresponding algebraic vector bundle, endowed with its flat Gauß-Manin connection; and $F^\bullet$ is the Hodge filtration on $\mathcal{V}$ induced by the stupid filtration on the algebraic De Rham complex $\Omega^\bullet_{X/S}$. In this situation one defines the locus of exceptional Hodge classes $\mathrm{Hod}(\mathcal{V}) \subset \mathcal{V}^{\mathrm{an}}$ as the set of Hodge classes $\lambda \in F^0\mathcal{V}^{\mathrm{an}} \cap \mathbb{V}_{\mathbb{Q}}$ whose orbit under monodromy is infinite, and the Hodge locus $\mathrm{HL}(S, \mathbb{V})$ as its projection in S^{an}. Thus $\mathrm{HL}(S, \mathbb{V})$ is the subset of points s in S^{an} for which the Hodge structure $H^{2i}(X_s, \mathbb{Z}(i))$ admits more Hodge classes than the very general fiber $H^{2i}(X_{s'}, \mathbb{Z}(i))$.

According to the Hodge conjecture each $\lambda \in \mathrm{Hod}(\mathcal{V})$ should be the cycle class of an exceptional algebraic cycle in the corresponding fiber of f. As algebraic subvarieties of the fibers are parametrised by a common Hilbert scheme, the Hodge conjecture and an easy countability argument implies the following (as noticed by Weil in [Weil79], where he asks for an unconditional proof):

($\star$) The locus of Hodge classes $\mathrm{Hod}(\mathcal{V})$ is a countable union of closed irreducible algebraic subvarieties of $\mathcal{V}$. The restriction of f to any such subvariety of $\mathcal{V}$ is finite over its image. In particular the Hodge locus $\mathrm{HL}(S, \mathbb{V})$ is a countable union of closed irreducible algebraic subvarieties of S.

More generally let $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ be any polarizable variation of $\mathbb{Z}$-Hodge structure ($\mathbb{Z}$VHS) on a smooth complex irreducible algebraic variety S. Thus $\mathbb{V}$ is a finite rank $\mathbb{Z}_{S^{\mathrm{an}}}$-local system on the complex manifold S^{an}; and $(\mathcal{V}, F^\bullet, \nabla)$ is the unique regular algebraic module with integrable connection on S whose analytification is $\mathbb{V} \otimes_{\mathbb{Z}_{S^{\mathrm{an}}}} \mathcal{O}_{S^{\mathrm{an}}}$ endowed with its Hodge filtration $F^\bullet$ and the holomorphic flat connection ∇^{an} defined by $\mathbb{V}$, see [Sc73, (4.13)]). We will abbreviate the $\mathbb{Z}$VHS $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ simply by $\mathbb{V}$. If we define the locus of exceptional Hodge classes $\mathrm{Hod}(\mathcal{V}) \subset \mathcal{V}$ and the Hodge locus $\mathrm{HL}(S, \mathbb{V}) \subset S$ as in the geometric case, Cattani, Deligne and Kaplan [CDK95] proved a vast generalisation of Weil's expectation:

Theorem 1.1. *(Cattani-Deligne-Kaplan) Let $\mathbb{V}$ be a polarizable $\mathbb{Z}$VHS on a smooth complex quasi-projective variety S. Then ($\star$) holds true.*

From now on we do not distinguish a complex algebraic variety X from its associated complex analytic space X^{an}, the meaning being clear from the context. It will be convenient for us to work in the following more general tensorial setting. Let $\mathbb{V}^\otimes$ be the infinite direct sum of $\mathbb{Z}$VHS $\bigoplus_{a,b\in\mathbb{N}} \mathbb{V}^{\otimes a} \otimes (\mathbb{V}^\vee)^{\otimes b}$, where $\mathbb{V}^\vee$ denotes the $\mathbb{Z}$VHS dual to $\mathbb{V}$; and let $(\mathcal{V}^\otimes, F^\bullet)$ be the corresponding filtered algebraic vector bundle of infinite rank. We denote by $\mathrm{Hod}(\mathcal{V}^\otimes) \subset \mathcal{V}^\otimes$ and $\mathrm{HL}(S, \mathbb{V}^\otimes) \subset S$ the corresponding locus of Hodge tensors and the tensorial Hodge locus respectively. Thus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is the subset of points s in S^{an} for which the Hodge structure $\mathbb{V}_s$ admits more Hodge *tensors* than the very general fiber $\mathbb{V}_{s'}$, equivalently where the Mumford-Tate group $\mathbf{G}_s$ of $\mathbb{V}_s$ is not of maximal dimension. Theorem 1.1 says that $\mathrm{Hod}(\mathcal{V}^\otimes)$ and $\mathrm{HL}(S, \mathbb{V}^\otimes)$ are countable unions of closed irreducible subvarieties of $\mathcal{V}^\otimes$ and S respectively, called *the special subvarieties of $\mathcal{V}^\otimes$ and S for* $\mathbb{V}$. We refer to [BKT18] for a simplified proof of the statement for $\mathrm{HL}(S, \mathbb{V}^\otimes)$ using o-minimal geometry.

1.2. **Fields of definition of Hodge loci.** The question we attack in this paper is the relation between the field of definition of the $\mathbb{Z}$VHS $\mathbb{V}$ and the fields of definition of the corresponding special subvarieties.

Once again the geometric case provides us with a motivation and a heuristic. Suppose that $f : X \to S$ is defined over a number field $L \subset \mathbb{C}$. In that case one easily checks, refining Weil's argument, that the Hodge conjecture implies, in addition to ($\star$):

($\star\star$) (a) each irreducible component of $\mathrm{Hod}(\mathcal{V})$, respectively $\mathrm{HL}(S, \mathbb{V})$, is defined over a finite extension of L;
(b) each of the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-conjugates of such a component is again an irreducible component of $\mathrm{Hod}(\mathcal{V})$, respectively $\mathrm{HL}(S, \mathbb{V})$.

Remark 1.2. Of course ($\star\star$) for $\mathrm{Hod}(\mathcal{V})$ implies ($\star\star$) for $\mathrm{HL}(S, \mathbb{V})$, and is a priori strictly stronger.

Remark 1.3. The full Hodge conjecture is not needed to deduce ($\star\star$). As proven by Voisin [Voi07, Lemma 1.4], the property ($\star\star$) for $\mathrm{Hod}(\mathcal{V})$ is equivalent to the conjecture that Hodge classes in the fibers of f are (de Rham) absolute Hodge classes. We won't use the notion of absolute Hodge classes in this article and refer the interested reader to [ChSc14] for a survey. Our methods, being primarily concerned with the geometric properties of the special subvarieties themselves, say little directly about Hodge classes.

Let us now turn to general $\mathbb{Z}$VHS.

Definition 1.4. *We say that a $\mathbb{Z}$VHS $\mathbb{V}$ is* defined over a number field $K \subset \mathbb{C}$ *if S, $\mathcal{V}$, $F^\bullet$ and ∇ are defined over K: $S = S_K \otimes_K \mathbb{C}$, $\mathcal{V} = \mathcal{V}_K \otimes_K \mathbb{C}$, $F^\bullet\mathcal{V} = (F^\bullet_K \mathcal{V}_K) \otimes_K \mathbb{C}$ and $\nabla = \nabla_K \otimes_K \mathbb{C}$ with the obvious compatibilities.*

In the same way the property $(\star)$, which is implied by the Hodge conjecture in the geometric case, was proven to be true for a general $\mathbb{Z}$VHS, we expect the property $(\star\star)$, which is implied by the Hodge conjecture in the geometric case, to hold true for any $\mathbb{Z}$VHS $\mathbb{V}$, namely:

Conjecture 1.5. *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS defined over a number field $L \subset \mathbb{C}$. Then:*

(a) any special subvariety of $\mathcal{V}^\otimes$ (resp. of S) for $\mathbb{V}$ is defined over a finite extension of L;

(b) any of the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-conjugates of a special subvariety of $\mathcal{V}^\otimes$ (resp. of S) for $\mathbb{V}$ is a special subvariety of $\mathcal{V}^\otimes$ (resp. of S) for $\mathbb{V}$.

Remark 1.6. Simpson's non-abelian period conjecture [Si90, "Standard conjecture" p.372] predicts that any $\mathbb{Z}$VHS defined over a number field $L \subset \mathbb{C}$ ought to be motivic: there should exist a $\overline{\mathbb{Q}}$-Zariski-open subset $U \subset S$ such that the restriction of $\mathbb{V}$ to U is a direct factor of a geometric $\mathbb{Z}$VHS on U. Thus Conjecture 1.5 would follow from Simpson's "standard conjecture" and $(\star\star)$ in the geometric case. Of course Simpson's standard conjecture seems unreachable with current techniques.

Let us mention the few results in the direction of Conjecture 1.5 we are aware of:

Suppose we are in the geometric situation of a morphism $f : X \to S$ defined over $\mathbb{Q}$. In [Voi07, Theor. 0.6] (see also [Voi13, Theor. 7.8]), Voisin proves the following:

(1) for $\mathrm{Hod}(\mathcal{V})$: let $Z \subset \mathcal{V}$ is an irreducible component of $\mathrm{Hod}(\mathcal{V})$ through a Hodge class $\alpha \in H^{2k}(X_0, \mathbb{Z}(k))_{\mathrm{prim}}$ such that the only constant sub-$\mathbb{Q}$VHS of the base change of $\mathbb{V}_\mathbb{Q}$ to Z is $\mathbb{Q} \cdot \alpha$. Then Z is defined over $\overline{\mathbb{Q}}$.

(2) for $\mathrm{HL}(S, \mathbb{V})$: Let Z be as in (1). Under the weaker assumption that any constant sub-$\mathbb{Q}$VHS of the base change of $\mathbb{V}_\mathbb{Q}$ to Z is purely of type $(0, 0)$, the projection of Z in S is an irreducible component of $\mathrm{HL}(S, \mathbb{V})$ defined over $\overline{\mathbb{Q}}$, and its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-translates are still irreducible components of $\mathrm{HL}(S, \mathbb{V})$.

In the case of a general $\mathbb{Z}$VHS Saito and Schnell [SaSc16] prove:

(1) for $\mathrm{Hod}(\mathcal{V})$: if $\mathbb{V}$ is defined over a number field then a special subvariety of $\mathcal{V}$ for $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$ if it contains a $\overline{\mathbb{Q}}$-point of $\mathcal{V}$.

(2) for $\mathrm{HL}(S, \mathbb{V}^\otimes)$: without assuming that $\mathcal{V}$ is defined over $\overline{\mathbb{Q}}$ but only assuming that S is defined over a number field L, then a special subvariety of S for $\mathbb{V}$ is defined over a finite extension of L if and only if it contains a $\overline{\mathbb{Q}}$-point of S. This generalizes the well-known fact that the special subvarieties of Shimura varieties are defined over $\overline{\mathbb{Q}}$ (as any special subvariety of a Shimura variety contains a CM-point, and CM-points are defined over $\overline{\mathbb{Q}}$).

Remark 1.7. These results seem to indicate a significant gap in difficulty between Conjecture 1.5 for $\mathrm{Hod}(\mathcal{V})$ and Conjecture 1.5 for $\mathrm{HL}(S, \mathbb{V})$. Saito and Schnell's result (2), which only requires S to be defined over $\overline{\mathbb{Q}}$, looks particularly surprising. They also seem to indicate that the statement (b) in Conjecture 1.5 goes deeper than (a). In particular Saito and Schnell's result (2) says nothing about Galois conjugates.

Remark 1.8. Voisin's and Saito-Schnell's criteria look difficult to check in practice. Even in explicit examples one usually knows very little about the geometry of a special variety Y. In Voisin's case one would need to control the Hodge structure on the cohomology of a smooth compactification of X base-changed to Z. In Saito-Schnell's case there is in general no natural source of $\overline{\mathbb{Q}}$-points (like the CM points in the Shimura case).

1.3. **Results.** All results in this paper concern Conjecture 1.5 for $\mathrm{HL}(S, \mathbb{V}^{\otimes})$. We provide a simple geometric criterion for a special subvariety of S for $\mathbb{V}$ to be defined over $\overline{\mathbb{Q}}$ and its Galois conjugates to be special. As this criterion is purely geometric we say nothing about fields of definitions of isolated points in the Hodge locus. In fact our Theorem 1.16 will reduce Conjecture 1.5 to its particular case for special points.

Let us first recall the notion of algebraic monodromy group.

Definition 1.9. *Let S be a smooth irreducible complex algebraic variety, let k be a field and $\mathbb{V}$ a k-local system (of finite rank) on S^{an} (in our case k will be $\mathbb{Q}$ or $\mathbb{C}$). Given an irreducible closed subvariety $Y \subset S$, the algebraic monodromy group $\mathbf{H}_Y$ of Y for $\mathbb{V}$ is the connected component of the identity of the Tannaka group of the Tannakian category $\langle \mathbb{V}_{|Y^{\mathrm{nor}}} \rangle^{\otimes}_{k\mathrm{Loc}}$ of k-local systems on the normalisation Y^{nor} of Y tensorially generated by the restriction of $\mathbb{V}$ and its dual.*

Equivalently $\mathbf{H}_Y$ is the connected component of the Zariski-closure of the monodromy $\rho : \pi_1(Y^{\mathrm{nor,an}}) \to \mathbf{GL}(V_k)$ of the local system $\mathbb{V}_{|Y^{\mathrm{nor}}}$.

Theorem 1.10. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety S, whose adjoint generic Mumford-Tate group $\mathbf{G}_S^{\mathrm{ad}}$ is simple. Then:*

(a) if S is defined over a number field L then any strict maximal special subvariety $Y \subset S$ satisfying $\mathbf{H}_Y \neq \{1\}$ is defined over $\overline{\mathbb{Q}}$.

(b) if $\mathbb{V}$ is moreover defined over L then the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-translates of such a special subvariety are special subvarieties of S for $\mathbb{V}$.

As an easy geometric illustration of Theorem 1.10 we obtain for instance:

Corollary 1.11. *Let $\mathbb{P}_{\mathbb{C}}^{N(n,d)}$ be the projective space parametrising the hypersurfaces X of $\mathbb{P}_{\mathbb{C}}^{n+1}$ of degree d. Let $U_{n,d} \subset \mathbb{P}_{\mathbb{C}}^{N(n,d)}$ be the Zariski-open subset parametrising the smooth hypersurfaces X and let $\mathbb{V} \longrightarrow U$ be the polarized variation of $\mathbb{Z}$-Hodge structure corresponding to the primitive cohomology $H^n(X, \mathbb{Z})_{\mathrm{prim}}$. Then any strict maximal special subvariety $Y \subset U_{n,d}$ for $\mathbb{V}$ with algebraic monodromy group $\mathbf{H}_Y \neq \{1\}$ is defined over $\overline{\mathbb{Q}}$; moreover its Galois conjugates are special.*

Remark 1.12. In Corollary 1.11 we can more generally replace $U_{n,d}$ with the space $U_{n,\mathbf{d}}$, with $U_{n,\mathbf{d}}$ the open subset of $\mathbb{P}_{\mathbb{C}}^{N_1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{N_r}$ parametrising smooth complete intersections of r hypersurfaces of degrees $\mathbf{d} = (d_1, \cdots, d_r)$ in $\mathbb{P}_{\mathbb{C}}^{n+1}$.

Theorem 1.10 is obtained as a corollary of a more general result, where we replace the condition $\mathbf{H}_Y \neq 1$ with a more general one:

Definition 1.13. *Let S be a smooth irreducible complex algebraic variety and $\mathbb{V}$ a k-local system on S^{an}. Let $Y \subset S$ be an irreducible closed subvariety. We say that Y is* weakly non-factor for $\mathbb{V}$ *if it is not contained in a closed irreducible $Z \subset S$ such that the k-algebraic monodromy group $\mathbf{H}_Y$ is a strict normal subgroup of $\mathbf{H}_Z$.*

Admittedly in the situation of Definition 1.13 it is not easy to decide whether or not a given irreducible closed subvariety $Y \subset S$ is weakly non-factor for $\mathbb{V}$. As explained in Section 2 the situation is much better when $\mathbb{V}$ is a $\mathbb{Z}$VHS and $Y \subset S$ is special for $\mathbb{V}$: in that case Y being non-factor roughly means that Y cannot be non-trivially Hodge-theoretically deformed inside a larger special subvariety.

The main result in this paper, from which Theorem 1.10 is deduced, is then the following:

Theorem 1.14. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety S.*

(a) if S is defined over a number field L then any special subvariety of S for $\mathbb{V}$ which is weakly non-factor for $\mathbb{V}_{\mathbb{Q}}$ is defined over a finite extension of L;
(b) if moreover $\mathbb{V}$ is defined over L then the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-translates of such a special subvariety are also special, weakly non-factor subvarieties of S for $\mathbb{V}$.

Remark 1.15. In the situation of Theorem 1.10[(b)] and more generally Theorem 1.14[(b)] we expect that the generic Mumford-Tate group remains constant in the Galois orbit of the special subvarieties we consider. However we cannot prove it. This illustrates how our method, which is not directly related to absolute Hodge classes, is different from Voisin's.

As another application of the ideas of Theorem 1.14, we are able to reduce the Conjecture 1.5(a) for $\mathrm{HL}(S, \mathbb{V})$ to the case of points:

Theorem 1.16. *Special subvarieties for $\mathbb{Z}$VHSs defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$ if and only if it holds true for special points.*

2. $\mathbb{Z}$VHS versus local systems, Mumford-Tate group versus monodromy, special versus weakly special

In this section we recall the geometric background providing the intuition for Theorem 1.14, namely the geometry of special subvarieties and their generalisation, the weakly special subvarieties. We refer to [K17] and [KO19] for details.

Let $\mathbb{V}_{\mathbb{Q}}$ be a $\mathbb{Q}$-local system on S and $Y \subset S$ an irreducible closed subvariety. In Definition 1.9 we recalled the definition of the algebraic monodromy group $\mathbf{H}_Y$ for $\mathbb{V}_{\mathbb{Q}}$. Suppose now that $\mathbb{V}_{\mathbb{Q}}$ underlies a $\mathbb{Z}$VHS $\mathbb{V}$ over S. In addition to $\mathbf{H}_Y$, which depends only on the underlying local system, one attaches a more subtle invariant to Y and $\mathbb{V}$: the generic Mumford-Tate group $\mathbf{G}_Y$ i.e. the (connected component of the identity of the) Tannaka group of the category $\langle \mathbb{V}_{|Y^{\mathrm{nor}}} \rangle^{\otimes}_{\mathbb{Q}\mathrm{VHS}}$ of $\mathbb{Q}$VHS on the normalisation of

Y tensorially generated by the restriction of $\mathbb{V}$ and its dual. One may check from the definiteness of a polarisation of the variation on Y that the monodromy on Y acts on Hodge tensors through a finite group, from which it follows that $\mathbf{H}_Y \subset \mathbf{G}_Y$. The Mumford-Tate group $\mathbf{G}_Y$ is usually much harder to compute than $\mathbf{H}_Y$ as its definition is not purely geometric. The $\mathbb{Z}$VHS $\mathbb{V}$ is completely described by its complex analytic period map $\Phi_S : S^{\mathrm{an}} \to X_S := \Gamma\backslash\mathcal{D}_S$. Here $\mathcal{D}_S$ denotes the Mumford-Tate domain associated to the generic Mumford-Tate group $\mathbf{G}_S$ of $(S, \mathbb{V})$, $\Gamma_S \subset \mathbf{G}_S(\mathbb{Q})$ is an arithmetic lattice and the complex analytic quotient X_S is called the Hodge variety associated to $\mathbb{V}$. The special subvarieties of the Hodge variety X_S and their generalisation, the weakly special subvarieties of X_S are defined purely in group-theoretic terms, see [KO19, Def. 3.1]. One proves that the special subvarieties of S for $\mathbb{V}$ are precisely the irreducible components of the Φ_S-preimage of the special subvarieties of X_S, thus obtaining the following characterisation, see [KO19, Def. 1.2].

Proposition 2.1. *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS on S. A* special subvariety *of S for $\mathbb{V}$ is a closed irreducible algebraic subvariety $Y \subset S$ maximal among the closed irreducible algebraic subvarieties of S with generic Mumford-Tate group $\mathbf{G}_Y$.*

Similarly, one defines a generalisation of the special subvarieties of X_S, the so-called *weakly special* subvarieties of X_S, purely in group-theoretic terms see [KO19, Def. 3.1]. The weakly special subvarieties of S for $\mathbb{V}$, which generalize the special ones, are defined as the irreducible components of the Φ_S-preimage of the weakly special subvarieties of X_S. Again one obtains the following characterisation, see [KO19, Cor. 3.14]:

Proposition 2.2. *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS on S. A* weakly special subvariety *$Y \subset S$ for $\mathbb{V}$ is a closed irreducible algebraic subvariety Y of S maximal among the closed irreducible algebraic subvarieties of S with algebraic monodromy group $\mathbf{H}_Y$.*

A posteriori Proposition 2.2 offers an alternative definition of the weakly special subvarieties of S for a $\mathbb{Z}$VHS $\mathbb{V}$. It is important for us to notice that this alternative definition of the weakly special subvarieties of S for $\mathbb{V}$ makes sense for $\mathbb{V}$ any k-local system on S^{an}, k a field:

Definition 2.3. *Let k be a field and let $\mathbb{V}$ be a k-local system on S. We define a* weakly special subvariety *$Y \subset S$ for $\mathbb{V}$ to be a closed irreducible algebraic subvariety Y of S maximal among the closed irreducible algebraic subvarieties of S with algebraic monodromy group $\mathbf{H}_Y$.*

Remark 2.4. If $\mathbb{V}$ is a k-local system on S, $Y \subset S$ is a closed irreducible subvariety, and k' is a field extension of k, the k'-algebraic monodromy group $\mathbf{H}_Y(\mathbb{V} \otimes_k k')$ is the base change $\mathbf{H}_Y(\mathbb{V}) \otimes_k k'$. Thus Y being weakly special for $\mathbb{V}$ is equivalent to Y being weakly special for $\mathbb{V} \otimes_k k'$.

For $\mathbb{V}$ a $\mathbb{Z}$VHS and $Y \subset S$ an irreducible closed subvariety there exists a unique weakly special subvariety $\langle Y \rangle_{\mathrm{ws}}$ with algebraic monodromy group $\mathbf{H}_Y$ and a unique special subvariety $\langle Y \rangle_{\mathrm{s}}$ with generic Mumford-Tate group $\mathbf{G}_Y$ containing Y, see [KO19, 2.1.4]:

$$Y \subset \langle Y \rangle_{\mathrm{ws}} \subset \langle Y \rangle_{\mathrm{s}} \subset S \quad .$$

When $\mathbb{V}$ is a mere local system there exists by definition a weakly special subvariety with algebraic monodromy group $\mathbf{H}_Y$ and containing Y but its uniqueness is not clear to us.

Let us now recall that for $\mathbb{V}$ a $\mathbb{Z}$VHS special subvarieties of S for $\mathbb{V}$ can be thought of as families of weakly special subvarieties. Indeed let $Y \subset S$ be a weakly special subvariety. A fundamental result of Deligne-André [An92, Theor.1] states that the group $\mathbf{H}_Y$ is normal in (the derived group of) $\mathbf{G}_Y$. Following [KO19, Prop. 2.13], the decomposition $\mathbf{G}_Y^{\mathrm{ad}} = \mathbf{H}_Y^{\mathrm{ad}} \times \mathbf{G}'^{\mathrm{ad}}_Y$ induces a product decomposition $X_Y = wX_Y \times X'_Y$, where X_Y is the smallest special subvariety of X_S containing $\Phi_S(Y)$ and Y is (an irreducible component of) $\Phi_S^{-1}(wX_Y \times \{x'_0\})$ for a certain point $x' \in X'_Y$ and a weakly special subvariety wX_Y of X_S. All the (irreducible components of) the preimages $\Phi_S^{-1}(wX_Y \times \{x'\})$, $x' \in X'_Y$, are weakly special subvarieties of S for $\mathbb{V}$ that can be thought as Hodge theoretic deformations of Y. In particular, there are only countably many special subvarieties of S for $\mathbb{V}$, while there are uncountably many weakly special ones, organised in countably many "product families".

We can now make a few remarks on the notion of *weakly non-factor* subvarieties defined in Definition 1.13:

(1) For $\mathbb{V}$ a local system a closed irreducible subvariety $Y \subset S$ is weakly non-factor if and only if any weakly special subvariety $Y \subset Z \subset S$ with $\mathbf{H}_Z = \mathbf{H}_Y$ is weakly non-factor. When $\mathbb{V}$ is a $\mathbb{Z}$VHS it amounts to saying that the weakly special closure $\langle Y \rangle_{\mathrm{ws}} \subset S$ is weakly non-factor.

(2) Let $\mathbb{V}$ be a $\mathbb{Z}$VHS. Given a closed irreducible subvariety $Y \subset S$, let $wX_Y \subset X_S$ be the smallest weakly special subvariety containing $\Phi_S(Y)$. It follows from the above description of the weakly special subvarieties that Y is weakly non-factor for $\mathbb{V}$ if and only if there does not exist $Y \subset Z \subset S$, with Z closed irreducible, such that $wX_Z = wX_Y \times wX' \subset X_S$ with wX' a weakly special subvariety of X_S with $\mathbf{H}_{wX'} \neq 1$. The "weakly non-factor" condition is thus a Hodge theoretic rigidity of Y. In particular one obtains the following:

Lemma 2.5. *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS on S. Any weakly non-factor, weakly special subvariety of S is special.*

(3) The terminology "weakly non-factor" generalizes the terminology "non-factor" introduced by Ullmo [Ull07] for special subvarieties of Shimura varieties.

(4) For $\mathbb{V}$ a non-isotrivial local system on S, it follows from the definition that for any weakly non-factor subvariety $Y \subset S$ the algebraic monodromy group $\mathbf{H}_Y$ is non-trivial. When $\mathbb{V}$ is moreover a $\mathbb{Z}$VHS this last condition is equivalent to saying that Y has *positive period dimension for* $\mathbb{V}$ in the sense of [KO19]: its image $\Phi_S(Y)$ is not a point.

Given S a smooth complex quasi-projective variety and $\mathbb{V}$ a complex local system, we say that $\mathbb{V}$ is defined over a number field $L \subset \mathbb{C}$ if both S and the algebraic module with integrable connection $(\mathcal{V}, \nabla)$ corresponding to $\mathbb{V}$ under the Deligne-Riemann-Hilbert correspondence (see (3.1) below) are defined over L. Theorem 1.14 then follows immediately from Lemma 2.5 and the general result on local systems:

Theorem 2.6. *Let S be a smooth complex quasi-projective variety and $\mathbb{V}$ a complex local system on S^{an}.*

(a) Suppose that S is defined over a number field L. Then any weakly special, weakly non-factor subvariety of S for $\mathbb{V}$ is defined over a finite extension of L;

(b) if moreover $\mathbb{V}$ is defined over L, then any $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-translates of a weakly special, resp. weakly non-factor, subvariety of S for $\mathbb{V}$ is a weakly special, resp. weakly non-factor, subvariety of S for $\mathbb{V}$.

3. Proof of the main results

3.1. Proof of Theorem 2.6(b).

Let S be a smooth complex quasi-projective variety, $\mathrm{Loc}_{\mathbb{C}}(S^{\mathrm{an}})$ the category of complex local systems of finite rank on S^{an}, $\mathrm{MIC}(S^{\mathrm{an}})$ the category of holomorphic modules with integrable connection on S^{an} and $\mathrm{MIC}_r(S)$ the category of algebraic modules with regular integrable connection on S. Following Deligne [De70, Theor.5.9], the analytification functor $\mathrm{MIC}_r(S) \to \mathrm{MIC}(S^{\mathrm{an}})$ is an equivalence of tensor categories. Composed with the Riemann-Hilbert correspondence this provide an equivalence of tensor categories

$$\mathrm{MIC}_r(S) \overset{\tau}{\simeq} \mathrm{Loc}_{\mathbb{C}}(S^{\mathrm{an}}) \quad . \tag{3.1}$$

Let $\mathbb{V} \in \mathrm{Loc}_{\mathbb{C}}(S^{\mathrm{an}})$. Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism. Let $S^\sigma := S \times_{\mathbb{C},\sigma} \mathbb{C}$ be the twist of SS under σ. We denote by $\mathbb{V}^\sigma \in \mathrm{Loc}_{\mathbb{C}}((S^\sigma)^{\mathrm{an}})$ the image of $\mathbb{V}$ under the composition of equivalence of (Tannakian) categories

$$\mathrm{Loc}_{\mathbb{C}}(S^{\mathrm{an}}) \overset{\tau^{-1}}{\sim} \mathrm{MIC}_r(S) \overset{\cdot \times_{\mathbb{C},\sigma}\mathbb{C}}{\sim} \mathrm{MIC}_r(S^\sigma) \overset{\tau}{\sim} \mathrm{Loc}_{\mathbb{C}}((S^\sigma)^{\mathrm{an}}) \quad . \tag{3.2}$$

Theorem 2.6(b) then follows immediately from the Proposition 3.1 below. □

Proposition 3.1. *Let S be a smooth complex quasi-projective variety and $\mathbb{V} \in \mathrm{Loc}_{\mathbb{C}}(S^{\mathrm{an}})$. Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism. Let $Y \subset S$ be a closed irreducible subvariety with Galois twist $Y^\sigma \subset S^\sigma$.*

(1) the complex algebraic monodromy group $\mathbf{H}_Y$ of Y with respect to $\mathbb{V}$ is canonically isomorphic to the complex algebraic monodromy group $\mathbf{H}_{Y^\sigma}$ of Y^σ with respect to $\mathbb{V}^\sigma$.

(2) Y is weakly special for $\mathbb{V}$ if and only if Y^σ is weakly special for $\mathbb{V}^\sigma$.

(3) Y is weakly non-factor for $\mathbb{V}$ if and only if Y^σ is weakly non-factor for $\mathbb{V}^\sigma$.

Proof. Let us first assume that Y is smooth. In that case the equivalence of tensor categories (3.2) $\mathrm{Loc}_{\mathbb{C}}(Y^{\mathrm{an}}) \overset{\tau}{\simeq} \mathrm{Loc}_{\mathbb{C}}((Y^\sigma)^{\mathrm{an}})$ restricts to an equivalence of tensor categories

$$\langle \mathbb{V}_{|Y} \rangle^{\otimes} \overset{\tau}{\simeq} \langle \mathbb{V}^\sigma_{|Y} \rangle^{\otimes} \quad .$$

Taking (the connected component of the identity of) their Tannaka groups we obtain a canonical isomorphism

$$\mathbf{H}_Y \simeq \mathbf{H}_{Y^\sigma} \quad ,$$

thus proving Proposition 3.1(1) in that case.

When Y is not smooth, we consider a desingularisation $Y^s \overset{p}{\to} Y^{\mathrm{nor}} \overset{\pi}{\to} Y$. Notice that $(Y^s)^\sigma$ is a desingularisation of $(Y^{\mathrm{nor}})^\sigma = (Y^\sigma)^{\mathrm{nor}}$. Notice moreover that the algebraic monodromy groups of $(p \circ \pi)^* \mathbb{V}_{|Y}$ and $\pi^* \mathbb{V}_Y$ coincides, as $p_* : \pi_1(Y^s) \to \pi_1(Y^{\mathrm{nor}})$ is surjective. Arguing as above for Y^s and $(Y^s)^\sigma$ proves Proposition 3.1(1) in general.

Suppose now that $Y \subset S$ is a closed irreducible subvariety. If Y^σ is not weakly special for $\mathbb{V}^\sigma$ there exists $Z \supset Y^\sigma$ a closed irreducible subvariety of S^σ containing Y^σ strictly and such that $\mathbf{H}_Z = \mathbf{H}_{Y^\sigma}$. But then $Z^{\sigma^{-1}}$ is a closed irreducible subvariety of S containing Y strictly, and such that $\mathbf{H}_{Z^{\sigma^{-1}}} = \mathbf{H}_Y$ by Proposition 3.1(1). It follows that Y is not weakly special. This proves Proposition 3.1(2).

The argument for Proposition 3.1(3) is similar. We are reduced to showing that for S a smooth complex quasi-projective variety, $\mathbb{V} \in \mathrm{Loc}_{\mathbb{C}}(S^{\mathrm{an}})$, $\sigma : \mathbb{C} \to \mathbb{C}$ a field automorphism and $Y \subset S$ a closed irreducible subvariety with Galois twist $Y^\sigma \subset S^\sigma$, then $\mathbf{H}_Y$ is normal in $\mathbf{H}_S$ if and only if $\mathbf{H}_{Y^\sigma}$ is normal in $\mathbf{H}_{S^\sigma}$. Consider the Tannakian subcategory $\mathcal{T}$ of $\langle \mathbb{V} \rangle^{\otimes}$ consisting of the local systems which are trivial in restriction to Y^{an}. Applying σ we obtain that $\mathcal{T}^\sigma$ is the Tannakian subcategory of $\langle \mathbb{V}^\sigma \rangle^{\otimes}$ of local systems that are trivial on $(Y^\sigma)^{\mathrm{an}}$. But as a result of the Tannakian formalism the normal closures of $\mathbf{H}_Y$ and $\mathbf{H}_{Y^\sigma}$ in $\mathbf{H}_S$ and $\mathbf{H}_{S^\sigma}$ respectively are the kernel of the canonical morphism from $\mathbf{H}_S$ to the Tannaka group of $\mathcal{T}$, resp. from $\mathbf{H}_{S^\sigma}$ to the Tannaka group of $\mathcal{T}^\sigma$. Hence the result. □

3.2. Proof of Theorem 1.14 when $\mathbb{V}$ is defined over a number field.

Although this is not necessary to prove the theorem in general, let us notice that Theorem 1.14 in the case where $\mathbb{V}$ is defined over a number field L follows from Theorem 2.6(b). Indeed when $\mathbb{V}$ is a $\mathbb{Z}$VHS, weakly special weakly non-factor subvarieties of S for $\mathbb{V}$ are special subvarieties of S for $\mathbb{V}$ by Lemma 2.5. Applying Theorem 2.6(b), it follows that the $\mathrm{Aut}(\mathbb{C}/L)$-translates of any special, weakly non-factor, subvariety of S for $\mathbb{V}$ is special (and weakly non-factor). But special subvarieties of S for $\mathbb{V}$ form a countable set. It follows immediately that any special, weakly non-factor, subvariety of S for $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$ (see for instance [Voi13, Claim p.25]). □

3.3. Proof of Theorem 2.6(a).

Let us now prove Theorem 2.6(a), hence finish the proof of Theorem 1.14. Let S be a complex irreducible smooth quasi-projective variety and $\mathbb{V}$ a complex local system on S^{an}. Suppose that S is defined over a number field $L \subset \mathbb{C}$. Let $Y \subset S$ be a weakly special subvariety of S for $\mathbb{V}$ which is weakly non-factor. Let us show that Y is defined over $\overline{\mathbb{Q}}$.

Let $Z \subset S$ be the $\overline{\mathbb{Q}}$-Zariski-closure of Y, i.e. the smallest closed subvariety of S defined over $\overline{\mathbb{Q}}$ and containing Y. Thus Z is irreducible.

The subset $Z^0 \subset Z$ of smooth points is $\overline{\mathbb{Q}}$-Zariski-open (meaning that $Z - Z^0$ is a closed subvariety of Z defined over $\overline{\mathbb{Q}}$) and dense. Notice that $Y \cap Z^0$ is Zariski-open in Y (otherwise Y would be contained in the closed subvariety $Z - Z^0$ defined over $\overline{\mathbb{Q}}$, in contradiction to the $\overline{\mathbb{Q}}$-Zariski-density of Y in Z); moreover the fact that $Y \subset S$ is weakly special, resp. weakly non-factor for $(S, \mathbb{V})$ implies that $Y^0 := Y \cap Z^0$ is weakly special, resp. weakly non-factor for $(Z^0, \mathbb{V}_{|Z^0})$. Replacing $Y \subset S$ by $Y^0 \subset Z^0$ if necessary, we can without loss of generality assume that Y is $\overline{\mathbb{Q}}$-Zariski-dense in S. We are reduced to proving that $Y = S$, or equivalently that $\mathbf{H}_Y = \mathbf{H}_S$. This follows immediately from the Proposition 3.2 below, of independent interest. $\square$

Proposition 3.2. *Let S be a smooth complex quasi-projective variety, $\mathbb{V}$ a complex local system on S^{an} and let $Y \subset S$ be a closed irreducible weakly non-factor subvariety for $\mathbb{V}$. Suppose that S is defined over $\overline{\mathbb{Q}}$ and that Y is $\overline{\mathbb{Q}}$-Zariski-dense in S. Then $\mathbf{H}_Y = \mathbf{H}_S$.*

Proof. Let $\mathcal{Y}$ be "the" spread of Y with respect to S. Let us recall its definition. Let $K \subset \mathbb{C}$ be the minimal field of definition of Y, see [Gro65, Cor. 4.8.11]. This is the smallest subfield $\overline{\mathbb{Q}} \subset K \subset \mathbb{C}$ such that Y is defined over K: there exists a K-scheme of finite type Y_K such that $Y = Y_K \otimes_K \mathbb{C}$. Let us choose $R \subset K$ a finitely generated $\overline{\mathbb{Q}}$-algebra whose field of fractions is K and let $\mathcal{Y}_R$ be an R-model of $Y_K = \mathcal{Y}_R \otimes_R K$. The morphism $\mathcal{Y}_R \to \operatorname{Spec} R$ induces a morphism of complex varieties $\mathcal{Y} := \mathcal{Y}_R \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to T := \operatorname{Spec}(R \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$, defined over $\overline{\mathbb{Q}}$. Notice that the complex dimension of T is the transcendence degree of K over $\overline{\mathbb{Q}}$. The natural closed immersion $Y_R \subset S \otimes_{\overline{\mathbb{Q}}} R$ makes $\mathcal{Y}$ a closed irreducible variety

$$\mathcal{Y} \subset S \times_{\mathbb{C}} T$$

defined over $\overline{\mathbb{Q}}$, with induced projections $p : \mathcal{Y} \to S$ and $\pi : \mathcal{Y} \to T$, both defined over $\overline{\mathbb{Q}}$, such that $\mathcal{Y}_{t_0} := \pi^{-1}(x_0) \simeq Y$ where $t_0 \in T(\mathbb{C})$ is the closed point given by $R \subset K \subset \mathbb{C}$. By construction the morphism p is dominant. The variety $\mathcal{Y}$ is called "the" spread of Y. It depends on the choice of R but different choices give rise to birational varieties $\mathcal{Y}$s. Shrinking $\operatorname{Spec} R$ if necessary, we can assume without loss of generality that T is smooth.

Let $\mathcal{Y}^0 \subset \mathcal{Y}$ be the $\overline{\mathbb{Q}}$-Zariski-open dense subset of smooth points. As p is dominant, the fact that $Y \subset S$ is weakly non-factor for $(S, \mathbb{V})$ implies that $Y^0 := \mathcal{Y}^0 \cap Y \subset \mathcal{Y}^0$ is weakly non-factor for $(\mathcal{Y}^0, p^{-1}(\mathbb{V})_{|\mathcal{Y}^0})$. As $\mathbf{H}_{Y^0} = \mathbf{H}_Y$ and $\mathbf{H}_{\mathcal{Y}^0} = \mathbf{H}_S$, to show that $\mathbf{H}_Y = \mathbf{H}_S$ we are reduced, replacing S by $\mathcal{Y}^0$ and Y by $\mathcal{Y}^0 \cap Y$ if necessary, to the situation where there exists a morphism $\pi : S \to T$ defined over $\overline{\mathbb{Q}}$ such that $Y = S_{t_0} \subset S$ and Y is weakly non-factor for $(S, \mathbb{V})$.

It follows from [GM88, Theorem p.57] that there exist finite Whitney stratifications (S_l) of S and $(T_l)_{l \leq d}$ of T by locally closed algebraic subsets T_l of dimension l ($d = \dim T$) such that for each connected component Z (a stratum) of T_l, $\pi^{-1}(Z)^{\mathrm{an}}$ is a topological fibre bundle over Z^{an}, and a union of connected components of strata of (S_j^{an}), each mapped submersively to Z^{an} (moreover, for all $t \in Z^{\mathrm{an}}$, there exists an open neighbourhood $U(t)$ in Z^{an} and a stratum preserving homeomorphism $h : \pi^{-1}(U) \simeq \pi^{-1}(t) \times U$ such that $\pi_{|\pi^{-1}(U)} = p_U \circ h$, where p_U denotes the projection to U). These Whitney stratifications can be chosen defined over $\overline{\mathbb{Q}}$ (meaning that the closure of each stratum is defined over $\overline{\mathbb{Q}}$): see [Tei82], [Ar13, 3.1.9].

It follows from the minimality of K that t_0 belongs to the unique open stratum T_d, $d = \dim T$. Without loss of generality we can and will assume from now on that $T = T_d$. In particular S^{an} is a topological fibre bundle over T^{an}.

If follows that the image of $\pi_1(Y^{\mathrm{an}})$ in $\pi_1(S^{\mathrm{an}})$ is a normal subgroup. Hence $\mathbf{H}_Y$ is a normal subgroup of $\mathbf{H}_S$. As $Y \subset S$ is weakly non-factor it follows that $\mathbf{H}_Y = \mathbf{H}_S$. $\square$

3.4. Proof of Theorem 1.10.

Let S, $\mathbb{V}$ and Y as in the statement of Theorem 1.10. Let us show that Y is weakly non-factor. Let $Z \subset S$ be a closed irreducible subvariety of S containing Y strictly, and such that $\mathbf{H}_Y$ is is a strict normal subgroup of $\mathbf{H}_Z$. As the special closure $\langle Z \rangle_{\mathrm{s}}$ of Z is a special subvariety of $(S, \mathbb{V})$ containing Y, it follows from the maximality of Y that $\langle Z \rangle_{\mathrm{s}} = S$. As $\mathbf{H}_Z$ is normal (see [An92, Theor.1]) in the algebraic group $\mathbf{G}_Z^{\mathrm{der}} = \mathbf{G}_S^{\mathrm{der}}$ which is assumed to be simple, it follows that either $\mathbf{H}_Z = \{1\}$ or $\mathbf{H}_Z = \mathbf{H}_S = \mathbf{G}_S^{\mathrm{der}}$. As $\mathbf{H}_Y$ is a strict normal subgroup of $\mathbf{H}_Z$, necessarily $\mathbf{H}_Y = \{1\}$ (and $\mathbf{H}_Z = \mathbf{H}_S$). This is impossible as $\mathbf{H}_Y \neq 1$ by assumption. Hence such a Z does not exist and Y is weakly non-factor. The conclusion then follows from Theorem 1.14. □

3.5. Proof of Corollary 1.11.

In the situation of Corollary 1.11 the variation $\mathbb{V}$ is clearly defined over $\mathbb{Q}$. Let $\mathbf{G}_{n,d}$ be the group of automorphisms of $H^n(X, \mathbb{Q})_{\mathrm{prim}}$ preserving the cup-product. When n is odd the primitive cohomology is the same as the cohomology. When n is even it is the orthogonal complement of $h^{n/2}$, where h is the hyperplane class. Thus $\mathbf{G}_{n,d}$ is either a symplectic or an orthogonal group depending on the parity of n, and is a simple $\mathbb{Q}$-algebraic group. A classical result of Beauville [Beau86, Theor.2, Theor.4] proves that the image of the monodromy representation for $\mathbb{V}$ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. In particular the algebraic monodromy group $\mathbf{H}_{U_{n,d}}$ coincides with the simple group $\mathbf{G}_{d,n}$. As $\mathbf{H}_{U_{n,d}} \subset \mathbf{G}_{U_{n,d}}^{\mathrm{ad}} \subset \mathbf{G}_{d,n}$ we deduce $\mathbf{G}_{U_{n,d}}^{\mathrm{ad}} = \mathbf{G}_{d,n}$. The result then follows from Theorem 1.10. □

3.6. Proof of Theorem 1.16.

Let us suppose that the special points for $\mathbb{Z}$VHS's defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$. Let $\mathbb{V} \to S^{\mathrm{an}}$ be a $\mathbb{Z}$VHS defined over $\overline{\mathbb{Q}}$ and let Y be a special subvariety of S for $\mathbb{V}$. Let us show that Y is defined over $\overline{\mathbb{Q}}$.

Suppose for the sake of contradiction that Y is not defined over $\overline{\mathbb{Q}}$. Let $Z \subset S$ be the $\overline{\mathbb{Q}}$-Zariski closure Z of Y in S. Again, replacing S by the $\overline{\mathbb{Q}}$-Zariski open subset of smooth points Z^0 of Z and Y by $Y^0 := Z^0 \cap Y$ we can without loss of generality assume that $Z = S$ is smooth. Arguing as in the proof of Theorem 2.6(a) we may assume that $\mathbf{H}_Y$ is a strict normal subgroup of $\mathbf{H}_S$. Because $\mathbf{G}_S$ is reductive and $\mathbf{H}_S$ is normal in the derived group $\mathbf{G}_S^{\mathrm{der}}$, it follows that $\mathbf{H}_Y$ is a product of simple factors of $\mathbf{G}_S^{\mathrm{der}}$, hence normal in $\mathbf{G}_S$.

It follows that there exist a finite collection of natural integers a_i, b_i, $1 \leq i \leq n$ such that the $\mathbb{Z}$VHS $\mathbb{V}' := (\bigoplus_{1 \leq i \leq n} \mathbb{V}^{\otimes a_i} \otimes (\mathbb{V}^\vee)^{\otimes b_i})^{\mathbf{H}_Y}$ consisting of the $\mathbf{H}_Y$-invariant vectors in $\bigoplus_{1 \leq i \leq n} \mathbb{V}^{\otimes a_i} \otimes (\mathbb{V}^\vee)^{\otimes b_i}$ has generic Mumford-Tate group $\mathbf{G}'_S = \mathbf{G}_S / \mathbf{H}_Y$ and algebraic monodromy group $\mathbf{H}'_S := \mathbf{H}_S / \mathbf{H}_Y$. Writing $(\mathbf{G}'_S = \mathbf{G}_S / \mathbf{H}_Y, \mathcal{D}'_S := \mathcal{D}_S / \mathbf{H}_Y)$ for the quotient Hodge datum of $(\mathbf{G}_S, \mathcal{D}_S)$ by $\mathbf{H}_Y$ and $\pi : X_S \twoheadrightarrow X'_S$ the induced projection of Hodge varieties, the period map for $\mathbb{V}'$ is $\Phi'_S := \pi \circ \Phi_S : S^{\mathrm{an}} \to X'_S$. The special subvariety Y of S for $\mathbb{V}$ is still a special subvariety of S for $\mathbb{V}'$ and its image $\Phi'_S(Y)$ is a point.

Passing to a finite cover if necessary and filling in finitely many punctures at infinity, we may assume that Φ'_S is proper. Following [BBT18, Theor.1.1] there exists a factorisation

$$\Phi'_S = \Psi \circ q \quad ,$$

where $q : S \to B$ is a proper morphism of quasi-projective varieties defined over $\overline{\mathbb{Q}}$ satisfying $q_*\mathcal{O}_S = \mathcal{O}_B$ and $\Psi : B \to X'$ is a quasi-finite period map. This means that $\mathbb{V}' = q^*\mathbb{V}'_B$ for a $\mathbb{Z}$VHS $\mathbb{V}'_B$, and that $b_0 := q(Y)$ is a special point of B for $\mathbb{V}'_B$.

It follows from Lemma 3.3 below that the $\mathbb{Z}$VHS $\mathbb{V}'$ can be defined over $\overline{\mathbb{Q}}$. It then follows from Lemma 3.4 below that $\mathbb{V}'_B$ is also defined over $\overline{\mathbb{Q}}$. Under our assumption that special points of $\mathbb{Z}$VHS defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$ one concludes that the special point b_0 of B for $\mathbb{V}'_B$ is defined over $\overline{\mathbb{Q}}$. But then the irreducible component Y of $q^{-1}(b_0)$ is also defined over $\overline{\mathbb{Q}}$, a contradiction.

This finishes the proof of Theorem 1.16. $\square$

Lemma 3.3. *Let $\mathbb{V}$ be a $\mathbb{Z}$VHS and $\mathbb{V}'$ a sub-$\mathbb{Z}$VHS. If $\mathbb{V}$ is definable over $K \subset \mathbb{C}$ then there exists a K-structure on $\mathbb{V}$ and $\mathbb{V}'$ such that the projection $\mathbb{V} \twoheadrightarrow \mathbb{V}'$ is defined over K.*

Proof. Let E be the finite dimensional K-algebra of ∇-flat $F^\bullet$-preserving algebraic sections over S of $\mathcal{V}_K \otimes \mathcal{V}_K^\vee$. Each invertible element of $E_\mathbb{C} := E \otimes_K \mathbb{C}$ defines a natural K-structure on $\mathcal{V}$, $F^\bullet$ and ∇, the original one $(\mathcal{V}_K, F^\bullet_K, \nabla_K)$ being preserved exactly by the invertible elements of E.

Let J be the Jacobson radical of E. Let us choose $T \subset E$ a (semisimple) splitting of the projection $E \to E/J$. As the category of polarizable $\mathbb{Q}$VHS is abelian semisimple the finite dimensional complex algebra $\mathrm{Hom}_{\mathbb{Z}\mathrm{VHS}}(\mathbb{V}, \mathbb{V}) \otimes_\mathbb{Z} \mathbb{C}$ is semisimple. Under the Riemann-Hilbert correspondence it identifies with a semisimple subalgebra $\mathcal{A} \subset E_\mathbb{C}$. Following a classical result of Wedderburn-Malcev there exists an element $j \in J_\mathbb{C} := J \otimes_K \mathbb{C}$ such that $(1+j)\mathcal{A}(1+j)^{-1} \subset T_\mathbb{C}$.

Let $e_\mathbb{C} \in \mathcal{A}$ be the idempotent corresponding to the projection of $\mathbb{Z}$VHS $\pi : \mathbb{V} \twoheadrightarrow \mathbb{V}'$ under the Riemann-Hilbert correspondence. As $T_\mathbb{C}$ is semisimple, hence a product of matrix algebras, any idempotent of $T_\mathbb{C}$ is conjugated to an idempotent in T. Thus there exist an invertible element $f \in T_\mathbb{C}$ and $e \in T$ such that $(1+j)e_\mathbb{C}(1+j)^{-1} = f^{-1}ef$.

If we endow $(\mathcal{V}, F^\bullet, \nabla)$ with the K-structure defined by the element $f(1+j) \in E_\mathbb{C}$ it follows that the image of $\pi : \mathbb{V} \twoheadrightarrow \mathbb{V}'$ under the Riemann-Hilbert correspondence is defined over K for this new K-structure. Hence the result. $\square$

Lemma 3.4. *Let $f : S \longrightarrow B$ be a proper morphism of K-varieties defined over $K \subset \mathbb{C}$, such that $f_*\mathcal{O}_S = \mathcal{O}_B$. Let $\mathbb{V}_B$ be a $\mathbb{Z}$VHS on B. If the $\mathbb{Z}$VHS $\mathbb{V}_S := f^*\mathbb{V}_B$ on S is definable over K then $\mathbb{V}_B$ is also definable over K.*

Proof. Let $(\mathcal{V}_S := f^*\mathcal{V}_B, F^\bullet_S := f^*F^\bullet_B, \nabla_S := f^*\nabla_B)$ be the De Rham incarnation of $\mathbb{V}_S$. It follows from the projection formula and the assumption $f_*\mathcal{O}_S = \mathcal{O}_B$ that

$$f_*\mathcal{V}_S = f_*(f^*\mathcal{V}_B \otimes_{\mathcal{O}_S} \mathcal{O}_S) = \mathcal{V}_B \otimes_{\mathcal{O}_B} f_*\mathcal{O}_S = \mathcal{V}_B \quad .$$

It follows easily that $F^\bullet_B = f_*F^\bullet_S$ and $\nabla_B = f_*\nabla_S$. As f, $F^\bullet_S$ and ∇_S are defined over K, it follows that $F^\bullet_B$ and ∇_B are defined over K. $\square$

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