THE ANDRÉ-OORT CONJECTURE.

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Abstract. In this paper we prove, assuming the Generalized Riemann Hypothesis, the André-Oort conjecture on the Zariski closure of sets of special points in a Shimura variety. In the case of sets of special points satisfying an additional assumption, we prove the conjecture without assuming the GRH.

Contents

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1. Introduction.

1.1. **The André-Oort conjecture.** The purpose of this paper is to prove, under certain assumptions, the André-Oort conjecture on special subvarieties of Shimura varieties.

Before stating the André-Oort conjecture we provide some motivation from algebraic geometry. Let Z be a smooth complex algebraic variety and let $\mathcal{F} \longrightarrow Z$ be a variation of polarizable $\mathbb{Q}$-Hodge structures on Z (for example $\mathcal{F} = R^i f_* \mathbb{Q}$ for a smooth proper morphism $f : Y \longrightarrow Z$). To every $z \in Z$ one associates a reductive algebraic $\mathbb{Q}$-group $\mathbf{MT}(z)$, called the Mumford-Tate group of the Hodge structure $\mathcal{F}_z$. This group is the stabiliser of the Hodge classes in the rational Hodge structures tensorially generated by $\mathcal{F}_z$ and its dual. A point $z \in Z$ is said to be Hodge generic if $\mathbf{MT}(z)$ is maximal. If Z is irreducible, two Hodge generic points of Z have the same Mumford-Tate group, called the generic Mumford-Tate group $\mathbf{MT}_Z$. The complement of the Hodge generic locus is a countable union of closed irreducible algebraic subvarieties of Z, each not contained in the union of the others. This is proved in [7]. Furthermore, it is shown in [40] that when Z is defined over $\overline{\mathbb{Q}}$ (and under certain simple assumptions) these components are also defined over $\overline{\mathbb{Q}}$. The irreducible components of the intersections of these subvarieties are called *special subvarieties* (or subvarieties of Hodge type) of Z relative to $\mathcal{F}$. Special subvarieties of dimension zero are called *special points*.

Example: Let Z be the modular curve $Y(N)$ (with $N \geq 4$) and let $\mathcal{F}$ be the variation of polarizable $\mathbb{Q}$-Hodge structures $R^1 f_* \mathbb{Q}$ of weight one on Z associated to the universal elliptic curve $f : E \longrightarrow Z$. Special points on Z parametrize elliptic curves with complex multiplication. The generic Mumford-Tate group on Z is $\mathbf{GL}_{2,\mathbb{Q}}$. The Mumford-Tate group of a special point corresponding to an elliptic curve with complex multiplication by a quadratic imaginary field K is the torus $\mathrm{Res}_{K/\mathbb{Q}} \mathbf{G}_{\mathbf{m},K}$ obtained by restriction of scalars from K to $\mathbb{Q}$ of the multiplicative group $\mathbf{G}_{\mathbf{m},K}$ over K.

The general Noether-Lefschetz problem consists in describing the geometry of these special subvarieties, in particular the distribution of special points. Griffiths transversality condition prevents, in general, the existence of moduli spaces for variations of polarizable $\mathbb{Q}$-Hodge structures. Shimura varieties naturally appear as solutions to such moduli problems with additional data (c.f. [11], [12], [23]). Recall that a $\mathbb{Q}$-Hodge structure on a $\mathbb{Q}$-vector space V is a structure of $\mathbf{S}$-module on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$, where $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_{\mathbf{m},\mathbb{C}}$. In other words it is a morphism of real algebraic groups $h \colon \mathbf{S} \longrightarrow \mathbf{GL}(V_{\mathbb{R}})$.

The Mumford-Tate group $\mathbf{MT}(h)$ is the smallest algebraic $\mathbb{Q}$-subgroup $\mathbf{H}$ of $\mathbf{GL}(V)$ such that h factors through $\mathbf{H}_{\mathbb{R}}$. A *Shimura datum* is a pair $(\mathbf{G}, X)$, with $\mathbf{G}$ a linear connected reductive group over $\mathbb{Q}$ and X a $\mathbf{G}(\mathbb{R})$-conjugacy class in the set of morphisms of real algebraic groups $\mathrm{Hom}(\mathbf{S}, \mathbf{G}_{\mathbb{R}})$, satisfying the "Deligne's conditions" [12, 1.1.13]. These conditions imply, in particular, that the connected components of X are Hermitian symmetric domains and that $\mathbb{Q}$-representations of $\mathbf{G}$ induce polarizable variations of $\mathbb{Q}$-Hodge structures on X. A morphism of Shimura data from $(\mathbf{G}_1, X_1)$ to $(\mathbf{G}_2, X_2)$ is a $\mathbb{Q}$-morphism $f : \mathbf{G}_1 \longrightarrow \mathbf{G}_2$ that maps X_1 to X_2.

Given a compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ (where $\mathbf{A}_{\mathrm{f}}$ denotes the ring of finite adèles of $\mathbb{Q}$) the set $\mathbf{G}(\mathbb{Q}) \backslash (X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K)$ is naturally the set of $\mathbb{C}$-points of a quasi-projective variety (a *Shimura variety*) over $\mathbb{C}$, denoted $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. The projective limit $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} = \varprojlim_K \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a $\mathbb{C}$-scheme on which $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ acts continuously by multiplication on the right (c.f. section 4.1.1). The multiplication by $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ on $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ induces an algebraic correspondence T_g on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, called a *Hecke correspondence*. One shows that a subvariety $V \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is *special* (with respect to some variation of Hodge structure associated to a faithful $\mathbb{Q}$-representation of $\mathbf{G}$) if and only if there is a Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$, a morphism of Shimura data $f : (\mathbf{H}, X_{\mathbf{H}}) \longrightarrow (\mathbf{G}, X)$ and an element $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ such that V is an irreducible component of the image of the morphism:

$$\mathrm{Sh}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} \stackrel{\mathrm{Sh}(f)}{\longrightarrow} \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \stackrel{\cdot g}{\longrightarrow} \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \ .$$

It can also be shown that the Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$ can be chosen in such a way that $\mathbf{H} \subset \mathbf{G}$ is the generic Mumford-Tate group on $X_{\mathbf{H}}$ (see Lemma 2.1 of [39]). A *special* point is a special subvariety of dimension zero. One sees that a point $\overline{(x, g)} \in \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ (where $x \in X$ and $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$) is *special* if and only if the group $\mathbf{MT}(x)$ is commutative (in which case $\mathbf{MT}(x)$ is a torus).

Given a special subvariety V of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, the set of special points of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ contained in V is dense in V for the strong (and in particular for the Zariski) topology. Indeed, one shows that V contains a special point, say s. Let $\mathbf{H}$ be a reductive group defining V and let $\mathbf{H}(\mathbb{R})^+$ denote the connected component of the identity in the real Lie group $\mathbf{H}(\mathbb{R})$. The fact that $\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$ is dense in $\mathbf{H}(\mathbb{R})^+$ implies that the "$\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$-orbit" of s, which is contained in V, is dense in V. This "orbit" (sometimes referred to as the Hecke orbit of s) consists of special points. The André-Oort conjecture is the converse statement.

Definition 1.1.1. *Given a set Σ of subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ we denote by $\mathbf{\Sigma}$ the subset $\cup_{V \in \Sigma} V$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.*

Conjecture 1.1.2 (André-Oort)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ a set of special points in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$. Then every irreducible component of the Zariski closure of $\mathbf{\Sigma}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a special subvariety.*

One may notice an analogy between this conjecture and the so-called Manin-Mumford conjecture (first proved by Raynaud) which asserts that irreducible components of the Zariski closure of a set of *torsion* points in an Abelian variety are translates of Abelian subvarieties by torsion points. There is a large (and constantly growing) number of proofs of the Manin-Mumford conjecture. A proof of the Manin-Mumford conjecture using a strategy similar to the one used in this paper was recently given by Ullmo and Ratazzi (see [38]).

1.2. **The results.** Our main result is the following:

Theorem 1.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ be a set of special points in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$. We make* **one** *of the following assumptions:*

(1) *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*

(2) *Assume that there exists a faithful representation* $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *such that with respect to this representation, the Mumford-Tate groups* $\mathbf{MT}_s$ *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class as* s *ranges through* Σ*.*

Then every irreducible component of the Zariski closure of $\mathbf{\Sigma}$ *in* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ *is a special subvariety.*

In fact we prove the following

Theorem 1.2.2. *Let* $(\mathbf{G}, X)$ *be a Shimura datum,* K *a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ *and let* Σ *be a set of special subvarieties in* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$*. We make* **one** *of the following assumptions:*

(1) *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*
(2) *Assume that there exists a faithful representation* $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *such that with respect to this representation, the generic Mumford-Tate groups* $\mathbf{MT}_V$ *of* V *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class as* V *ranges through* Σ*.*

Then every irreducible component of the Zariski closure of $\mathbf{\Sigma}$ *in* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ *is a special subvariety.*

The case of theorem 1.2.2 where Σ is a set of special points is theorem 1.2.1.

1.3. **Some remarks on the history of the André-Oort conjecture.** For history and results obtained before 2002, we refer to the introduction of [17]. We just mention that conjecture 1.1.2 was stated by André in 1989 in the case of an irreducible curve in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a Zariski dense set of special points, and in 1995 by Oort for irreducible subvarieties of moduli spaces of polarised Abelian varieties containing a Zariski-dense set of special points.

Let us mention some results we will use in the course of our proof.

In [9] (further generalized in [36] and [39]), the conclusion of the theorem 1.2.2 is proved for sets Σ of *strongly special* subvarieties in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ without assuming (1) or (2) (cf. section 2). The statement is proved using ergodic theoretic techniques.

Using Galois-theoretic techniques and geometric properties of Hecke correspondences, Edixhoven and the second author (see [18]) proved the conjecture for curves in Shimura varieties containing infinite sets of special points satisfying our assumption (2). Subsequently, the second author (in [43]) proved the André-Oort conjecture for curves in Shimura varieties assuming the GRH. The main new ingredient in [43] is a theorem on lower bounds for Galois orbits of special points. In the work [16], Edixhoven proves, assuming the GRH, the André-Oort conjecture for products of modular curves. In [42], the second author proves the André-Oort conjecture for sets of special points satisfying an additional condition.

The authors started working together on this conjecture in 2003 trying to generalize the Edixhoven-Yafaev strategy to the general case of the André-Oort conjecture. In the process two main difficulties occur. One is the question of irreducibility of transforms of subvarieties under Hecke correspondences. This problem is dealt with in sections 7 and 8. The other difficulty consists in dealing with higher dimensional special subvarieties. Our strategy is to proceed by induction on the generic dimension of elements of Σ. The main ingredient for controlling the induction was the discovery by Ullmo and the second author in [39] of a possible combination of Galois theoretic and ergodic techniques. It took form while the second author was visiting the University of Paris-Sud in January-February 2005.

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1.5. **Conventions.** Let F be a field. An F-algebraic variety is a reduced separated scheme over F, not necessarily irreducible. It is of finite type over F unless mentioned. A subvariety is always assumed to be a closed subvariety.

Let $F \subset \mathbb{C}$ be a number field, Y_F an F-algebraic variety and $Z \subset Y := Y_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$ a $\mathbb{C}$-subvariety. We will use the following common abuse of notation: Z is said to be F-irreducible if $Z = Z_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$, where $Z_F \subset Y_F$ is an irreducible closed subvariety.

1.6. **Organization of the paper.** Sections 2 and 3 of the paper explain how to reduce the theorem 1.2.2 to the more geometric theorem 3.1.1 using the Galois/ergodic alternative proven in [39]. In these sections we freely use notations recalled in section 4 and 5, which consist in preliminaries. In addition to fixing notations we prove there the crucial corollary 5.3.10 comparing the degrees of subvarieties under morphisms of Shimura varieties. The sections 6, 7, 8, 9, 10 contain the proof of theorem 3.1.1. Their role and their organisation is described in details in section 3.4.

2. Equidistribution and Galois orbits.

In this section we recall a crucial ingredient in the proof of the theorem 1.2.2: the Galois/ ergodic alternative from [39].

2.1. Some definitions.

2.1.1. *Shimura subdata defining special subvarieties.*

Definition 2.1.1. *Let $(\mathbf{G}, X)$ be a Shimura datum where $\mathbf{G}$ is the generic Mumford-Tate group on X. Let X^+ be a connected component of X and let K be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We denote by $S_K(\mathbf{G}, X)_{\mathbb{C}}$ the connected component of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Thus $S_K(\mathbf{G}, X)_{\mathbb{C}} = \Gamma_K \backslash X^+$, where $\Gamma_K = \mathbf{G}(\mathbb{Q})_+ \cap K$ is a neat arithmetic subgroup of the stabiliser $\mathbf{G}(\mathbb{Q})_+$ of X^+ in $\mathbf{G}(\mathbb{Q})$.*

Definition 2.1.2. *Let V be a special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$. We say that a Shimura subdatum $(\mathbf{H}_V, X_V)$ of $(\mathbf{G}, X)$ defines V if $\mathbf{H}_V$ is the generic Mumford-Tate group on X_V and there exists a connected component X_V^+ of X_V contained in X^+ such that V is the image of $X_V^+ \times \{1\}$ in $S_K(\mathbf{G}, X)_{\mathbb{C}}$.*

From now on, when we say that a Shimura subdatum $(\mathbf{H}_V, X_V)$ defines V, the choice of the component $X_V^+ \subset X^+$ will always be tacitly assumed.

Given a special subvariety V of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ there exists a Shimura subdatum $(\mathbf{H}_V, X_V)$ defining V by [39, lemma 2.1]. Notice that as an abstract $\mathbb{Q}$-algebraic group $\mathbf{H}_V$ is uniquely defined by V whereas the embedding $\mathbf{H}_V \hookrightarrow \mathbf{G}$ is uniquely defined by V up to conjugation by Γ_K.

2.1.2. *The measure μ_V.* Let $(\mathbf{G}, X)$ be a Shimura datum, K a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and X^+ a connected component of X. Let $(\mathbf{H}_V, X_V)$ be a Shimura subdatum of $(\mathbf{G}, X)$ defining a special subvariety V of $S_K(\mathbf{G}, X)_{\mathbb{C}}$. Thus there exists a neat arithmetic group Γ_V of the stabiliser $\mathbf{H}_V(\mathbb{Q})_+$ of X_V^+ in $\mathbf{H}_V(\mathbb{Q})$ and a (finite) morphism

$$f : \Gamma_V \backslash X_V^+ \longrightarrow S_K(\mathbf{G}, X)_{\mathbb{C}}$$

whose image is V.

Definition 2.1.3. *We define μ_V to be the probability measure on $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ supported on V, push-forward by f of the standard probability measure on the Hermitian locally symmetric space $\Gamma_V \backslash X_V^+$ induced by the Haar measure on $\mathbf{H}_V(\mathbb{R})_+$.*

Remark 2.1.4. Notice that the measure μ_V depends only on V, not on the choice of the embedding $\mathbf{H}_V \hookrightarrow \mathbf{G}$.

2.1.3. $\mathbf{T}$-*special subvarieties.*

Definition 2.1.5. *Let $(\mathbf{G},X)$ be a Shimura datum and let $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ be the canonical morphism. Fix a (possibly trivial) $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}^{\mathrm{ad}}$. A $\mathbf{T}$-special subdatum $(\mathbf{H},X_{\mathbf{H}})$ of $(\mathbf{G},X)$ is a Shimura subdatum such that $\mathbf{H}$ is the generic Mumford-Tate group of $X_{\mathbf{H}}$ and $\mathbf{T}$ is the connected centre of $\lambda(\mathbf{H})$.*

Let X^+ be a connected component of X and let K be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. A special subvariety V of $S_K(\mathbf{G},X)_{\mathbb{C}}$ is $\mathbf{T}$-special if there exists a $\mathbf{T}$-special subdatum $(\mathbf{H},X_{\mathbf{H}})$ of $(\mathbf{G},X)$ such that V is an irreducible component of the image of $\mathrm{Sh}_{K\cap\mathbf{H}(\mathbf{A}_{\mathrm{f}})}(\mathbf{H},X_{\mathbf{H}})_{\mathbb{C}}$ in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$.

In the case where $\mathbf{T}$ is trivial, we call V strongly *special.*

Remarks 2.1.6.
(a) If moreover $(\mathbf{H},X_{\mathbf{H}})$ defines V then V is said to be $\mathbf{T}$-special *standard* in [39].
(b) The definition of *strongly special* given in [9] requires that $\mathbf{H}_V$ is not contained in a proper parabolic subgroup of $\mathbf{G}$ but as explained in [36, rem. 3.9] this last condition is automatically satisfied.

2.2. **The rough alternative.** With these definitions, the alternative from [39] can roughly be stated as follows.

Let $(\mathbf{G},X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type, X^+ a connected component of X, let K be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and E a number field over which $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ admits a canonical model (cf. section 4.1.2). Let $Z \subset S_K(\mathbf{G},X)_{\mathbb{C}}$ be an irreducible subvariety containing a Zariski-dense union $\cup_{n\in\mathbb{N}} V_n$ of special subvarieties V_n of $S_K(\mathbf{G},X)_{\mathbb{C}}$.

- either there exists an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}$ and a subset $\Sigma \subset \mathbb{N}$ such that each V_n, $n \in \Sigma$, is $\mathbf{T}$-special and $\mathbf{\Sigma} = \cup_{n\in\Sigma} V_n$ is Zariski-dense in Z. Then one can choose Σ so that the sequence (after possibly replacing by a subsequence) of probability measures $(\mu_{V_n})_{n\in\Sigma}$ weakly converges to the probability measure μ_V of some special subvariety V and for n large, V_n is contained in V. This implies that $Z = V$ is special (cf. theorem 2.3.1).

- otherwise the function $\deg_{L_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/E)\cdot V_n)$ is an unbounded function of n as n ranges through Σ and we can use Galois-theoretic methods to study Z (cf. definition 5.3.3 for the definition of the degree $\deg_{L_K}$) .

We now explain this alternative in more details.

2.3. **Equidistribution results.** Ratner's classification of probability measures on homogeneous spaces of the form $\Gamma\backslash\mathbf{G}(\mathbb{R})^+$ (where Γ denotes a lattice in $\mathbf{G}(\mathbb{R})^+$), ergodic under some unipotent flows [31], and Dani-Margulis recurrence lemma [10] enable Clozel and Ullmo [9] to prove the following equidistribution result in the strongly special case, generalized by Ullmo and Yafaev [39, theorem 3.8 and corollary 3.9] to the $\mathbf{T}$-special case:

Theorem 2.3.1 (Clozel-Ullmo, Ullmo-Yafaev)**.** *Let $(\mathbf{G},X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type, X^+ a connected component of X and K a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$. Let $\mathbf{T}$ be an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus of $\mathbf{G}$. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbf{T}$-special subvarieties of $S_K(\mathbf{G},X)_\mathbb{C}$. Let μ_{V_n} be the canonical probability measure on $\mathrm{Sh}_K(\mathbf{G},X)_\mathbb{C}$ supported on V_n. There exists a $\mathbf{T}$-special subvariety V of $S_K(\mathbf{G},X)_\mathbb{C}$ and a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ weakly converging to μ_V. Furthermore V contains V_{n_k} for all k sufficiently large. In particular, the irreducible components of the Zariski closure of a set of $\mathbf{T}$-special subvarieties of $S_K(\mathbf{G},X)_\mathbb{C}$ are special.*

Remarks 2.3.2.
(1) Note that a special point of $S_K(\mathbf{G},X)_\mathbb{C}$, whose Mumford-Tate group is a non-central torus, is not strongly special. Moreover, given an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}$, the connected Shimura variety $S_K(\mathbf{G},X)_\mathbb{C}$ contains only a finite number of $\mathbf{T}$-special points (cf. [39, lemma 3.7]). Thus theorem 2.3.1 says nothing *directly* on the André-Oort conjecture.
(2) In fact the conclusion of the theorem 2.3.1 is simply not true for special points: they are dense for the Archimedian topology in $S_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$, so just consider a sequence of special points converging to a non-special point in $S_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$ (or diverging to a cusp if $S_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$ is non-compact). In this case the corresponding sequence of Dirac delta measures will converge to the Dirac delta measure of the non-special point (respectively escape to infinity).
(3) There is a so-called equidistribution conjecture which implies the André-Oort conjecture and much more. A sequence (x_n) of points of $S_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$ is called *strict* if for any proper special subvariety V of $\mathrm{Sh}_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$, the set
$$\{n : x_n \in V\}$$
is finite. Let E be a field of definition of a canonical model of $\mathrm{Sh}_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$. To any special point x, one associates a probability measure Δ_x on $\mathrm{Sh}_K(\mathbf{G},X)_\mathbb{C}(\mathbb{C})$

as follows :

$$\Delta_x = \frac{1}{|\mathrm{Gal}(\overline{E}/E)\cdot x|}\sum_{y\in \mathrm{Gal}(\overline{E}/E)\cdot x}\delta_y$$

where δ_y is the Dirac measure at the point y and $|\mathrm{Gal}(\overline{E}/E)\cdot x|$ denotes the cardinality of the Galois orbit $\mathrm{Gal}(\overline{E}/E)\cdot x$. The equidistribution conjecture predicts that if (x_n) is a strict sequence of special points, then the sequence of measures Δ_{x_n} weakly converges to the canonical probability measure attached to $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$. This statement implies the André-Oort conjecture. The equidistribution conjecture is known for modular curves and is open in general. There are some recent conditional results for Hilbert modular varieties due to Zhang (see [44]). For more on this, we refer to the survey [37].

2.4. **Lower bounds for Galois orbits.** In this section, we recall the lower bound obtained in [39] for the degree of the Galois orbit of a special subvariety which is not strongly special.

2.4.1. *Data associated to a special subvariety.*

Definition 2.4.1. *Let $(\mathbf{G},X)$ be a Shimura datum and X^+ a connected component of X. Let $K=\prod_{p \text{ prime}} K_p$ be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$. Let $(\mathbf{H}_V,X_V)$ be a Shimura subdatum of $(\mathbf{G},X)$ defining a special subvariety V of $S_K(\mathbf{G},X)_{\mathbb{C}}$.*

We denote by:

- *E_V the reflex field of $(\mathbf{H}_V,X_{\mathbf{H}_V})$.*
- *$\mathbf{T}_V$ the connected centre of $\mathbf{H}_V$. It is a (possibly trivial) torus.*
- *$K^{\mathrm{m}}_{\mathbf{T}_V}=\prod_{p \text{ prime}} K^{\mathrm{m}}_{\mathbf{T}_V,p}$ the maximal compact open subgroup of $\mathbf{T}_V(\mathbf{A}_\mathrm{f})$, where $K^{\mathrm{m}}_{\mathbf{T}_V,p}$ denotes the maximal compact open subgroup of $\mathbf{T}_V(\mathbb{Q}_p)$.*
- *$K_{\mathbf{T}_V}$ the compact open subgroup $\mathbf{T}_V(\mathbf{A}_\mathrm{f})\cap K\subset K^{\mathrm{m}}_{\mathbf{T}_V}$. Thus $K_{\mathbf{T}_V}=\prod_{p \text{ prime}} K_{\mathbf{T}_V,p}$, where $K_{\mathbf{T}_V,p}:=\mathbf{T}_V(\mathbb{Q}_p)\cap K_p$.*
- *$\mathbf{C}_V$ the torus $\mathbf{H}_V/\mathbf{H}_V^{\mathrm{der}}$ isogenous to $\mathbf{T}_V$.*
- *$d_{\mathbf{T}_V}$ the absolute value of the discriminant of the splitting field L_V of $\mathbf{C}_V$, and n_V the degree of L_V over $\mathbb{Q}$.*
- *$\beta_V:=\log(d_{\mathbf{T}_V})$.*

Remark 2.4.2. Notice that the group $K_{\mathbf{T}_V}$ depends on the particular embedding $\mathbf{H}_V\hookrightarrow\mathbf{G}$ (which is determined by V up to conjugation by $\Gamma=\mathbf{G}(\mathbb{Q})_+\cap K$). On the other hand the other quantities defined above, and also the indices $|K^{\mathrm{m}}_{\mathbf{T}_V,p}/K_{\mathbf{T}_V,p}|$, p prime, depend on V but not on the particular embedding $\mathbf{H}_V\hookrightarrow\mathbf{G}$.

We will frequently make use of the following lemma:

Lemma 2.4.3. *With the above notations assume moreover that the group* $\mathbf{G}$ *is semisimple of adjoint type. Then the* $\mathbb{Q}$*-torus* $\mathbf{T}_V$ *is* $\mathbb{R}$*-anisotropic.*

Proof. As $\mathbf{T}_V$ is the connected centre of the generic Mumford-Tate group $\mathbf{H}_V \subset \mathbf{G}$ of V the group $\mathbf{T}_V(\mathbb{R})$ fixes some point x of X. As $\mathbf{G}$ is semisimple of adjoint type the stabiliser of x in $\mathbf{G}(\mathbb{R})$ is compact. $\square$

2.4.2. *The lower bound.* One of the main ingredients of our proof of theorem 1.2.2 is the following lower bound for the degree of Galois orbits obtained in [39, theorem 2.19] (we refer to the section 5 for the definition of the degree function $\deg_{L_K}$):

Theorem 2.4.4 (Ullmo-Yafaev)**.** *Assume the GRH for CM fields. Let* $(\mathbf{G}, X)$ *be a Shimura datum with* $\mathbf{G}$ *semisimple of adjoint type and let* X^+ *be a fixed connected component of* X*.*

Fix positive integers R *and* N*. There exist a positive real number* B *depending only on* $\mathbf{G}$*,* X *and* R *and a positive constant* $C(N)$ *depending on* $\mathbf{G}$*,* X*,* R *and* N *such that the following holds.*

Let $K = \prod_{p \text{ prime}} K_p$ *be a neat compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$*. Let* V *be a special subvariety of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$ *and* $(\mathbf{H}_V, X_V)$ *a Shimura subdatum of* $(\mathbf{G}, X)$ *defining* V*. Let* F *be an extension of* $\mathbb{Q}$ *of degree at most* R *containing the reflex field* E_V *of* $(\mathbf{H}_V, X_V)$*. Let* $K_{\mathbf{H}_V} := K \cap \mathbf{H}_V(\mathbf{A}_f)$*. Then:*

$$\deg_{L_{K_{\mathbf{H}_V}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V) > C(N) \cdot \Big(\prod_{\substack{p \text{ prime} \\ K^{\mathrm{m}}_{\mathbf{T}_V,p} \neq K_{\mathbf{T}_V,p}}} \max(1, B \cdot |K^{\mathrm{m}}_{\mathbf{T}_V,p}/K_{\mathbf{T}_V,p}|) \Big) \cdot \beta_V^N \quad . \tag{2.1}$$

Furthermore, if one fixes a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *and one considers only the subvarieties* V *such that the associated tori* $\mathbf{T}_V$ *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class, then the assumption of the GRH can be dropped.*

Remark 2.4.5. The lower bound (2.1) still holds if we replace V by Y an irreducible subvariety of V defined over $\overline{\mathbb{Q}}$ whose Galois orbits are “sufficiently similar” to those of V. For simplicity we refer to [39, theor.2.19] for this refined statement, which we will use in the proof of the lemma 9.2.3.

2.5. **The precise alternative.** Throughout the paper we will be using the following notations.

Definition 2.5.1. *Let* $(\mathbf{G}, X)$ *be a Shimura datum with* $\mathbf{G}$ *semisimple of adjoint type. Let* X^+ *be a fixed connected component of* X*.*

We fix R a positive integer such that for any Shimura subdatum $(\mathbf{H}, X_{\mathbf{H}})$ of $(\mathbf{G}, X)$ there exists an extension F of $\mathbb{Q}$ of degree at most R containing the Galois closure of the reflex field $E_{\mathbf{H}}$ of $(\mathbf{H}, X_{\mathbf{H}})$. Such an R exists by [39, lemma 2.5].

Let $K = \prod_{p \text{ prime}} K_p$ be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let V be a special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ and $(\mathbf{H}_V, X_V)$ a Shimura subdatum of $(\mathbf{G}, X)$ defining V.

With the notations of definition 2.4.1 and with B as in theorem 2.4.4 we define:

$$\alpha_V := \prod_{\substack{p \,\text{prime} \\ K^m_{\mathbf{T}_V,p} \neq K_{\mathbf{T}_V,p}}} \max(1, B \cdot |K^m_{\mathbf{T}_V,p}/K_{\mathbf{T}_V,p}|) \ .$$

Remark 2.5.2. By remark 2.4.2 the quantity α_V depends only on V and not on the particular embedding $\mathbf{H}_V \hookrightarrow \mathbf{G}$.

The alternative roughly explained in the introduction to section 2 can now be formulated in the following theorem ([39, theorem 3.10]).

Theorem 2.5.3 (Ullmo-Yafaev)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum with $\mathbf{G}$ semisimple of adjoint type. Let X^+ be a fixed connected component of X. Fix R a positive integer as in definition 2.5.1.*

Let $K = \prod_{p \text{ prime}} K_p$ be a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ be a set of special subvarieties V of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ such that $\alpha_V \beta_V$ is bounded as V ranges through Σ.

There exists a finite set $\{\mathbf{T}_1, \cdots, \mathbf{T}_r\}$ of $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtori of $\mathbf{G}$ such that any V in Σ is $\mathbf{T}_i$-special for some $i \in \{1, \cdots, r\}$.

3. Reduction and strategy.

From now on we will use the following convenient terminology:

Definition 3.0.4. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let Σ be a set of special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. A subset Λ of Σ is called a modification of Σ if $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ have the same Zariski closure in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ (recall, cf. definition 1.1.1, that $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ denote the unions of subvarieties in Λ and Σ respectively).*

3.1. First reduction.

We first have the following reduction of the proof of theorem 1.2.2:

Theorem 3.1.1. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let Z be an irreducible subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\mathbf{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma) < \dim Z$.*

We make **one** *of the following assumptions:*

1. *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*

(2) *Assume that there is a faithful representation* $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *such that with respect to this representation, the connected centres* $\mathbf{T}_V$ *of the generic Mumford-Tate groups* $\mathbf{H}_V$ *of* V *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class as* V *ranges through* Σ.

Then

(a) *The variety* Z *contains a Zariski dense set* Σ' *of special subvarieties of constant dimension* $n(\Sigma') > n(\Sigma)$.

(b) *Furthermore, if* Σ *satisfies the condition* (2)*, one can choose* Σ' *also satisfying* (2).

Proposition 3.1.2. *Theorem 3.1.1 implies the main theorem 1.2.2.*

Proof. Let $\mathbf{G}$, X, K and Σ as in the main theorem 1.2.2. Without loss of generality one can assume that the Zariski closure Z of $\boldsymbol{\Sigma}$ is irreducible. Moreover by Noetherianity one can assume that all the $V \in \Sigma$ have the same dimension $n(\Sigma)$.

Notice that the assumption (2) of the theorem 1.2.2 implies the assumption (2) of the theorem 3.1.1. We then apply theorem 3.1.1,(a) to Σ: the subvariety Z contains a Zariski-dense set $\boldsymbol{\Sigma}'$ of special subvarieties V', $V' \in \Sigma'$, of constant dimension $n(\Sigma') > n(\Sigma)$.

By theorem 3.1.1,(b) one can replace Σ by Σ'. Applying this process recursively and as $n(\Sigma') \leq \dim(Z)$, we conclude that Z is special. $\square$

3.2. **Second reduction.** Part (b) of theorem 3.1.1 will be dealt with in section 6. Part (a) of theorem 3.1.1 can itself be reduced to the following theorem:

Theorem 3.2.1. *Let* $(\mathbf{G}, X)$ *be a Shimura datum with* $\mathbf{G}$ *semisimple of adjoint type and let* X^+ *be a connected component of* X*. Fix* R *a positive integer as in the definition 2.5.1.*

Let $K = \prod_{p \text{ prime}} K_p$ *be a neat compact open subgroup of* $\mathbf{G}(\mathbf{A}_\mathrm{f})$*. Let* Z *be a Hodge generic geometrically irreducible subvariety of the connected component* $S_K(\mathbf{G}, X)_\mathbb{C}$ *of* $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$*. Suppose that* Z *contains a Zariski dense set* $\boldsymbol{\Sigma}$*, which is a union of special subvarieties* V*,* $V \in \Sigma$*, all of the same dimension* $n(\Sigma)$ *and such that for any modification* Σ' *of* Σ *the set* $\{\alpha_V \beta_V,\ V \in \Sigma'\}$ *is unbounded (with the notations of definitions 2.4.1 and 2.5.1).*

We make **one** *of the following assumptions:*

(1) *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*

(2) *Assume that there is a faithful representation* $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *such that with respect to this representation, the connected centres* $\mathbf{T}_V$ *of the generic Mumford-Tate groups* $\mathbf{H}_V$ *of* V *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class as* V *ranges through* Σ.

After possibly replacing Σ *by a modification, for every* V *in* Σ *there exists a special subvariety* V' *such that* $V \subsetneq V' \subset Z$.

Proposition 3.2.2. *Theorem 3.2.1 implies theorem 3.1.1* (a).

Proof. Let $(\mathbf{G}, X)$, K, Z and Σ be as in theorem 3.1.1.

First let us reduce the proof of theorem 3.1.1 to the case where in addition Z satisfies the assumptions of theorem 3.2.1.

Notice that the image of a special subvariety by a morphism of Shimura varieties deduced from a morphism of Shimura data is a special subvariety. Conversely any irreducible component of the preimage of a special subvariety by such a morphism is special. This implies that if $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is a compact open subgroup and if $K' \subset K$ is a finite index subgroup then theorem 3.1.1(a) is true at level K if and only if it is true at level K'. In particular we can assume without loss of generality that K is a product $\prod_{p \text{ prime}} K_p$ and that K is neat.

We can assume that the variety Z in theorem 3.1.1 is Hodge generic. To fulfill this condition, replace $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ by the smallest special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing Z (cf. [18, prop.2.1]). This comes down to replacing $\mathbf{G}$ with the generic Mumford-Tate group on Z.

Let $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the Shimura datum adjoint to $(\mathbf{G}, X)$ and $\lambda : (\mathbf{G}, X) \longrightarrow (\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ the natural morphism of Shimura data. For $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ sufficiently small let K^{ad} be a neat compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbf{A}_{\mathrm{f}})$ containing $\lambda(K)$. Consider the finite morphism of Shimura varieties $f : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Let Σ^{ad} be the set of special subvarieties $f(V)$ of $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$, $V \in \Sigma$. In order to be able to replace $\mathbf{G}$ by $\mathbf{G}^{\mathrm{ad}}$, we need to check that if Σ satisfies the assumption (2), then Σ^{ad} also satisfies the assumption (2). For V in Σ, let $(\mathbf{H}_V, X_V)$ be the Shimura datum defining V and $\mathbf{T}_V$ be the connected centre of $\mathbf{H}_V$. Then the tori $\mathbf{T}_V$ (and hence the tori $\lambda(\mathbf{T}_V)$) are split by the same field. Choose a faithful representation $\mathbf{G}^{\mathrm{ad}} \hookrightarrow \mathbf{GL}_m$. By [39], lemma 3.13, part (i), the tori $\lambda(\mathbf{T}_V)$ lie in finitely many $\mathbf{GL}_m(\mathbb{Q})$-conjugacy classes. It follows that, after replacing Σ^{ad} by a modification, the assumption (2) for $f(Z)$ is satisfied. Applying our first remark to the morphism f we obtain that theorem 3.1.1 (a) for $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ implies theorem 3.1.1 (a) for $(\mathbf{G}, X)$. Thus we reduced the proof of theorem 3.1.1 (a) to the case where $\mathbf{G}$ is semisimple of adjoint type.

We can also assume that Z is contained in $S_K(\mathbf{G}, X)_{\mathbb{C}}$ as proving theorem 3.1.1 for Z is equivalent to proving theorem 3.1.1 for any irreducible component of its image under some Hecke correspondence. In particular the quantities α_V and β_V, $V \in \Sigma$, are well-defined.

If for some modification Σ' of Σ the set $\{\alpha_V \beta_V,\ V \in \Sigma'\}$ is bounded, by theorem 2.5.3 and by Noetherianity there exists an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}$ and a modification of Σ such that any element of this modification is $\mathbf{T}$-special. Applying theorem 2.3.1 one obtains that Z is special.

Finally we can assume that Z satisfies the hypothesis of theorem 3.2.1: we have reduced the proof of theorem 3.1.1 to the case where in addition Z satisfies the assumptions of theorem 3.2.1.

Let Σ' be the set of the special subvarieties V' obtained from theorem 3.2.1 applied to Z. Thus Z contains the Zariski-dense set $\mathbf{\Sigma}' = \cup_{V' \in \Sigma'} V'$. After possibly replacing Σ' by a modification, we can assume by Noetherianity of Z that the subvarieties in Σ' have the same dimension $n(\Sigma') > n(\Sigma)$. This proves the theorem 3.1.1 (a) assuming theorem 3.2.1. $\square$

3.3. **Sketch of the proof of the André-Oort conjecture in the case where Z is a curve.** The strategy for proving theorem 3.2.1 is fairly complicated. We first recall the strategy developed in [18] in the case where Z is a curve. In the next section we explain why this strategy *cannot* be directly generalized to higher dimensional cases.

As already noticed in the proof of proposition 3.2.2 one can assume without loss of generality that the group $\mathbf{G}$ is semisimple of adjoint type, Z is Hodge generic (i.e. its generic Mumford-Tate group is equal to $\mathbf{G}$), and Z is contained in the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. The proof of the theorem 1.2.1 in the case where Z is a curve then relies on three ingredients.

3.3.1. The first one is a geometric *criterion* for a Hodge generic subvariety Z to be special in terms of Hecke correspondences. Given a Hecke correspondence T_m, $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ (cf. section 4.1.1) we denote by T_m^0 the correspondence it induces on $S_K(\mathbf{G}, X)_{\mathbb{C}}$. Let q_i, $1 \leq i \leq n$, be elements of $\mathbf{G}(\mathbb{Q})_+ \cap KmK$ defined by the equality

$$\mathbf{G}(\mathbb{Q})_+ \cap KmK = \coprod_{1 \leq i \leq n} \Gamma_K q_i^{-1} \Gamma_K \ .$$

Let T_{q_i}, $1 \leq i \leq n$, denote the correspondence on $S_K(\mathbf{G}, X)_{\mathbb{C}}$ induced by the action of q_i on X^+ (in general it does not coincide with the correspondence on $S_K(\mathbf{G}, X)_{\mathbb{C}}$ induced by the Hecke correspondence T_{q_i} on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$). The correspondence T_m^0 decomposes as $T_m^0 = \sum_{1 \leq i \leq n} T_{q_i}$.

Theorem 3.3.1. [18, theorem 7.1] *Let $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ be a Shimura variety, with $\mathbf{G}$ semisimple of adjoint type. Let $Z \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ be a Hodge generic subvariety of the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose there exist a prime l and an element $m \in \mathbf{G}(\mathbb{Q}_l)$ such that the neutral component $T_m^0 = \sum_{i=1}^{n} T_{q_i}$ of the Hecke correspondence T_m associated with m has the following properties:*

(1) $Z \subset T_m^0 Z$.

(2) For any $i \in \{1, \cdots n\}$, the varieties $T_{q_i} Z$ and $T_{q_i^{-1}} Z$ are irreducible.

(3) For any $i \in \{1, \cdots n\}$ the $T_{q_i} + T_{q_i^{-1}}$-orbit is dense in $S_K(\mathbf{G}, X)$.

Then $Z = S_K(\mathbf{G}, X)$, in particular Z is special.

From (1) and (2) one deduces the existence of one index i such that $Z = T_{q_i}Z = T_{q_i^{-1}}Z$. It follows that Z contains a $T_{q_i} + T_{q_i^{-1}}$-orbit. The equality $Z = S_K$ follows from (3).

In the case where Z is a curve one proves the existence of a prime l and of an element $m \in \mathbf{G}(\mathbb{Q}_l)$ satisfying these properties as follows. The property (3) is easy to obtain: it is satisfied by any m such that for each simple factor $\mathbf{G}_j$ of $\mathbf{G}$, the projection of m to $\mathbf{G}_j(\mathbb{Q}_l)$ is not contained in a compact subgroup (see [18], Theorem 6.1). The property (2), which is crucial for this strategy, is obtained by showing that for any prime l outside a finite set of primes $\mathcal{P}_Z$ and any $q \in \mathbf{G}(\mathbb{Q})^+ \cap (\mathbf{G}(\mathbb{Q}_l) \times \prod_{p \neq l} K_p)$, the variety T_qZ is irreducible. This is a corollary of a result due independently to Weisfeiler and Nori (cf. theorem 4.2.3) applied to the Zariski closure of the image of the monodromy representation. This result implies that for all l except those in a finite set $\mathcal{P}_Z$, the closure in $\mathbf{G}(\mathbb{Q}_l)$ of the image of the monodromy representation for the $\mathbb{Z}$-variation of Hodge structure on the smooth locus Z^{sm} of Z coincides with the closure of $K \cap \mathbf{G}(\mathbb{Q})^+$ in $\mathbf{G}(\mathbb{Q}_l)$. To prove the property (1) one uses Galois orbits of special points contained in Z and the fact that Hecke correspondences commute with the Galois action. First one notices that Z is defined over a number field F, finite extension of the reflex field $E(\mathbf{G}, X)$ (cf. section 4.1.2). If $s \in Z$ is a special point, r_s the associated reciprocity morphism and $m \in \mathbf{G}(\mathbb{Q}_l)$ belongs to $r_s((\mathbb{Q}_l \otimes F)^*) \subset \mathbf{MT}(s)(\mathbb{Q}_l)$ then the Galois orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot s$ is contained in the intersection $Z \cap T_mZ$. If this intersection is proper its cardinality $Z \cap T_mZ$ is bounded above by a uniform constant times the degree $[K_l : K_l \cap mK_lm^{-1}]$ of the correspondence T_m. To find l and m such that $Z \subset T_mZ$ it is then enough to exhibit an $m \in r_s((\mathbb{Q}_l \otimes F)^*)$ such that the cardinality $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s|$ is larger than $[K_l : K_l \cap mK_lm^{-1}]$. This is dealt with by the next two ingredients.

3.3.2. The second ingredient claims the existence of "unbounded" Hecke correspondences of controlled degree defined by elements in $r_s((\mathbb{Q}_l \otimes F)^*)$:

Theorem 3.3.2. [18, corollary 7.4.4] *There exists an integer k such that for all $s \in \Sigma$ and for any prime l splitting $MT(s)$ such that $\mathbf{MT}(s)_{\mathbb{F}_l}$ is a torus, there exists an $m \in r_s((\mathbb{Q}_l \otimes F)^*) \subset \mathbf{MT}(s)(\mathbb{Q}_l)$ such that*

(1) for any simple factor $\mathbf{G}_i$ of $\mathbf{G}$ the image of m in $\mathbf{G}_i(\mathbb{Q}_l)$ is not in a compact subgroup.

(2) $[K_l : K_l \cap mK_lm^{-1}] \ll l^k$.

3.3.3. The third ingredient is a lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot s|$ due to Edixhoven, and improved in theorem 2.4.4.

3.3.4. Finally using this lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot s|$ and the effective Chebotarev theorem consequence of the GRH one proves the existence for any special point $s\in\Sigma$ with a sufficiently big Galois orbit of a prime l outside $\mathcal{P}_Z$, splitting $\mathbf{MT}(s)$, such that $\mathbf{MT}(s)_{\mathbb{F}_l}$ is a torus and such that $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s|\gg l^k$. Effective Chebotarev is not needed under the assumption that the $\mathbf{MT}(s)$, $s\in\Sigma$, are isomorphic. The reason being that in this case, the splitting field of the $\mathbf{MT}(s)$ is constant and the classical Chebotarev theorem provides us with a suitable l.

We then choose an m satisfying the conditions of the theorem 3.3.2. As $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s|\gg [K_l : K_l\cap mK_lm^{-1}]$ one obtains $Z\subset T_mZ$ and by the criterion 3.3.1 the subvariety Z is special.

3.4. **Strategy for proving the theorem 3.2.1: the general case.** Let $\mathbf{G}$, X, X^+, K, Z and Σ be as in the statement of the theorem 3.2.1.

Notice that the idea of the proof of [18] generalizes to the case where $\dim Z = n(\Sigma)+1$ (cf. section 9.2.1). In the general case, for a V in Σ with $\alpha_V\beta_V$ sufficiently large we want to exhibit V' special subvariety in Z containing V properly.

Our first step (section 7) is geometric: we give a criterion (theorem 7.2.1) similar to criterion 3.3.1 saying that an inclusion $Z\subset T_mZ$, for a prime l and an element $m\in\mathbf{H}_V(\mathbb{Q}_l)$ satisfying certain conditions, implies that V is properly contained in a special subvariety V' of Z.

The criterion we need has to be much more subtle than the one in [18]. In the characterization of [18], in order to obtain the irreducibility of T_mZ the prime l must be outside some finite set $\mathcal{P}_Z$ of primes. It seems impossible to make the set of bad primes $\mathcal{P}_Z$ explicit in terms of numerical invariants of Z, except in a few cases where the Chow ring of the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ is easy to describe (like the case considered by Edixhoven, where $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ is a product $\prod_{i=1}^n X_i$ of modular curves, and where he shows that for a k-dimensional subvariety Z dominant on all factors X_i, $1\le i\le n$, the bad primes $p\in\mathcal{P}_Z$ are smaller than the supremum of the degree of the projections of Z on the k-factors $X_{i_1}\times\cdots\times X_{i_k}$ of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$). In particular that characterization is not suitable for our induction.

Our criterion 7.2.1 for an irreducible subvariety Z containing a special subvariety V which is not strongly special and satisfying $Z\subset T_mZ$ for some $m\in\mathbf{T}_V(\mathbb{Q}_l)$ to contain a special subvariety V' containing V properly does no longer require the irreducibility of T_mZ. In particular it is valid for *any* prime l, outside $\mathcal{P}_Z$ or not. Instead we notice that the inclusion $Z \subset T_m Z$ implies that Z contains the image Z' in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ of the $\langle K'_l, (k_1 m k_2)^n \rangle$-orbit of (one irreducible component of) the preimage of V in the pro-l-covering of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Here k_1 and k_2 are some elements of K_l, n some positive integer and K'_l the l-adic closure of the image of the monodromy of Z. If the group $\langle K'_l, (k_1 m k_2)^n \rangle$ is not compact, then the irreducible component of Z' containing V contains a special subvariety V' of Z containing V properly.

The main problem with this criterion is that the group $\langle K'_l, k_1 m k_2 \rangle$ can be compact, containing K'_l with very small index. This is the case in Edixhoven's counter-example [15, Remark 7.2]. In this case $\mathbf{G} = \mathrm{PGL}_2 \times \mathrm{PGL}_2$, $K'_l := \Gamma_0(l) \times \Gamma_0(l)$ and $k_1 m k_2$ is $w_l \times w_l$, the product of two Atkin-Lehner involutions. The index $[\langle K'_l, k_1 m k_2 \rangle : K'_l]$ is four.

Our second step (section 8) consists in getting rid of this problem and is purely group-theoretic. We notice that if K_l is *not a maximal* compact open subgroup but is contained in a well-chosen *Iwahori* subgroup of $\mathbf{G}(\mathbb{Q}_l)$, then for "many" m in $\mathbf{T}_V(\mathbb{Q}_l)$ the element $k_1 m k_2$ is not contained in a compact subgroup for any k_1 and k_2 in K_l. This is our theorem 8.1 about the existence of adequate Hecke correspondences. The proof relies on simple properties of the Bruhat-Tits decomposition of $\mathbf{G}(\mathbb{Q}_l)$.

Our third step (section 9) is Galois-theoretic and geometric. We use theorem 2.4.4, theorem 7.2.1, theorem 8.1 to show (under one of the assumptions of theorem 3.1.1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a special but not strongly special subvariety V to contain a special subvariety V' containing V properly. The proof is a nice geometric induction on $r = \dim Z - \dim V$.

Our last step (section 10) is number-theoretic: we complete the proof of the theorem 3.2.1 and hence of theorem 1.2.2 by exhibiting, using effective Chebotarev under the GRH (or usual Chebotarev under the second assumption of theorem 1.2.2), a prime l satisfying our desiderata. For this step it is crucial that both the index of an Iwahori subgroup in a maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$ and the degree of the correspondence T_m are bounded by a uniform power of l.

4. Preliminaries.

4.1. **Shimura varieties.** In this section we define some notations and recall some standard facts about Shimura varieties that we will use in this paper. We refer to [11], [12], [23] for details.

As far as groups are concerned, reductive algebraic groups are assumed to be connected. The exponent ^{0} denotes the algebraic neutral component and the exponent $^+$ the topological neutral component. Thus if $\mathbf{G}$ is a $\mathbb{Q}$-algebraic group $\mathbf{G}(\mathbb{R})^+$ denotes the topological neutral component of the real Lie group of $\mathbb{R}$-points $\mathbf{G}(\mathbb{R})$. We also denote by $\mathbf{G}(\mathbb{Q})^+$ the intersection $\mathbf{G}(\mathbb{R})^+ \cap \mathbf{G}(\mathbb{Q})$.

When $\mathbf{G}$ is reductive we denote by $\mathbf{G}^{\mathrm{ad}}$ the adjoint group of $\mathbf{G}$ (the quotient of $\mathbf{G}$ by its center) and by $\mathbf{G}(\mathbb{R})_+$ the preimage in $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$. The notation $\mathbf{G}(\mathbb{Q})_+$ denotes the intersection $\mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q})$. In particular when $\mathbf{G}$ is adjoint then $\mathbf{G}(\mathbb{Q})^+ = \mathbf{G}(\mathbb{Q})_+$.

For any topological space Z, we denote by $\pi_0(Z)$ the set of connected components of Z.

4.1.1. *Definition.* Let $(\mathbf{G}, X)$ be a Shimura datum. We fix X^+ a connected component of X. Given K a compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$ one obtains the homeomorphic decomposition

$$\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C} = \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbf{A}_\mathrm{f})/K \simeq \coprod_{g\in\mathcal{C}} \Gamma_g\backslash X^+ \quad , \tag{4.1}$$

where $\mathcal{C}$ denotes a set of representatives for the (finite) double coset space $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbf{A}_\mathrm{f})/K$, and Γ_g denotes the arithmetic subgroup $gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ of $\mathbf{G}(\mathbb{Q})_+$. We denote by Γ_K the group Γ_e corresponding to the identity element $e \in \mathcal{C}$ and by $S_K(\mathbf{G}, X)_\mathbb{C} = \Gamma_K\backslash X^+$ the corresponding connected component of $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$.

The Shimura variety $\mathrm{Sh}(\mathbf{G}, X)_\mathbb{C}$ is the $\mathbb{C}$-scheme projective limit of the $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$ for K ranging through compact open subgroups of $\mathbf{G}(\mathbf{A}_\mathrm{f})$. The group $\mathbf{G}(\mathbf{A}_\mathrm{f})$ acts continuously on the right on $\mathrm{Sh}(\mathbf{G}, X)_\mathbb{C}$. The set of $\mathbb{C}$-points of $\mathrm{Sh}(\mathbf{G}, X)_\mathbb{C}$ is

$$\mathrm{Sh}(\mathbf{G}, X)_\mathbb{C}(\mathbb{C}) = \frac{\mathbf{G}(\mathbb{Q})}{\mathbf{Z}(\mathbb{Q})}\backslash(X \times \mathbf{G}(\mathbf{A}_\mathrm{f})/\overline{\mathbf{Z}(\mathbb{Q})}) \quad ,$$

where $\mathbf{Z}$ denotes the centre of $\mathbf{G}$ and $\overline{\mathbf{Z}(\mathbb{Q})}$ denotes the closure of $\mathbf{Z}(\mathbb{Q})$ in $\mathbf{G}(\mathbf{A}_\mathrm{f})$ [12, prop.2.1.10]. The action of $\mathbf{G}(\mathbf{A}_\mathrm{f})$ on the right is given by: $\overline{(x, h)} \xrightarrow{\cdot g} \overline{(x, h \cdot g)}$. For $m \in \mathbf{G}(\mathbf{A}_\mathrm{f})$, we denote by T_m the Hecke correspondence

$$\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C} \longleftarrow \mathrm{Sh}(\mathbf{G}, X)_\mathbb{C} \xrightarrow{\cdot m} \mathrm{Sh}(\mathbf{G}, X)_\mathbb{C} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C} \quad .$$

4.1.2. *Reciprocity morphisms and canonical models.* Given $(\mathbf{G}, X)$ a Shimura datum, where X is the $\mathbf{G}(\mathbb{R})$-conjugacy class of some $h : \mathbf{S} \longrightarrow \mathbf{G}_\mathbb{R}$, we denote by $\mu_h : \mathbf{G}_{m,\mathbb{C}} \longrightarrow \mathbf{G}_\mathbb{C}$ the $\mathbb{C}$-morphism of $\mathbb{Q}$-groups obtained by composing the embedding of tori

$$\begin{array}{ccc} \mathbf{G}_{m,\mathbb{C}} & \longrightarrow & \mathbf{S}_\mathbb{C} \\ z & \longrightarrow & (z, 1) \end{array}$$

with $h_\mathbb{C}$. Let $E(\mathbf{G}, X)$ be the field of definition of the $\mathbf{G}(\mathbb{C})$-conjugacy class of μ_h, it is called the reflex field of $(\mathbf{G}, X)$. In the case where $\mathbf{G}$ is a torus $\mathbf{T}$ and $X = \{h\}$ we denote by

$$r_{(\mathbf{T},\{h\})} : \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \longrightarrow \mathbf{T}(\mathbf{A}_{\mathrm{f}})/\overline{\mathbf{T}(\mathbb{Q})}$$

(where $\overline{\mathbf{T}(\mathbb{Q})}$ is the closure of $\mathbf{T}(\mathbb{Q})$ in $\mathbf{T}(\mathbf{A}_{\mathrm{f}})$) the reciprocity morphism defined in [12, 2.2.3] for any field $E \subset \mathbb{C}$ containing $E(\mathbf{T}, \{h\})$. Let $x = \overline{(h, g)}$ be a special point in $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ image of the pair $(h : \mathbf{S} \longrightarrow \mathbf{T} \subset \mathbf{G}, g) \in X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})$. The field $E(h) = E(\mathbf{T}, \{h\})$ depends only on h and is an extension of $E(\mathbf{G}, X)$ [12, 2.2.1]. The Shimura variety $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ admits a unique model $\mathrm{Sh}(\mathbf{G}, X)$ over $E(\mathbf{G}, X)$ such that the $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$-action on the right is defined over $E(\mathbf{G}, X)$, the special points are algebraic and if $x = \overline{(h, g)}$ is a special point of $\mathrm{Sh}(\mathbf{G}, X)(\mathbb{C})$ then an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E(h)) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{G}, X))$ acts on x by $\sigma(x) = \overline{(h, \tilde{r}(\sigma)g)}$, where $\tilde{r}(\sigma) \in \mathbf{T}(\mathbf{A}_{\mathrm{f}})$ is any lift of $r_{(\mathbf{T},\{h\})}(x) \in \mathbf{T}(\mathbf{A}_{\mathrm{f}})/\overline{\mathbf{T}(\mathbb{Q})}$, cf. [12, 2.2.5]. This is called the canonical model of $\mathrm{Sh}(\mathbf{G}, X)$. For any compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$, one obtains the canonical model for $\mathrm{Sh}_K(\mathbf{G}, X)$ over $E(\mathbf{G}, X)$. For details on this definition, sketches of proofs of the existence and uniqueness and all the relevant references we refer the reader to Chapters 12-14 of [23] as well as [12].

For $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ the Hecke correspondence T_m is defined over $E(\mathbf{G}, X)$. We will denote by $\pi_K : \mathrm{Sh}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ the natural projection.

4.1.3. *The tower of Shimura varieties at a prime l.* Let l be a prime. Suppose $K^l \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}}^l)$ is a compact open subgroup, where $\mathbf{A}_{\mathrm{f}}^l$ denotes the ring of finite adèles outside l.

Definition 4.1.1. *We denote by* $\mathrm{Sh}_{K^l}(\mathbf{G}, X)$ *the* $E(\mathbf{G}, X)$*-scheme* $\varprojlim \mathrm{Sh}_{K^l \cdot U_l}(\mathbf{G}, X)$ *where* U_l *runs over all compact open subgroups of* $\mathbf{G}(\mathbb{Q}_l)$.

The scheme $\mathrm{Sh}_{K^l}(\mathbf{G}, X)$ is the quotient $\mathrm{Sh}(\mathbf{G}, X)/K^l$. It admits a continuous $\mathbf{G}(\mathbb{Q}_l)$-action on the right. Given a compact open subgroup $U_l \subset \mathbf{G}(\mathbb{Q}_l)$ we denote by π_{U_l} : $\mathrm{Sh}_{K^l}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K^l U_l}(\mathbf{G}, X)$ the canonical projection.

4.1.4. *Neatness.* Let $\mathbf{G} \subset \mathbf{GL}_n$ be a linear algebraic group over $\mathbb{Q}$. We recall the definition of *neatness* for subgroups of $\mathbf{G}(\mathbb{Q})$ and its generalization to subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We refer to [3] and [28, 0.6] for more details.

Given an element $g \in \mathbf{G}(\mathbb{Q})$ let $\mathrm{Eig}(g)$ be the subgroup of $\overline{\mathbb{Q}}^*$ generated by the eigenvalues of g. We say that $g \in \mathbf{G}(\mathbb{Q})$ is *neat* if the subgroup $\mathrm{Eig}(g)$ is torsion-free. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is neat if any element of Γ is neat. In particular such a group is torsion-free.

Remark 4.1.2. The notion of neatness is independent of the embedding $\mathbf{G} \subset \mathbf{GL}_n$.

Given an element $g_p \in \mathbf{G}(\mathbb{Q}_p)$ let $\mathrm{Eig}_p(g_p)$ be the subgroup of $\overline{\mathbb{Q}_p}^*$ generated by all eigenvalues of g_p. Let $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}_p}$ be some embedding and consider the torsion part $(\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g_p))_{\mathrm{tors}}$. Since every subgroup of $\overline{\mathbb{Q}}^*$ consisting of roots of unity is normalized by

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, this group does not depend on the choice of the embedding $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}_p}^*$. We say that g_p is *neat* if

$$(\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g_p))_{\mathrm{tors}} = \{1\} \ .$$

We say that $g = (g_p)_p \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is neat if

$$\bigcap_p (\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g_p))_{\mathrm{tors}} = \{1\} \ .$$

A subgroup $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is neat if any element of K is neat. Of course if the projection K_p of K in $\mathbf{G}(\mathbb{Q}_p)$ is neat then K is neat. Notice that if K is a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ then all of the Γ_g in the decomposition (4.1) are.

Neatness is preserved by conjugacy and intersection with an arbitrary subgroup. Moreover if $\rho : \mathbf{G} \longrightarrow \mathbf{H}$ is a $\mathbb{Q}$-morphism of linear algebraic $\mathbb{Q}$-groups and $g \in \mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$) is neat then its image $\rho(g)$ is also neat.

We recall the following well-known lemma:

Lemma 4.1.3. *Let $K = \prod_p K_p$ be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let l be a prime number. There exists an open subgroup K'_l of K_l such that the subgroup $K' := K'_l \times \prod_{p\neq l} K_p$ of K is neat.*

Proof. As noticed above if K'_l is neat then $K' := K'_l \times \prod_{p\neq l} K_p$ is neat. As a subgroup of a neat group is neat, it is enough to show that a special maximal compact open subgroup $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ contains a neat subgroup K'_l with finite index. By [28, p.5] one can take, $K'_l = K_l^{(1)}$ the first congruence kernel. □

4.1.5. *Integral structures.* Let $(\mathbf{G}, X)$ be a Shimura datum and $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a neat compact open subgroup. We can fix a $\mathbb{Z}$-structure on $\mathbf{G}$ and its subgroups by choosing a finitely generated free $\mathbb{Z}$-module W, a faithful representation $\xi \colon \mathbf{G} \hookrightarrow \mathbf{GL}(W_{\mathbb{Q}})$ and taking the Zariski closures in the $\mathbb{Z}$-group-scheme $\mathbf{GL}(W)$. If we choose the representation ξ in such a way that K is contained in $\mathbf{GL}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W)$ (i.e. K stabilizes $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W$) and ξ factors through $\mathbf{G}^{\mathrm{ad}}$, this induces canonically a $\mathbb{Z}$-variation of Hodge structure on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$: cf. [18, section 3.2]. If $K = \prod_{p \text{ prime}} K_p$ then for almost all primes l the group K_l is a hyperspecial maximal compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$ which coincides with $\mathbf{G}(\mathbb{Z}_l)$.

4.1.6. *Good position with respect to a torus.*

Definition 4.1.4. *Let l be a prime number, $\mathbf{G}$ a reductive $\mathbb{Q}_l$-group and $\mathbf{T} \subset \mathbf{G}$ a split torus. A compact open subgroup U_l of $\mathbf{G}(\mathbb{Q}_l)$ is said to be in good position with respect to $\mathbf{T}$ if $U_l \cap \mathbf{T}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{T}(\mathbb{Q}_l)$.*

If $\mathbf{G}$ *is a reductive* $\mathbb{Q}$*-group,* $\mathbf{T} \subset \mathbf{G}$ *a torus and* l *a prime number splitting* $\mathbf{T}$*, we say that a compact open subgroup* U_l *of* $\mathbf{G}(\mathbb{Q}_l)$ *is in good position with respect to* $\mathbf{T}$ *if it is in good position with respect to* $\mathbf{T}_{\mathbb{Q}_l}$*.*

Lemma 4.1.5. *Suppose that* $(\mathbf{G}, X)$ *is a Shimura datum,* $K = \prod_{p \text{ prime}} K_p$ *is a neat open compact subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ *and* $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}_n$ *is a faithful rational representation such that* K *is contained in* $\mathbf{GL}_n(\hat{\mathbb{Z}})$*. Let* $\mathbf{T} \subset \mathbf{G}$ *be a torus and* l *be a prime number splitting* $\mathbf{T}$ *such that* $\mathbf{T}_{\mathbb{F}_l}$ *is an* $\mathbb{F}_l$*-torus. Then the group* $\mathbf{G}(\mathbb{Z}_l)$ *is in good position with respect to* $\mathbf{T}$*.*

Proof. Let $\mathbf{T}'$ be the scheme-theoretic closure of $\mathbf{T}_{\mathbb{Q}_l}$ in $(\mathbf{GL}_n)_{\mathbb{Z}_l}$. The scheme $\mathbf{T}'$ is a flat group scheme affine and of finite type over $\mathbb{Z}_l$ whose fibers $\mathbf{T}_{\mathbb{F}_l}$ over $\mathbb{F}_l$ and $\mathbf{T}_{\mathbb{Q}_l}$ over $\mathbb{Q}_l$ are tori. Hence by [13, Exp.X, cor.4.9] the group scheme $\mathbf{T}'$ is a torus over $\mathbb{Z}_l$. As its generic fiber $\mathbf{T}_{\mathbb{Q}_l}$ is split, $\mathbf{T}'$ is split by [13, Exp.X, cor.1.2]. Hence $\mathbf{G}(\mathbb{Z}_l) \cap \mathbf{T}(\mathbb{Q}_l) = \mathbf{T}'(\mathbb{Z}_l)$ is a maximal compact subgroup of $\mathbf{T}(\mathbb{Q}_l) = \mathbf{T}'(\mathbb{Q}_l)$ and the result follows. □

4.2. ***p*-adic closure of Zariski-dense groups.** We will use the following well-known result (we provide a proof for completeness):

Proposition 4.2.1. *Let* H *be a subgroup of* $\mathbf{GL}_n(\mathbb{Z})$ *and let* $\mathbf{H}$ *be the Zariski closure of* H *in* $\mathbf{GL}_{n,\mathbb{Z}}$*. Suppose that* $\mathbf{H}^0_{\mathbb{Q}}$ *is semisimple. Then for any prime number* p *the closure of* H *in* $\mathbf{H}(\mathbb{Z}_p)$ *is open.*

Proof. The case when H is finite is obvious. Suppose that H is infinite. Since $\mathbf{H}(\mathbb{Z}_p)$ is compact and H is infinite, the closure H_p of H in $\mathbf{H}(\mathbb{Z}_p)$ is not discrete. Then it is a p-adic analytic group and it has a Lie algebra L which is a Lie subalgebra of the Lie algebra Lie $\mathbf{H}$ of $\mathbf{H}$ and projects non-trivially on any factor of Lie $\mathbf{H}$. By construction L is invariant under the adjoint action of H, thus also under the adjoint action of the Zariski closure $\mathbf{H}$ of H. As $\mathbf{H}^0_{\mathbb{Q}}$ is semisimple one deduces $L_{\mathbb{Q}} = \text{Lie } \mathbf{H}_{\mathbb{Q}}$, which implies that H_p is open in $\mathbf{H}(\mathbb{Z}_p)$. □

Remark 4.2.2. The easy proposition 4.2.1 can be strengthened to the following remarkable theorem, due independently to Weisfeiler and Nori, which was used in [18] but which we will not need:

Theorem 4.2.3 ([41], [27])**.** *Let* H *be a finitely generated subgroup of* $\mathbf{GL}_n(\mathbb{Z})$ *and let* $\mathbf{H}$ *be the Zariski closure of* H *in* $\mathbf{GL}_{n,\mathbb{Z}}$*. Suppose that* $\mathbf{H}(\mathbb{C})$ *has finite fundamental group. Then the closure of* H *in* $\mathbf{GL}_n(\mathbf{A}_{\mathrm{f}})$ *is open in the closure of* $\mathbf{H}(\mathbb{Z})$ *in* $\mathbf{GL}_n(\mathbf{A}_{\mathrm{f}})$*.*

5. Degrees on Shimura varieties.

In this section we recall the results we will need on projective geometry of Shimura varieties and prove the crucial corollary 5.3.10 which compares the degrees of a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)$ with respect to two different line bundles.

5.1. **Degrees.** We will need only basics on numerical intersection theory as recalled in [22, chap.1, p.15-17]. Let X be a complete irreducible complex variety and L a line bundle on X with topological first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. Given $V \subset X$ an irreducible subvariety we define the degree of V with respect to L by

$$\deg_L V = c_1(L)^{\dim V} \cap [V] \in H_0(X, \mathbb{Z}) = \mathbb{Z} \quad ,$$

where $[V] \in H_{2\dim V}(X, \mathbb{Z})$ denotes the fundamental class of V and $\cap$ denotes the cap product between $H^{2\dim V}(X, \mathbb{Z})$ and $H_{2\dim V}(X, \mathbb{Z})$. We also write $\deg_L V = \int_V c_1(L)^{\dim V}$. It satisfies the projection formula: given $f : Y \longrightarrow X$ a generically finite surjective proper map one has

$$\deg_{f^*L} Y = (\deg f) \deg_L X \quad .$$

When the subvariety V is not irreducible, let $V = \cup_i V_i$ be its decomposition into irreducible components. We define

$$\deg_L V = \sum_i \deg_L V_i \quad .$$

When the variety X is a disjoint union of irreducible components X_i, $1 \leq i \leq n$, the function $\deg_L$ is defined as the sum $\sum_{i=1}^n \deg_{L_{|X_i}}$.

5.2. **Nefness.** Recall (cf. [22, def. 1.4.1]) that a line bundle L on a complete scheme X is said to be *nef* if $\deg_L C \geq 0$ for every irreducible curve $C \subset X$. We will need the following basic result (cf. [22, theor.1.4.9]):

Theorem 5.2.1 (Kleiman)**.** *Let L be a line bundle on a complete complex scheme X. Then L is nef if and only if for every irreducible subvariety $V \subset X$ one has* $\deg_L V \geq 0$.

5.3. **Baily-Borel compactification.**

Definition 5.3.1. *Let $(\mathbf{G}, X)$ be a Shimura datum and $K \subset \mathbf{G}(\mathbf{A}_\mathrm{f})$ a neat compact open subgroup. We denote by $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}}$ the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$, cf.* [2].

The Baily-Borel compactification $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}}$ is a normal projective variety. Its boundary $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}} \setminus \mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$ has complex codimension > 1 if and only if $\mathbf{G}$ has no split $\mathbb{Q}$-simple factors of dimension 3. The following proposition summarizes basic properties of $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}}$ that we will use.

Proposition 5.3.2. *(1) The line bundle of holomorphic forms of maximal degree on X descends to $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ and extends uniquely to an ample line bundle L_K on $\overline{\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}}$ such that, at the generic points of the boundary components of codimension one, it is given by forms with logarithmic poles. Let K_1 and K_2 be neat compact open subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and g in $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ such that $K_2 \subset gK_1g^{-1}$. Then the morphism from $\mathrm{Sh}_{K_2}(\mathbf{G},X)_{\mathbb{C}}$ to $\mathrm{Sh}_{K_1}(\mathbf{G},X)_{\mathbb{C}}$ induced by g extends to a morphism $f : \overline{\mathrm{Sh}_{K_2}(\mathbf{G},X)_{\mathbb{C}}} \longrightarrow \overline{\mathrm{Sh}_{K_1}(\mathbf{G},X)_{\mathbb{C}}}$, and the line bundle $f^*L_{K_1}$ is canonically isomorphic to L_{K_2} .*

(2) The canonical model $\mathrm{Sh}_K(\mathbf{G},X)$ of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ over the reflex field $E(\mathbf{G},X)$ admits a unique extension to a model $\overline{\mathrm{Sh}_K(\mathbf{G},X)}$ of $\overline{\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}}$ over $E(\mathbf{G},X)$. The line bundle L_K is naturally defined over $E(\mathbf{G},X)$.

(3) Let $\varphi : (\mathbf{H},Y) \longrightarrow (\mathbf{G},X)$ be a morphism of Shimura data and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbf{A}_{\mathrm{f}})$, $K_{\mathbf{G}} \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ neat compact open subgroups with $\varphi(K_{\mathbf{H}}) \subset K_{\mathbf{G}}$. Then the canonical map $\phi : \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},Y) \longrightarrow \mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},X)$ induced by φ extends to a morphism still denoted by $\phi : \overline{\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H},Y)} \longrightarrow \overline{\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G},X)}$.

Proof. The first statement is [2, lemma 10.8] and [28, prop.8.1, sections 8.2, 8.3]. The second one is [28, theor.12.3.a]. The third statement is [32, theorem p.231] (over $\mathbb{C}$) and [28, theor. 12.3.b] (over $E(\mathbf{G},X)$). □

Definition 5.3.3. *Given a complex subvariety $Z \subset \mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ we will denote by $\deg_{L_K} Z$ the degree of the compactification $\overline{Z} \subset \overline{\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}}$ with respect to the line bundle L_K. We will write $\deg Z$ when it is clear to which line bundle we are referring to.*

Remark 5.3.4. Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group of Hermitian type (and of non-compact type) with associated Hermitian domain X. Recall that a subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called an arithmetic lattice if Γ is commensurable to $\mathbf{G}(\mathbb{Q}) \cap \mathbf{GL}_n(\mathbb{Z})$, where we fixed a faithful $\mathbb{Q}$-representation $\xi : \mathbf{G} \hookrightarrow \mathbf{GL}_n$. This definition is independent of the choice of ξ. If $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a neat arithmetic lattice the quotient $\Gamma\backslash X$ is a smooth quasi-projective variety, which is projective if and only if $\mathbf{G}$ is $\mathbb{Q}$-anisotropic (cf. [3]). The Baily-Borel compactification $\overline{\Gamma\backslash X}$ of the quasi-projective complex variety $\Gamma\backslash X$ and the bundle L_Γ on $\overline{\Gamma\backslash X}$ are well-defined (cf. [2]).

5.3.1. *Comparison of degrees for Shimura subdata.*

Proposition 5.3.5. *Let $\phi : \mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}',X')_{\mathbb{C}}$ be a morphism of Shimura varieties associated to a Shimura subdatum $\varphi : (\mathbf{G},X) \longrightarrow (\mathbf{G}',X')$, a neat compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and a neat compact open subgroup K' of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$ containing $\varphi(K)$. Then the line bundle*

$$\Lambda_{K,K'} := \phi^* L_{K'} \otimes L_K^{-1}$$

on $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ is nef.

This proposition is a corollary of the following

Proposition 5.3.6. *Let $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ be a $\mathbb{Q}$-morphism of connected semisimple algebraic $\mathbb{Q}$-groups of Hermitian type (and of non-compact type) inducing a holomorphic totally geodesic embedding of the associated Hermitian domains $\phi : X^+ \longrightarrow X'^+$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic lattice and $\Gamma' \subset \mathbf{G}'(\mathbb{Q})$ a neat arithmetic lattice containing $\varphi(\Gamma)$. Then the line bundle*

$$\Lambda_{\Gamma,\Gamma'} := \phi^* L_{\Gamma'} \otimes L_{\Gamma}^{-1}$$

on $\overline{\Gamma\backslash X^+}$ is nef.

Proposition 5.3.6 implies the proposition 5.3.5. Let $C \subset \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ be an irreducible curve. To prove that $\deg_{\Lambda_{K,K'}} C \geq 0$ one can assume without loss of generality that C is contained in the connected component $\overline{S_K} = \overline{\Gamma_K\backslash X^+}$ and that $\phi : \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}} \longrightarrow \overline{\mathrm{Sh}_{K'}(\mathbf{G}', X')_{\mathbb{C}}}$ maps $\overline{S_K}$ to $\overline{S_{K'}} = \overline{\Gamma_{K'}\backslash X'^+}$. The morphism of reductive $\mathbb{Q}$-groups $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ induces a $\mathbb{Q}$-morphism $\overline{\varphi} : \mathbf{G}^{\mathrm{der}} \longrightarrow \mathbf{G}'^{\mathrm{ad}}$ of semisimple $\mathbb{Q}$-groups. Let Γ denote the neat lattice $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \cap K \subset \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and Γ' the neat lattice of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})$ image of $\Gamma_{K'}$. Notice that $\Gamma'\backslash X'^+ = \Gamma_{K'}\backslash X'^+$. Consider the diagram

(5.1)
$$\begin{tikzcd}
\overline{\Gamma\backslash X^+} \arrow[d, "\pi"] \arrow[dr, "\phi\circ\pi"] & \\
\overline{\Gamma_K\backslash X^+} \arrow[r, "\phi"'] & \overline{\Gamma'\backslash X'^+}
\end{tikzcd}$$

with π the natural finite map. The proposition 5.3.2 (1) extends to this setting:

$$\pi^*(L_{\Gamma_K}) = L_{\Gamma} \quad .$$

Thus

$$\pi^* \Lambda_{K,K'} = \Lambda_{\Gamma,\Gamma'} \quad .$$

Let d denote the degree of π. By the projection formula one obtains:

$$\deg_{\Lambda_{K,K'}} C = \frac{1}{d} \deg_{\Lambda_{\Gamma,\Gamma'}} \pi^{-1}(C) \quad .$$

Now $\deg_{\Lambda_{\Gamma,\Gamma'}} \pi^{-1}(C) \geq 0$ by proposition 5.3.6.

□

Proof of the proposition 5.3.6. Let $C \subset \overline{\Gamma\backslash X^+}$ be an irreducible curve. We want to show that $\deg_{\Lambda_{\Gamma,\Gamma'}} C \geq 0$. First notice that by the projection formula and by proposition 5.3.2 (1), we can assume that the group $\mathbf{G}$ is simply connected and the group $\mathbf{G}'$ is adjoint.

Let $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ be the decomposition of $\mathbf{G}$ into $\mathbb{Q}$-simple factors. Let $\varphi_i : \mathbf{G}_i \longrightarrow \mathbf{G}'$, $1 \leq i \leq r$ denote the components of $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$. If $\Gamma_1 \subset \Gamma$ is a finite index subgroup and $p : \overline{\Gamma_1\backslash X^+} \longrightarrow \overline{\Gamma\backslash X^+}$ is the corresponding finite morphism, by proposition 5.3.2 the line bundle $\Lambda_{\Gamma_1,\Gamma'}$ corresponding to $\phi \circ p$ is isomorphic to $p^*\Lambda_{\Gamma,\Gamma'}$. The fact that $\deg_{\Lambda_{\Gamma,\Gamma'}} C \geq 0$ is once more implied by $\deg_{\Lambda_{\Gamma_1,\Gamma'}} p^{-1}(C) \geq 0$. Thus we can assume that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$, with Γ_i a neat arithmetic subgroup of $\mathbf{G}_i(\mathbb{Q})$. The variety $\overline{\Gamma\backslash X^+}$ decomposes into a product

$$\overline{\Gamma\backslash X^+} = \overline{\Gamma_1\backslash X_1^+} \times \cdots \times \overline{\Gamma_r\backslash X_r^+}$$

and the line bundle $\Lambda_{\Gamma,\Gamma'}$ on $\overline{\Gamma\backslash X^+}$ decomposes as

$$\Lambda_{\Gamma,\Gamma'} = \Lambda_{\Gamma_1,\Gamma'} \boxtimes \cdots \boxtimes \Lambda_{\Gamma_r,\Gamma'} \quad ,$$

with $\Lambda_{\Gamma_i,\Gamma'} = \phi_i^* L_{\Gamma'} \otimes L_{\Gamma_i}^{-1}$ the corresponding line bundle on $\overline{\Gamma_i\backslash X_i^+}$. Let $p_i : \overline{\Gamma\backslash X^+} \longrightarrow \overline{\Gamma_i\backslash X_i^+}$ be the natural projection. As

$$\deg_{\Lambda_{\Gamma,\Gamma'}} C = \sum_{i=1}^{r} \deg_{p_i^*\Lambda_{\Gamma_i,\Gamma'}} C \quad ,$$

we have reduced the proof of the proposition to the case where $\mathbf{G}$ is $\mathbb{Q}$-simple. It then follows from the more precise following proposition 5.3.7. □

Proposition 5.3.7. *Assume that $\mathbf{G}$ is $\mathbb{Q}$-simple.*

(1) If $\mathbf{G}$ is $\mathbb{Q}$-anisotropic then the line bundle $\Lambda_{\Gamma,\Gamma'}$ on the smooth complex projective variety $\Gamma\backslash X^+$ admits a metric of non negative curvature.
(2) If $\mathbf{G}$ is $\mathbb{Q}$-isotropic then either the line bundle $\Lambda_{\Gamma,\Gamma'}$ on $\overline{\Gamma\backslash X^+}$ is trivial or it is ample.

Proof. Let $\mathbf{G}' = \mathbf{G}'_1 \times \cdots \times \mathbf{G}'_{r'}$ be the decomposition of $\mathbf{G}'$ into $\mathbb{Q}$-simple factors and $\varphi_j : \mathbf{G} \longrightarrow \mathbf{G}'_j$, $1 \leq j \leq r'$, the components of $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$. By naturality of L_Γ and $L_{\Gamma'}$ (cf. proposition 5.3.2) one can assume that $\Gamma' = \Gamma'_1 \times \cdots \Gamma'_{r'}$. Accordingly one has

$$\Gamma'\backslash X'^+ = \Gamma'_1\backslash X'^+_1 \times \cdots \times \Gamma'_{r'}\backslash X'^+_{r'} \quad .$$

As $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ is injective and $\mathbf{G}$ is $\mathbb{Q}$-simple we can without loss of generality assume that $\varphi_1 : \mathbf{G} \longrightarrow \mathbf{G}'_1$ is injective. As

$$\Lambda = (\phi_1^* L_{\Gamma'_1} \otimes L_\Gamma^{-1}) \otimes \phi_2^* L_{\Gamma'_2} \otimes \cdots \phi_{r'}^* L_{\Gamma'_r} \quad ,$$

and the $L_{\Gamma'_j}$, $j \geq 2$, are ample on $\overline{\Gamma'_j \backslash X'^+_j}$ it is enough to prove the statement replacing $\Lambda_{\Gamma,\Gamma'}$ by $\phi_1^* L_{\Gamma'_1} \otimes L_{\Gamma}^{-1}$. Thus we can assume $\mathbf{G}'$ is $\mathbb{Q}$-simple.

By the adjunction formula the line bundle $\Lambda_{\Gamma,\Gamma'|\Gamma\backslash X^+}$ restriction of $\Lambda_{\Gamma,\Gamma'}$ coincides with $\Lambda^{\max} N^*$, where N denotes the automorphic bundle on $\Gamma\backslash X^+$ associated to the normal bundle of X in X' and N^* denotes its dual. As X is totally geodesic in X' the curvature form on N is the restriction to N of the curvature form on TX'. As X' is non-positively curved, the automorphic bundle N^* and thus also the automorphic line bundle $\Lambda_{\Gamma,\Gamma'|\Gamma\backslash X^+}$ admits a Hermitian metric of non-negative curvature. This concludes the proof of the proposition in the case $\mathbf{G}$ is $\mathbb{Q}$-anisotropic.

Suppose now $\mathbf{G}$ is $\mathbb{Q}$-isotropic. For simplicity we denote $\Lambda_{\Gamma,\Gamma'}$ by Λ from now on. We have to prove that the boundary components of $\overline{\Gamma\backslash X^+}$ do not essentially modify the positivity of $\Lambda_{|\Gamma\backslash X^+}$. We use the notation and the results of Dynkin [14], Ihara [21] and Satake [33]. Let $X = X_1 \times \cdots \times X_r$ (resp. $X' = X'_1 \times \cdots \times X'_{r'}$) be the decomposition of X (resp. X') into irreducible factors. Each X_i (resp. X'_j) is the Hermitian symmetric domain associated to an $\mathbb{R}$-isotropic $\mathbb{R}$-simple factor $\mathbf{G}_i$ (resp. $\mathbf{G}'_j$) of $\mathbf{G}_\mathbb{R}$ (resp. $\mathbf{G}'_\mathbb{R}$). The group $\mathbf{G}_\mathbb{R}$ (resp. $\mathbf{G}'_\mathbb{R}$) decomposes as $\mathbf{G}_0 \times \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ (resp. $\mathbf{G}'_0 \times \mathbf{G}'_1 \times \cdots \times \mathbf{G}'_{r'}$) with $\mathbf{G}_0$ (resp. $\mathbf{G}'_0$) an $\mathbb{R}$-anisotropic group. Let $\mathbf{m}$ (resp. $\mathbf{m}'$) be the r-tuple (resp. r'-tuple) of non-negative integers defining the automorphic line bundle L_K (resp. $L_{K'}$) (cf. [33, lemma 2]) and M_ϕ be the $r' \times r$-matrix with integral coefficients associated to $\varphi : \mathbf{G} \hookrightarrow \mathbf{G}'$ (cf. [33, section 2.1]). The automorphic line bundle $\Lambda_{|\Gamma\backslash X^+}$ on $\Gamma\backslash X^+$ is associated to the r-tuple of integers $\boldsymbol{\lambda} = \mathbf{m}' M_\varphi - \mathbf{m}$ (where $\mathbf{m}$ and $\mathbf{m}'$ are seen as row vectors). It admits a locally homogeneous Hermitian metric of non-negative curvature if and only if $\lambda_i \geq 0$, $1 \leq i \leq r$ (in which case we say that $\boldsymbol{\lambda}$ is non-negative).

Lemma 5.3.8. *The row vector $\boldsymbol{\lambda}$ is non-negative.*

Proof. As $\mathbf{G}$ and $\mathbf{G}'$ are defined over $\mathbb{Q}$, both $\mathbf{m}$ and $\mathbf{m}'$ are of rational type by [33, p.301]. So $m_i = m$ for all i, $m'_j = m'$ for all j. The equality $\boldsymbol{\lambda} = \mathbf{m}' M_\varphi - \mathbf{m}$ can be written in coordinates

$$\forall i \in \{1, \cdots, r\}, \quad \lambda_i = \sum_{1 \leq j \leq r'} m_{j,i}\, m' - m \quad , \tag{5.2}$$

with $M_\varphi = (m_{j,i})$. Fix i in $\{1, \cdots r\}$ and let us prove that $\lambda_i \geq 0$. As the $m_{i,j}$'s and m' are non-negative, it is enough to exhibit one j, $1 \leq j \leq r'$, with $m_{j,i}\, m' - m \geq 0$. Choose j such that the component $\varphi_{i,j} : X_i \longrightarrow X'_j$ of the map $\varphi : X_1 \times \cdots \times X_r \longrightarrow X'_1 \times \cdots \times X'_{r'}$ induced by $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ is an embedding. Recall that with the notation of [33, p.290] one has

$$m_i =< H_{1,i}, H_{1,i} >_i \quad ,$$

where $\mathfrak{h}_i$ denotes the chosen Cartan subalgebra of $\mathfrak{g}_i(\mathbb{R})$ and $<,>_i$ denotes the canonical scalar product on $\sqrt{-1}\mathfrak{h}_i$. This gives the equality:

$$m_{j,i}\, m'_j - m_i =< \phi_j(H_{1,i}), \phi_j(H_{1,i}) >_j - < H_{1,i}, H_{1,i} >_i \quad . \tag{5.3}$$

As $\mathbf{G}_i$ is $\mathbb{R}$-simple, any two invariant non-degenerate forms on $\sqrt{-1}\mathfrak{h}_i$ are proportional: there exists a positive real constant $c_{i,j}$ (called by Dynkin [14, p.130] the index of $\varphi_{i,j} : \mathbf{G}_i \longrightarrow \mathbf{G}_j$) such that

$$\forall X, Y \in \sqrt{-1}\mathfrak{h}_i, \;\; < \phi_j(X), \phi_j(Y) >_j = c_{i,j} < X, Y >_i \quad .$$

Equation (5.3) thus gives:

$$m_{j,i}\, m'_j - m_i = (c_{i,j} - 1) < H_{1,i}, H_{1,i} >_i \quad . \tag{5.4}$$

By [14, theorem 2.2. p.131] the constant $c_{i,j}$ is a positive integer. Thus $m_{j,i}\, m'_j - m_i$ is non-negative and this finishes the proof that $\boldsymbol{\lambda}$ is non-negative. □

By [33, cor.2 p.298] the sum $M = \sum_{1\leq j \leq r'} m_{j,i}$ is independent of i ($1 \leq i \leq r$). This implies that $\boldsymbol{\lambda}$ is of rational type: one of the λ_i is non-zero if and only if all are. In this case $\boldsymbol{\lambda}$ is positive of rational type and Λ is ample on $\overline{\Gamma\backslash X^+}$ by [33, theor.1].

If $\boldsymbol{\lambda} = 0$, the line bundle $\Lambda_{|\Gamma\backslash X^+}$ is trivial. As $\mathbf{G}$ is $\mathbb{Q}$-simple, if $\mathbf{G}$ is not locally isomorphic to $\mathbf{SL}_2$ the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial.

The last case is treated in the following lemma:

Lemma 5.3.9. *If $\boldsymbol{\lambda} = 0$ and $\mathbf{G}$ is locally isomorphic to $\mathbf{SL}_2$, then $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is a local isomorphism and the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial.*

Proof. It follows from the equation (5.2) that there exists a unique integer j such that the morphism $\varphi_j : \mathbf{G}_{\mathbb{R}} \longrightarrow \mathbf{G}_j$ is non trivial. In particular $\mathbf{G}'$ is $\mathbb{R}$-simple. Moreover the equation (5.4) implies that index c of $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is equal to 1. Thus by [14, theorem 6.2 p.152] the Lie algebra $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{g}'$. If $\mathbf{G}'_{\mathbb{R}}$ is classical, the equality [14, (2.36) p.136] shows that necessarily $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is a local isomorphism. In particular the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial. If the group $\mathbf{G}'_{\mathbb{R}}$ is an exceptional simple Lie group of Hermitian type (thus E_6 or E_7), Dynkin shows in [14, Tables 16, 17 p.178-179] that there is a unique realization of $\mathfrak{g}$ as a regular subalgebra of $\mathfrak{g}'$ of index 1. However this realization is not of Hermitian type: the coefficient $\alpha'_1(\varphi(H_1))$ is zero. Thus this case is impossible. □

This finishes the proof of proposition 5.3.7.

□

From the nefness of $\Lambda_{K,K'}$ we now deduce the following crucial corollary:

Corollary 5.3.10. *Let $\phi : \mathrm{Sh}_K(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')$ be a morphism of Shimura varieties associated to a Shimura subdatum $\varphi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}', X')$. Assume that $\mathbf{Z}(\mathbb{R})$ is compact (where $\mathbf{Z}$ denotes the centre of $\mathbf{G}$). Let K' a neat compact open subgroup of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$ and denote by K the compact open subgroup $K' \cap \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ of $\mathbf{G}(\mathbf{A}_f)$. Then for any irreducible Hodge generic subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)$ one has $\deg_{L_K} Z \leq \deg_{L_{K'}} \phi(Z)$.*

Proof. As the irreducible components of Z are Hodge generic in $\mathrm{Sh}_K(\mathbf{G}, X)$ and as $\mathbf{Z}(\mathbb{R})$ is compact we know by lemma 2.2 in [39] (and its proof) that $\phi_{|Z} : Z \longrightarrow Z' := \phi(Z)$ is generically injective. In particular by the projection formula one has

$$\deg_{L_{K'}} Z' = \deg_{\phi^* L_{K'}} Z \quad .$$

So the inequality $\deg_{L_K} Z \leq \deg_{L_{K'}} Z'$ is equivalent to the inequality $\deg_{\phi^* L_{K'}} Z \geq \deg_{L_K} Z$.

As $\phi^* L_{K'} = L_K \otimes \Lambda_{K,K'}$ one has

$$\deg_{\phi^* L_{K'}} Z = \sum_{i=0}^{\dim Z} \binom{\dim Z}{i} \int_Z c_1(L_K)^i \wedge c_1(\Lambda_{K,K'})^{\dim Z - i} \quad .$$

The inequality $\deg_{\phi^* L'_K} Z \geq \deg_{L_K} Z$ thus follows if we show:

$$\forall i \ \ , 0 \leq i \leq \dim Z - 1, \qquad \int_Z c_1(L_K)^i \wedge c_1(\Lambda_{K,K'})^{\dim Z - i} \geq 0 \quad .$$

As L_K is ample it follows from the nefness of $\Lambda_{K,K'}$ and Kleiman's theorem 5.2.1. □

6. Inclusion of Shimura subdata.

In this section we prove a proposition which implies part (b) of the theorem 3.1.1. We also prove two auxiliary lemmas on inclusion of Shimura data.

Lemma 6.1. *Let $(\mathbf{H}, X_{\mathbf{H}}) \subset (\mathbf{H}', X_{\mathbf{H}'})$ be an inclusion of Shimura data. We assume that $\mathbf{H}$ and $\mathbf{H}'$ are the generic Mumford-Tate groups on $X_{\mathbf{H}}$ and $X_{\mathbf{H}'}$ respectively. Suppose that the connected centre $\mathbf{T}$ of $\mathbf{H}$ is split by a number field L. Then the connected centre $\mathbf{T}'$ of $\mathbf{H}'$ is split by L.*

Proof. Let $\mathbf{C}' := \mathbf{H}'/\mathbf{H}'^{\mathrm{der}}$. Then there is an isogeny between $\mathbf{T}'$ and $\mathbf{C}'$ induced by the quotient $\pi' \colon \mathbf{H}' \longrightarrow \mathbf{C}'$. The splitting fields of $\mathbf{T}'$ and $\mathbf{C}'$ are therefore the same.

We claim that for any $\alpha \in X_{\mathbf{H}'}$, the Mumford-Tate group of $\pi'\alpha$ is $\mathbf{C}'$. Indeed, as $\mathbf{C}'$ is commutative, and $X_{\mathbf{H}'}$ is an $\mathbf{H}'(\mathbb{R})$-conjugacy class, $\pi'\alpha$ does not depend on α. Let $\alpha \in X_{\mathbf{H}'}$ be Hodge generic and let $\mathbf{C}_1$ be the Mumford-Tate group of $\pi'\alpha$. Then α factors through $\pi'^{-1}(\mathbf{C}_1) = \mathbf{H}'$. It follows that $\mathbf{C}_1 = \mathbf{C}'$.

Let β be a Hodge generic point of $X_{\mathbf{H}}$. As $\mathbf{H} = \mathbf{T}\mathbf{H}^{\mathrm{der}}$ and $\mathbf{H}^{\mathrm{der}} \subset \mathbf{H}'^{\mathrm{der}}$, we have

$$\pi'(\mathbf{H}) = \pi'(\mathbf{T}) \ .$$

As $\pi'(\mathbf{H})$ is the Mumford-Tate group of $\pi'\beta$ (because $\mathbf{H}$ is the Mumford-Tate group of β), we see that

$$\pi'(\mathbf{T}) = \mathbf{C}' \ .$$

As the torus $\mathbf{T}$ is split by L, the torus $\mathbf{C}'$ and therefore also the torus $\mathbf{T}'$ are split by L. □

Lemma 6.2. *Let $(\mathbf{H}, X_{\mathbf{H}}) \subset (\mathbf{H}', X_{\mathbf{H}'})$ be an inclusion of Shimura data. We assume that $\mathbf{H}$ and $\mathbf{H}'$ are the generic Mumford-Tate groups on $X_{\mathbf{H}}$ and $X_{\mathbf{H}'}$ respectively. Let $\mathbf{T}$ and $\mathbf{T}'$ be the connected centres of $\mathbf{H}$ and $\mathbf{H}'$ respectively.*

Suppose that $\mathbf{T} \subset \mathbf{T}'$. Then $\mathbf{T} = \mathbf{T}'$.

Proof. We write

$$\mathbf{H}' = \mathbf{T}'\mathbf{H}'^{\mathrm{der}} \ .$$

We have $(\mathbf{T}' \cap \mathbf{H})^0 \subset \mathbf{T}$. On the other hand, by assumption $\mathbf{T} \subset (\mathbf{T}' \cap \mathbf{H})^0$, hence

$$\mathbf{T} = (\mathbf{T}' \cap \mathbf{H})^0 \ .$$

Write

$$\mathbf{H} = (\mathbf{T}' \cap \mathbf{H})^0\mathbf{H}^{\mathrm{der}} \ .$$

Fix α an element of $X_{\mathbf{H}}$. As $X_{\mathbf{H}'}$ is the $\mathbf{H}'(\mathbb{R})$-conjugacy class of α, any element $x \in X_{\mathbf{H}'}$ is of the form $g\alpha g^{-1}$ for some g of $\mathbf{H}'(\mathbb{R})$. Thus x factors through

$$g(\mathbf{T}' \cap \mathbf{H})^0_{\mathbb{R}} g^{-1} . \, g\mathbf{H}'^{\mathrm{der}}_{\mathbb{R}} g^{-1} = (\mathbf{T}' \cap \mathbf{H})^0_{\mathbb{R}} \, \mathbf{H}'^{\mathrm{der}}_{\mathbb{R}} \ .$$

It follows that the Mumford-Tate group of x is contained in $(\mathbf{T}' \cap \mathbf{H})^0\mathbf{H}'^{\mathrm{der}}$. For x Hodge generic, we obtain

$$(\mathbf{T}' \cap \mathbf{H})^0\mathbf{H}'^{\mathrm{der}} = \mathbf{H}' \ .$$

Hence $(\mathbf{T}' \cap \mathbf{H})^0 = \mathbf{T}'$ and $\mathbf{T} = \mathbf{T}'$. □

Proposition 6.3. *Suppose that the set Σ in the theorem 3.2.1 is such that with respect to a faithful representation $\rho\colon \mathbf{G} \longrightarrow \mathbf{GL}_n$ the centres $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as V ranges through Σ.*

We suppose that, after replacing Σ by a modification Σ', every V in Σ' is strictly *contained in a special subvariety $V' \subset Z$.*

Then the set Σ' admits a modification Σ'' such that the centres $\mathbf{T}_{V'}$ of the generic Mumford-Tate groups $\mathbf{H}_{V'}$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as V' ranges through Σ''.

Proof. First note that an inclusion of special subvarieties $V \subset V'$ corresponds to an inclusion of Shimura data $(\mathbf{H}_V, X_{\mathbf{H}_V}) \subset (\mathbf{H}_{V'}, X_{\mathbf{H}_{V'}})$ with $\mathbf{H}_V$ and $\mathbf{H}_{V'}$ the generic Mumford-Tate groups on $X_{\mathbf{H}_V}$ and $X_{\mathbf{H}_{V'}}$ respectively.

By assumption the connected centre $\mathbf{T}_V$ of $\mathbf{H}_V$ lie in the $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class of a fixed $\mathbb{Q}$-torus as V ranges through Σ. Hence the tori $\mathbf{T}_V$, $V \in \Sigma$, are split by the same field L. By lemma 6.1 the tori $\mathbf{T}_{V'}$ connected centers of the $\mathbf{H}_{V'}$, $V' \in \Sigma'$, are all split by L. By [39], lemma 3.13, part (i), the tori $\mathbf{T}_{V'}$ lie in finitely many $\mathbf{GL}_n(\mathbb{Q})$-conjugacy classes. The conclusion of the proposition follows. $\square$

7. The geometric criterion.

In this section we show that given a subvariety Z of a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a special subvariety V and satisfying certain assumptions, the existence of a suitable element $m \in \mathbf{G}(\mathbb{Q}_l)$ such that $Z \subset T_m Z$ implies that Z contains a special subvariety V' containing V properly.

7.1. **Hodge genericity.**

Definition 7.1.1. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a neat compact open subgroup, $F \subset \mathbb{C}$ a number field containing the reflex field $E(\mathbf{G}, X)$ and $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ an F-irreducible subvariety. We say that Z is Hodge generic if one of its geometrically irreducible components is Hodge generic in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.*

Lemma 7.1.2. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a neat compact open subgroup, $F \subset \mathbb{C}$ a number field containing the reflex field $E(\mathbf{G}, X)$ and $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ a Hodge generic F-irreducible subvariety. Let $Z = Z_1 \cup \cdots \cup Z_n$ be the decomposition of Z into geometrically irreducible components. Then each irreducible component Z_i, $1 \leq i \leq n$, is Hodge generic.*

Proof. As Z is Hodge generic, at least one of its irreducible components, say Z_1, is Hodge generic. Writing $Z = Z_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$ with $Z_F \subset \mathrm{Sh}_K(\mathbf{G}, X)_F$ irreducible, any irreducible component Z_j, $1 \leq j \leq n$, is of the form Z_1^σ for some element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. As the conjugate under any element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is still special, one gets the result. This is a consequence of a theorem of Kazhdan. See [24] for a comprehensive exposition of the proof in full generality and the relevant references. $\square$

7.2. **The criterion.** Our main theorem in this section is the following:

Theorem 7.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum, X^+ a connected component of X and $K = \prod_{p \text{ prime}} K_p \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ an open compact subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We assume that there* *exists a prime p_0 such that the compact open subgroup $K_{p_0} \subset \mathbf{G}(\mathbb{Q}_{p_0})$ is neat. Let $F \subset \mathbb{C}$ be a number field containing the reflex field $E(\mathbf{G}, X)$.*

Let V be a special but not strongly special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ contained in a Hodge generic F-irreducible subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.

Let $l \neq p_0$ be a prime number splitting $\mathbf{T}_V$ and m an element of $\mathbf{T}_V(\mathbb{Q}_l)$.

Suppose that Z and m satisfy the following conditions:

1. *$Z \subset T_m Z$.*
2. *Let $\lambda \colon \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ be the natural morphism. For every k_1 and k_2 in K_l, the element $\lambda(k_1 m k_2)$ generates an unbounded (i.e. not relatively compact) subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$.*

Then Z contains a special subvariety V' containing V properly.

Proof.

Lemma 7.2.2. *If the conclusion of the theorem 7.2.1 holds for all Shimura data $(\mathbf{G}, X)$ with $\mathbf{G}$ semisimple of adjoint type then it holds for all Shimura data.*

Proof. Let $\mathbf{G}$, X, K, p_0, F, V, Z, l and m be as in the statement of theorem 7.2.1. In particular $Z = Z_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$ with $Z_F \subset \mathrm{Sh}_K(G, X)_F$ an irreducible subvariety. Let $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the adjoint Shimura datum attached to $(\mathbf{G}, X)$ and $(X^{\mathrm{ad}})^+$ be the image of X^+ under the natural morphism $X \longrightarrow X^{\mathrm{ad}}$. Let $K^{\mathrm{ad}} = \prod_{p \text{ prime}} K_p^{\mathrm{ad}}$ be the compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbf{A}_{\mathrm{f}})$ defined as follows:

1. $K_{p_0}^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_{p_0})$ is the compact open subgroup image of K_{p_0} by λ.
2. $K_l^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ is the compact open subgroup image of K_l by λ.
3. If $p \notin \{p_0, l\}$, the group K_p^{ad} is a maximal compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_p)$ containing the image of K_p by λ.

The group K^{ad} is neat because K_{p_0}, and therefore $K_{p_0}^{\mathrm{ad}}$, is. As the reflex field $E(\mathbf{G}, X)$ contains the reflex field $E(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ and K^{ad} contains $\lambda(K)$ there is a finite morphism of Shimura varieties $f : \mathrm{Sh}_K(\mathbf{G}, X)_F \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$.

We define the irreducible subvariety Z_F^{ad} of $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$ to be the image of Z_F in $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$ by this morphism. Its base-change $Z^{\mathrm{ad}} := Z_F^{\mathrm{ad}} \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$, which is F-irreducible, coincides with $f_{\mathbb{C}}(Z)$, where $f_{\mathbb{C}} : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$ is the base change of f.

Let V^{ad} be the image $f_{\mathbb{C}}(V)$. As V is special but not strongly special, V^{ad} is a special but not strongly special subvariety of $S_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Thus $\mathbf{T}_{V^{\mathrm{ad}}} = \lambda(\mathbf{T}_V)$ is a non-trivial torus.

Let $m^{\mathrm{ad}} := \lambda(m)$. The inclusion $Z \subset T_m Z$ implies that $Z^{\mathrm{ad}} \subset T_{m^{\mathrm{ad}}} Z^{\mathrm{ad}}$. As $K_l^{\mathrm{ad}} = \lambda(K_l)$ the condition (2) for m and K_l implies the condition (2) for m^{ad} and K_l^{ad}.

Thus $\mathbf{G}^{\mathrm{ad}}$, X^{ad}, K^{ad}, p_0, F, V^{ad}, Z^{ad}, l and m^{ad} satisfy the assumptions of theorem 7.2.1. As irreducible components of the preimage of a special subvariety by a finite morphism of Shimura varieties are special, it is enough to show that Z^{ad} contains a special subvariety V'^{ad} containing V^{ad} properly to conclude that Z contains a special subvariety V' containing V properly. $\square$

For the rest of the proof of theorem 7.2.1, we are assuming the group $\mathbf{G}$ *to be semisimple of adjoint type. Moreover we will drop the label* $(\mathbf{G}, X)$ *when it is obvious which Shimura datum we are referring to.*

We fix a $\mathbb{Z}$-structure on $\mathbf{G}$ and its subgroups by choosing a finitely generated free $\mathbb{Z}$-module W, a faithful representation $\xi \colon \mathbf{G} \hookrightarrow \mathbf{GL}(W_{\mathbb{Q}})$ and taking the Zariski closures in the $\mathbb{Z}$-group-scheme $\mathbf{GL}(W)$. We choose the representation ξ in such a way that K is contained in $\mathbf{GL}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W)$ (i.e. K stabilizes $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W$). This induces canonically a $\mathbb{Z}$-variation of Hodge structure $\mathcal{F}$ on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ (cf. [18, section 3.2]), in particular on its irreducible component $S_K(\mathbf{G}, X)_{\mathbb{C}}$.

Let Z_1 be a geometrically irreducible component of Z containing V. Let z be a Hodge generic point of the smooth locus Z_1^{sm} of Z_1. Let $\pi_1(Z_1^{\mathrm{sm}}, z)$ be the topological fundamental group of Z_1^{sm} at the point z. We choose a point z^+ of X^+ lying above z. This choice canonically identifies the fibre at z of the locally constant sheaf underlying $\mathcal{F}$ with the $\mathbb{Z}$-module W. The action of $\pi_1(Z_1^{\mathrm{sm}}, z)$ on this fibre is described by the monodromy representation

$$\rho \colon \pi_1(Z_1^{\mathrm{sm}}, z) \longrightarrow \Gamma_K = \pi_1(S_K(\mathbf{G}, X)_{\mathbb{C}}, z) = \mathbf{G}(\mathbb{Q})^+ \cap K \xrightarrow{\xi} \mathbf{GL}(W) \ .$$

By proposition 7.1.2 the subvariety Z_1 is Hodge generic in $S_K(\mathbf{G}, X)_{\mathbb{C}}$. Hence the Mumford-Tate group of $\mathcal{F}_{|Z_1^{\mathrm{sm}}}$ at x is $\mathbf{G}$. It follows from [25, theor. 1.4] and the fact that the group $\mathbf{G}$ is adjoint that the group $\rho(\pi_1(Z_1^{\mathrm{sm}}, z))$ is Zariski-dense in $\mathbf{G}$.

Let l be a prime as in the statement. The proposition 4.2.1 implies that the l-adic closure of $\rho(\pi_1(Z_1^{\mathrm{sm}}, z))$ in $\mathbf{G}(\mathbb{Q}_l)$ is a compact open subgroup $K'_l \subset K_l$.

Write $K = K^l K_l$ with $K^l = \prod_{p \neq l} K_p$. Let $\pi_{K_l} \colon \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ be the Galois pro-étale cover with group K_l as defined in section 4.1.1. Let $\widetilde{Z_1}$ be an irreducible component of the preimage of Z_1 in Sh_{K^l} and let $\widetilde{V}$ be an irreducible component of the preimage of V in $\widetilde{Z_1}$.

The idea of the proof is to show that the inclusion $Z \subset T_m Z$ implies that $\widetilde{Z_1}$ is stabilized by a "big" group and then consider the orbit of $\widetilde{V}$ under the action of this group.

Lemma 7.2.3. *The variety* $\widetilde{Z_1}$ *is stabilized by the group* K'_l. *The set of irreducible components of* $\pi_{K_l}^{-1}(Z_1)$ *naturally identifies with the finite set* K_l/K'_l.

Proof. Let $\widetilde{z}$ be a geometric point of $\widetilde{Z_1^{\mathrm{sm}}}$ lying over z. As $\pi_{K_l} : \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ is pro-étale, the set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$ naturally identifies with the set of connected components of $\pi_{K_l}^{-1}(Z_1^{\mathrm{sm}})$. This set identifies with the quotient $K_l/\rho_{\mathrm{alg}}(\varpi_1(Z_1^{\mathrm{sm}}, z))$ where $\varpi_1(Z_1^{\mathrm{sm}}, z)$ denotes the algebraic fundamental group of Z_1^{sm} at z and $\rho_{\mathrm{alg}} \colon \varpi_1(Z_1^{\mathrm{sm}}, z) \longrightarrow K_l \subset \mathbf{G}(\mathbb{Q}_l)$ denotes the (continuous) monodromy representation of the K_l-pro-étale cover $\pi_{K_l} : \pi_{K_l}^{-1}(Z_1^{\mathrm{sm}}) \longrightarrow Z_1^{\mathrm{sm}}$. The group $\varpi_1(Z_1^{\mathrm{sm}}, z)$ naturally identifies with the profinite completion of $\pi_1(Z_1^{\mathrm{sm}}, z)$. One has the commutative diagram

$$\begin{CD} \pi_1(Z_1^{\mathrm{sm}}, z) @>{\rho}>> \mathbf{G}(\mathbb{Q}) \\ @V{i}VV @VV{j}V \\ \varpi_1(Z_1^{\mathrm{sm}}, z) @>>{\rho_{\mathrm{alg}}}> \mathbf{G}(\mathbb{Q}_l) \end{CD} \tag{7.1}$$

where $i : \pi_1(Z_1^{\mathrm{sm}}, z) \longrightarrow \varpi_1(Z_1^{\mathrm{sm}}, z)$ and $j : \mathbf{G}(\mathbb{Q}) \longrightarrow \mathbf{G}(\mathbb{Q}_l)$ denote the natural homomorphisms. As $i(\pi_1(Z_1^{\mathrm{sm}}, z))$ is dense in $\varpi_1(Z_1^{\mathrm{sm}}, z)$ and ρ_{alg} is continuous one deduces that $\rho_{\mathrm{alg}}(\varpi_1(Z_1^{\mathrm{sm}}, z)) = K_l'$. Thus the set of irreducible components of $\pi_{K_l}^{-1}(Z_1^{\mathrm{sm}})$ identifies with K_l/K_l' and $\widetilde{Z_1^{\mathrm{sm}}}$ is K_l'-stable. □

Lemma 7.2.4. *There exist elements k_1, k_2 of K_l and an integer $n \geq 1$ such that*

$$\widetilde{Z_1} = \widetilde{Z_1} \cdot (k_1 m k_2)^n$$

Proof. Let Z_i, $2 \leq i \leq n$, be the geometrically irreducible components of Z different from Z_1. For each $i \in \{2, \ldots, n\}$, let us fix a geometrically irreducible component $\widetilde{Z_i}$ of $\pi_{K_l}^{-1}(Z_i)$. The inclusion $Z \subset T_m Z$ implies that, for $i \in \{1, \ldots, n\}$, the component $\widetilde{Z_i}$ of $\pi_{K_l}^{-1}(Z_i)$ is also a geometrically irreducible component of $\pi_{K_l}^{-1}(T_m Z)$. As the geometrically irreducible components of $\pi_{K_l}^{-1}(T_m Z)$ are of the form $\widetilde{Z_i} \cdot (k_1 m k_2)$, $k_1, k_2 \in K_l$, there exists an index i, $1 \leq i \leq n$, and two elements k_1, k_2 in K_l such that

$$\widetilde{Z_1} = \widetilde{Z_i} \cdot k_1 m k_2 \quad . \tag{7.2}$$

As Z is F-irreducible there exists σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $Z_i = \sigma(Z_1)$. As the morphism $\pi_{K_l} : \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ is defined over F, the subvariety $\sigma(\widetilde{Z_1})$ of Sh_{K^l} satisfies $\pi_{K_l}(\sigma(\widetilde{Z_1})) = Z_i$. Hence the subvarieties $\sigma(\widetilde{Z_1})$ and $\widetilde{Z_i}$ of Sh_{K^l} are both irreducible components of $\pi_{K_l}^{-1}(Z_i)$. Thus there exists an element k of K_l such that

$$\widetilde{Z_i} = \sigma(\widetilde{Z_1}) \cdot k \quad . \tag{7.3}$$

From (7.2) and (7.3) and replacing k_1 with kk_1, we obtain k_1, k_2 in K_l such that

$$\widetilde{Z_1} = \sigma(\widetilde{Z_1}) \cdot (k_1 m k_2) \quad . \tag{7.4}$$

As the $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$-action is defined over F, the previous equation implies:

$$\forall\, j \in \mathbb{N}, \quad \widetilde{Z_1} = \sigma^j(\widetilde{Z_1}) \cdot (k_1 m k_2)^j \quad . \tag{7.5}$$

As the set of irreducible components of Z is finite, there exists a positive integer m such that $\sigma^m(Z_1) = Z_1$. Thus the Abelian group $(\sigma^m)^{\mathbf{Z}}$ acts on the set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$. By lemma 7.2.3 this set is finite. So there exists a positive integer n (multiple of m) such that $\sigma^n(\widetilde{Z_1}) = \widetilde{Z_1}$. The equality (7.5) applied to $j = n$ concludes the proof of the lemma. □

From the lemmas 7.2.3 and 7.2.4 one obtains the

Corollary 7.2.5. *Let U_l be the group $\langle K_l', (k_1 m k_2)^n \rangle$. The variety $\widetilde{Z_1}$ is stabilized by U_l.*

We now conclude the proof of theorem 7.2.1. Let $\mathbf{G} = \prod_{i=1}^{s} \mathbf{G}_i$ be the decomposition of the semisimple $\mathbb{Q}$-group of adjoint type $\mathbf{G}$ into $\mathbb{Q}$-simple factors and $X = \prod_{i=1}^{s} X_i$ (resp. $X^+ = \prod_{i=1}^{s} X_i^+$) the associated decomposition of X (respectively X^+). The Shimura datum $(\mathbf{G}, X)$ is the product of the Shimura data $(\mathbf{G}_i, X_i)$, $1 \leq i \leq s$, where each $\mathbf{G}_i$ is simple of adjoint type. Let $p_i : \mathbf{G} \longrightarrow \mathbf{G}_i$, $1 \leq i \leq s$, denote the natural projections. Let $(\mathbf{G}_{>1}, X_{>1})$ be the Shimura datum $(\prod_{i=2}^{s} \mathbf{G}_i, \prod_{i=2}^{s} X_i)$.

By the assumption made on m, the group U_l is unbounded in $\mathbf{G}(\mathbb{Q}_l)$. After possibly renumbering the factors, we can assume that $p_1(U_l)$ is unbounded in $\mathbf{G}_1(\mathbb{Q}_l)$. In particular the torus $\mathbf{T}_{V,1} := p_1(\mathbf{T}_V)$ is non-trivial. Indeed if it was trivial, then the group $p_1(U_l)$ would be contained in $p_1(K_l)$ which is compact.

Let $\mathbf{G}_{1,\mathbb{Q}_l} = \prod_{j=1}^{r} \mathbf{H}_j$ be the decomposition of $\mathbf{G}_{1,\mathbb{Q}_l}$ into $\mathbb{Q}_l$-simple factors. Again, up to renumbering we can assume that the image of U_l under the projection $h_1 : \mathbf{G}_{\mathbb{Q}_l} \longrightarrow \mathbf{H}_1$ is unbounded in $\mathbf{H}_1(\mathbb{Q}_l)$. Let $\mathbf{H}_{>1} = \prod_{j=2}^{r} \mathbf{H}_j$. Let $\tau : \widetilde{\mathbf{G}_{\mathbb{Q}_l}} \longrightarrow \mathbf{G}_{\mathbb{Q}_l}$ (resp. $\tau_1 : \widetilde{\mathbf{H}_1} \longrightarrow \mathbf{H}_1$) be the universal cover of $\mathbf{G}_{\mathbb{Q}_l}$ (resp. $\mathbf{H}_1$).

Lemma 7.2.6. *The group $U_l \cap \mathbf{H}_1(\mathbb{Q}_l)$ contains the group $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$ with finite index.*

Proof. Let $\widetilde{h_1} : \widetilde{\mathbf{G}_{\mathbb{Q}_l}} \longrightarrow \widetilde{\mathbf{H}_1}$ be the canonical projection. Let $\widetilde{U_l} = \tau^{-1}(U_l) \subset \widetilde{\mathbf{G}_{\mathbb{Q}_l}}(\mathbb{Q}_l)$. As U_l is an open non-compact subgroup of $\mathbf{G}_{\mathbb{Q}_l}(\mathbb{Q}_l)$, the group $\widetilde{U_l}$ is open non-compact in $\widetilde{\mathbf{G}_{\mathbb{Q}_l}}(\mathbb{Q}_l)$. As $h_1(U_l)$ is non-compact in $\mathbf{H}_1(\mathbb{Q}_l)$ the projection $\widetilde{h_1}(\widetilde{U_l})$ is open non-compact in the group $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$.

Notice that the group $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ is normalized by the subgroup $\widetilde{h_1}(\widetilde{U_l})$ of $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. Indeed, given $h \in \widetilde{h_1}(\widetilde{U_l})$, let $g \in \widetilde{U_l}$ satisfying $\widetilde{h_1}(g) = h$. As $\widetilde{\mathbf{H}_1}$ is a direct factor of $\widetilde{\mathbf{G}_{\mathbb{Q}_l}}$ one obtains:

$$(\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l))^h = (\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l))^g = (\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)) \ .$$

As $\widetilde{h_1}(\widetilde{U_l})$ is open non-compact and normalizes $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$, it follows from [29, theor.2.2] that $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l) = \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. As τ_1 is an isogeny of algebraic groups, we get that $U_l \cap \mathbf{H}_1(\mathbb{Q}_l)$ contains $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$ with finite index. $\square$

Define $K_{1,l}$ as the compact open subgroup $p_1(K_l)$ of $\mathbf{G}_{1,\mathbb{Q}_l}$ and $K_{>1,l}$ as the compact open subgroup $(p_2 \times \cdots \times p_s)(K)$ of $\mathbf{G}_{>1,\mathbb{Q}_l}$. As U_l is an open subgroup of $\mathbf{G}_{\mathbb{Q}_l}(\mathbb{Q}_l)$ it contains a compact open subgroup of $\mathbf{G}_{1,\mathbb{Q}_l}(\mathbb{Q}_l) = \prod_{j=1}^{r} \mathbf{H}_j(\mathbb{Q}_l)$, in particular a compact open subgroup $U_{1,l}$ of $K_{1,l} \cap \mathbf{H}_{>1}(\mathbb{Q}_l)$. Similarly U_l contains a compact open subgroup $U_{>1,l}$ of $K_{>1,l}$. The previous lemma shows that U_l contains the unbounded open subgroup $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)) \cdot U_{l,1} \cdot U_{l,>1}$.

Definition 7.2.7. *We replace U_l by its subgroup $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)) \cdot U_{1,l} \cdot U_{>1,l}$. We denote by V' the Zariski closure $\overline{\pi_{K_l}(\widetilde{V} \cdot U_l)}^{\mathrm{Zar}}$.*

As $\widetilde{Z_1}$ is stabilised by U_l, the variety V' is a subvariety of Z.

Lemma 7.2.8. *The subvariety V' of Z is special.*

Proof. Define $K_i := p_i(K)$, $1 \leq i \leq s$, and $\mathcal{K} := \prod_{i=1}^{s} K_i$. As the group K_{p_0} is neat its projections K_{i,p_0}, $1 \leq i \leq s$, are also neat, hence $\mathcal{K}$ is neat. Let $f : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, X)_{\mathbb{C}}$ be the natural finite morphism, $\mathcal{Z} := f(Z)$, $\mathcal{V} = f(V)$ and $\mathcal{V}' = f(V')$. As f is a finite morphism it follows that $\mathcal{V}'$ is also the Zariski closure $\overline{(f \circ \pi_{K_l})(\widetilde{V} \cdot U_l)}^{\mathrm{Zar}}$ of $(f \circ \pi_{K_l})(\widetilde{V} \cdot U_l)$ in $\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, X)_{\mathbb{C}}$.

As in the proof of lemma 7.2.2 it is enough to show that $\mathcal{V}'$ is special to conclude that V' is special.

Let $K_{>1}$ be the compact open subgroup $\prod_{i=2}^{s} K_i$ of $\mathbf{G}_{>1}(\mathbf{A}_{\mathrm{f}})$. The connected component $S_{\mathcal{K}}(\mathbf{G}, X)_{\mathbb{C}}$ of the Shimura variety $\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, X)_{\mathbb{C}}$ decomposes as a product

$$S_{\mathcal{K}}(\mathbf{G}, X)_{\mathbb{C}} = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}}$$

with $S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}} = \prod_{i=2}^{s} S_{K_i}(\mathbf{G}_i, X_i)_{\mathbb{C}}$.

Let $\mathcal{V}_{>1}$ denote the special subvariety of $S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}}$ projection of $\mathcal{V}$. Thanks to the definition 7.2.7 of U_l the inclusion

$$\mathcal{V}' \subset S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times \mathcal{V}_{>1} \tag{7.6}$$

holds.

For an element $q \in \mathbf{G}_1(\mathbb{Q})^+$, we let Γ_q be the subgroup of $\mathbf{G}_1(\mathbb{Q})^+$ generated by $\Gamma := K_1 \cap \mathbf{G}_1(\mathbb{Q})^+$ and q. We claim that we can choose $q \in \mathbf{G}_1(\mathbb{Q})^+ \cap U_l$ such that the index of Γ in Γ_q is infinite.

Indeed let $g \in \widetilde{\mathbf{H}}_1(\mathbb{Q}_l)$ be an element contained in a split subtorus of $\widetilde{\mathbf{H}}_{1,\mathbb{Q}_l}$ but not in the maximal compact subgroup of this subtorus. Then g is not contained in any compact subgroup of $\widetilde{\mathbf{H}}_1(\mathbb{Q}_l)$, hence its image $h := \tau(g) \in U_l$ is not contained in any compact subgroup of $\mathbf{G}_1(\mathbb{Q}_l)$. As $\mathbf{G}_1$ is simple and adjoint it has the weak approximation property [30, theorem 7.8]: the group $\mathbf{G}_1(\mathbb{Q})$ is dense in $\mathbf{G}_1(\mathbb{Q}_l)$. Let $\overline{\Gamma}^l$ denote the l-adic closure of Γ in $\mathbf{G}_1(\mathbb{Q}_l)$, this is a compact open subgroup of $\mathbf{G}_1(\mathbb{Q}_l)$ by proposition 4.2.1. As $\mathbf{G}_1(\mathbb{Q})^+$ has finite index in $\mathbf{G}_1(\mathbb{Q})$, the l-adic closure $\overline{\mathbf{G}_1(\mathbb{Q})^+}^l$ of $\mathbf{G}_1(\mathbb{Q})^+$ in $\mathbf{G}_1(\mathbb{Q}_l)$ is an open subgroup of finite index of $\mathbf{G}_1(\mathbb{Q}_l)$. By replacing h by a suitable positive power, we may assume that $h \in \overline{\mathbf{G}_1(\mathbb{Q})^+}^l$. The group $\overline{\Gamma}^l \cap U_l$ is an open subgroup of $\overline{\mathbf{G}_1(\mathbb{Q})^+}^l$, therefore there exist elements q of $\mathbf{G}_1(\mathbb{Q})^+$ and k of $\overline{\Gamma}^l \cap U_l$ such that $h = qk$. It follows that $q \in \mathbf{G}_1(\mathbb{Q})^+ \cap U_l$. We claim that Γ has infinite index in Γ_q. Suppose the contrary. Then the l-adic closure $\overline{\Gamma_q}^l$ of Γ_q in $\mathbf{G}_1(\mathbb{Q}_l)$ contains $\overline{\Gamma}^l$ with finite index, hence is compact. But, by construction, $h \in \overline{\Gamma_q}^l$ and h is not contained in any compact subgroup of $\mathbf{G}_1(\mathbb{Q}_l)$. This gives a contradiction.

Let us show that Γ_q is dense in $\mathbf{G}_1(\mathbb{R})^+$ (for the Archimedian topology). Let H be the Lie subgroup of $\mathbf{G}_1(\mathbb{R})^+$ closure of Γ_q and let H^+ be its connected component of the identity. First notice that the group Γ normalizes H^+, hence its Lie algebra. As $\mathbf{G}_1$ is $\mathbb{R}$-isotropic, it follows from [30, theor. 4.10] that Γ is Zariski-dense in $\mathbf{G}_{1,\mathbb{R}}$. Hence H^+ is a product of simple factors of $\mathbf{G}_1(\mathbb{R})^+$. The $\mathbb{Q}$-simple group $\mathbf{G}_1$ can be written as the restriction of scalars $\mathrm{Res}_{L/\mathbb{Q}}\mathbf{G}'_1$, with L a number field and $\mathbf{G}'_1$ an absolutely almost simple algebraic group over L. As $H^+ \cap \mathbf{G}_1(\mathbb{Q})$ is dense in H^+ it follows that $H^+ = \mathbf{G}_1(\mathbb{R})^+$ as soon as H^+ is non-trivial. If H^+ were trivial the group Γ_q would be discrete in $\mathbf{G}_1(\mathbb{R})^+$. As Γ_q contains the lattice Γ of $\mathbf{G}_1(\mathbb{R})^+$, necessarily Γ_q would also be a lattice of $\mathbf{G}_1(\mathbb{R})^+$, containing Γ with finite index. This contradicts the fact that Γ_q contains Γ with infinite index.

Let $x = (x_1, x_{>1}) \in X_1^+ \times X_{>1}^+$ be any point whose projection in $S_{K_1}(\mathbf{G}_1, X_1)_\mathbb{C} \times S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_\mathbb{C}$ lies in $\mathcal{V}$. Let $\mathcal{O} := (\Gamma_q \cdot x_1, x_{>1})$ be the Γ_q-orbit of x in $X_1^+ \times X_{>1}^+$. By definition of the group Γ_q the closure of $\mathcal{O}$ in $X_1^+ \times X_{>1}^+$ is mapped to $\mathcal{V}'$ under the uniformization map

$$X_1^+ \times X_{>1}^+ \longrightarrow S_{K_1}(\mathbf{G}_1, X_1)_\mathbb{C} \times S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_\mathbb{C} \ .$$

As Γ_q is dense in $\mathbf{G}_1(\mathbb{R})^+$ this closure is nothing else than $X_1^+ \times x_{>1}$. Thus:

$$\mathcal{V}' \supset S_{K_1}(\mathbf{G}_1, X_1)_\mathbb{C} \times \mathcal{V}_{>1} \ . \qquad (7.7)$$

Finally it follows from (7.6) and (7.7) that:

$$\mathcal{V}' = S_{K_1}(\mathbf{G}_1, X_1)_\mathbb{C} \times \mathcal{V}_{>1} \ .$$

In particular $\mathcal{V}'$ is special. Hence V' is special. $\square$

Lemma 7.2.9. *The subvariety V' of Z contains V properly.*

Proof. Obviously V' contains V. Let us show that $V' \neq V$. Once more it is enough to show that $\mathcal{V}' \neq \mathcal{V}$.

As the generic Mumford-Tate group $\mathbf{H}_V$ of V, hence of $\mathcal{V}$, centralizes the torus $\mathbf{T}_V$, the projection $\mathbf{H}_{V,1}$ of $\mathbf{H}_V$ on $\mathbf{G}_1$ centralizes the non-trivial torus $\mathbf{T}_{V,1}$ projection of $\mathbf{T}_V$ on $\mathbf{G}_1$. In particular $\mathbf{H}_{V,1}$ is a proper algebraic subgroup of $\mathbf{G}_1$. But as

$$\mathcal{V}' = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times \mathcal{V}_{>1} \ ,$$

the group $\mathbf{G}_1$ is a direct factor of the generic Mumford-Tate group of $\mathcal{V}'$. $\square$

This finishes the proof of theorem 7.2.1. $\square$

8. Existence of suitable Hecke correspondences.

In this section we prove, under some assumptions on the compact open subgroup K_l, the existence of Hecke correspondences of small degree candidates for applying theorem 7.2.1. Our main result is the following:

Theorem 8.1. *Let $(\mathbf{G}', X')$ be a Shimura datum with $\mathbf{G}'$ semisimple of adjoint type, X'^+ a connected component of X' and $K' = \prod_{p \text{ prime}} K'_p$ a neat open compact subgroup of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$. We fix a faithful rational representation $\rho : \mathbf{G}' \hookrightarrow \mathbf{GL}_n$ such that K' is contained in $\mathbf{GL}_n(\hat{\mathbb{Z}})$.*

There exist positive integers k and f such that the following holds.

Let $(\mathbf{G}, X)$ be a Shimura subdatum of $(\mathbf{G}', X')$, let X^+ be a connected component of X contained in X'^+ and $K := K' \cap G(\mathbf{A}_{\mathrm{f}})$. Let V be a special but not strongly special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ defined by a Shimura subdatum $(\mathbf{H}_V, X_V)$ of $(\mathbf{G}, X)$. Let $\mathbf{T}_V$ be the connected centre of $\mathbf{H}_V$ and E_V the reflex field of $(\mathbf{H}_V, X_V)$.

Let l be a prime number such that K'_l is a hyperspecial maximal compact open subgroup in $\mathbf{G}'(\mathbb{Q}_l)$ which coincides with $\mathbf{G}'(\mathbb{Z}_l)$, the prime l splits $\mathbf{T}_V$ and $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a torus.

There exist a compact open subgroup $I_l \subset K_l := K'_l \cap \mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$ and an element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ satisfying the following conditions:

1. *$[K_l : I_l] \leq l^f$.*
2. *Let $I \subset K$ be the compact open subgroup $K^l I_l$ of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ (where $K^l := K' \cap \mathbf{G}(\mathbf{A}_{\mathrm{f}}^l)$) and $\tau : \mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ be the natural morphism. Let $\widetilde{V} \subset S_I(\mathbf{G}, X)_{\mathbb{C}}$ be an irreducible component of $\tau^{-1}(V)$. There exists an element σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ such that $\sigma\widetilde{V} \subset T_m(\widetilde{V})$.*

(3) *For every* $k_1, k_2 \in I_l$ *the image of* $k_1 m k_2$ *in* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ *generates an unbounded subgroup of* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$.

(4) $[I_l : I_l \cap m I_l m^{-1}] < l^k$.

Remarks 8.0.10. (a) As noticed in the introduction, conclusion (3) in theorem 8.1 can not be ensured if we stay at a level K_l which is a maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$ and do not lift the situation to a smaller level I_l. For explicit counterexamples see remark 7.2 of [15].

(b) As already noticed in section 4.1.5 the condition that K'_l is a hyperspecial maximal compact open subgroup in $\mathbf{G}'(\mathbb{Q}_l)$ which coincides with $\mathbf{G}'(\mathbb{Z}_l)$ is satisfied for almost all primes l.

8.1. **Iwahori subgroups.** We refer to [5], [6] and [20] for more details about buildings, Iwahori subgroups and Iwahori-Hecke algebras.

8.1.1. We first recall the definition of an Iwahori subgroup. Let l be a prime number. Let $\mathbf{G}$ be a reductive linear algebraic isotropic $\mathbb{Q}_l$-group and $\mathbf{A} \subset \mathbf{G}$ a maximal split torus of $\mathbf{G}$. We denote by $\mathbf{M} \subset \mathbf{G}$ the centraliser of $\mathbf{A}$ in $\mathbf{G}$. Let $\mathcal{X}$ be the (extended) Bruhat-Tits building of $\mathbf{G}$ and $\mathcal{A} \subset \mathcal{X}$ the apartment of $\mathcal{X}$ associated to $\mathbf{A}$. Let $K_l^{\mathrm{m}} \subset \mathbf{G}(\mathbb{Q}_l)$ be a special maximal compact subgroup (c.f [5, (I), def. 1.3.7 p.22, def. 4.4.1 p.79]) of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{A}$ (cf. section 4.1.6 for the notion of "good position"). We denote by $x_0 \in \mathcal{A}$ the unique K_l^{m}-fixed vertex in $\mathcal{X}$. We choose C a chamber of $\mathcal{A}$ containing x_0 in its closure, we denote by $\mathcal{I}_l \subset K_l^{\mathrm{m}}$ the Iwahori subgroup fixing C pointwise and by $\mathcal{C} \subset \mathcal{A}$ the unique Weyl chamber with apex at x_0 containing C.

All Iwahori subgroups of $\mathbf{G}(\mathbb{Q}_l)$ are conjugate, cf. [35, 3.7].

Remark 8.1.1. Strictly speaking (i.e. with the notations of Bruhat-Tits [5]) the group $\mathcal{I}_l$ as defined above is an Iwahori subgroup only in the case where the group $\mathbf{G}^{\mathrm{der}}$ is simply-connected. Our terminology is a well-established abuse of notations.

8.1.2. *Iwahori subgroups and unboundedness.*

Definition 8.1.2. *We denote by* $\mathrm{ord}_{\mathbf{M}} : \mathbf{M}(\mathbb{Q}_l) \longrightarrow X_*(\mathbf{M})$ *the homomorphism characterized by*

$$\forall\, \alpha \in X^*(\mathbf{M}), \quad < \mathrm{ord}_{\mathbf{M}}(m), \alpha >= \mathrm{ord}_{\mathbb{Q}_l}(\alpha(m)) \ ,$$

where $\mathrm{ord}_{\mathbb{Q}_l}$ *denotes the normalized (additive) valuation on* $\mathbb{Q}_l^*$ *and* $X_*(\mathbf{M})$ *(resp.* $X^*(\mathbf{M})$*) denotes the group of cocharacters (resp. characters) of* $\mathbf{M}$. *We denote by* $\Lambda \subset X_*(\mathbf{M})$ *the free* $\mathbf{Z}$*-module* $\mathrm{ord}_{\mathbf{M}}(\mathbf{M}(\mathbb{Q}_l))$.

The group $\mathbf{M}(\mathbb{Q}_l)$ (in particular the group $\mathbf{A}(\mathbb{Q}_l)$) acts on $\mathcal{A}$ via Λ-translations.

Definition 8.1.3. *Let $\Lambda^+ \subset \Lambda$ be the positive cone associated to the Weyl chamber $\mathcal{C}$.*

Elements of Λ^+ acting on $\mathcal{A}$ map $\mathcal{C}$ to $\mathcal{C}$.

Proposition 8.1.4. *Let m be an element of $\mathbf{A}(\mathbb{Q}_l)$ with non-trivial image $\mathrm{ord}_{\mathbf{M}}(m) \in \Lambda^+$. Then for any elements $i_1, i_2 \in \mathcal{I}_l$, the element $i_1 m i_2 \in \mathbf{G}(\mathbb{Q}_l)$ is not contained in a compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$.*

Proof. Let W_0 be the finite Weyl group of $\mathbf{G}$, let W be the modified affine Weyl group associated to $\mathcal{A}$ and Ω the finite subgroup of W taking the chamber C to itself. Let $\Delta = \{\alpha_1, \dots, \alpha_m\}$ be the set of affine roots on $\mathcal{A}$ which are positive on C and whose null set H_α is a wall of C. For $\alpha \in \Delta$ we denote by S_α the reflexion of $\mathcal{A}$ along the wall H_α. The group W is generated by Ω and the S_α's, $\alpha \in \Delta$. It identifies with the semi-direct product $W_0 \ltimes \Lambda$ (cf. [6, p.140]).

Recall the Bruhat-Tits decomposition:

$$\mathbf{G}(\mathbb{Q}_l) = \mathcal{I}_l \cdot W \cdot \mathcal{I}_l \quad , \tag{8.1}$$

where by abuse of notations we still write W for a set of representatives of W in $\mathbf{G}(\mathbb{Q}_l)$. Let $r : \mathbf{G}(\mathbb{Q}_l) \longrightarrow W$ be the map sending $g \in \mathbf{G}(\mathbb{Q}_l)$ to the unique $r(g) \in W$ such that $r(g) \in \mathcal{I}_l g \mathcal{I}_l$. Geometrically speaking the map r essentially coincides with the retraction $\rho_{\mathcal{A},C}$ of the Bruhat-Tits building $\mathcal{X}$ with centre the chamber C onto the apartment $\mathcal{A}$ ([5, I, theor.2.3.4]).

Let $\mathcal{H}(\mathbf{G}, \mathcal{I}_l)$ be the Hecke algebra (for the convolution product) of bi-$\mathcal{I}_l$-invariant compactly supported continuous complex functions on $\mathbf{G}(\mathbb{Q}_l)$. By the equation (8.1) this is an associative algebra with a vector space basis $T_w = 1_{\mathcal{I}_l w \mathcal{I}_l}$, $w \in W$, where $1_{\mathcal{I}_l w \mathcal{I}_l}$ denotes the characteristic function of the double coset $\mathcal{I}_l w \mathcal{I}_l$. A presentation of the algebra $\mathcal{H}(\mathbf{G}, \mathcal{I}_l)$ with generators T_ω, $\omega \in \Omega$, and T_α, $\alpha \in \Delta$, is given in [6, theorem 3.6 p.142] (or [4, p.242-243]). Given $w \in W$ let $l(w) \in \mathbb{N}$ be the number of hyperplanes H_α separating the two chambers C and wC. One obtains in particular (cf. [6, theorem 3.6 (b)] or [3, section 3.2, 1) and 6)]):

$$\forall\, w, w' \in W, \quad T_w \cdot T_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w') \ . \tag{8.2}$$

Let $\delta \in X^*(\mathbf{M})$ be the determinant of the adjoint action of $\mathbf{M}$ on the Lie algebra of $\mathbf{N}$. For $\lambda \in \Lambda^+ \subset W$ one shows the equality (cf. [20, (1.11)]):

$$l(\lambda) = \langle \delta, \lambda \rangle \ . \tag{8.3}$$

In particular any two elements λ, μ in $\Lambda^+ \subset W$ satisfy $l(\lambda \cdot \mu) = l(\lambda) + l(\mu)$ (where the additive law of Λ, seen as a subgroup of W, is written mutiplicatively). Thus the equation (8.2) implies the relation:

$$T_\lambda T_\mu = T_{\lambda\cdot\mu} \quad . \tag{8.4}$$

Remark 8.1.5. Equality (8.4) is stated in [20, (1.15)] for the Iwahori-Hecke algebra of a split adjoint group, but the proof generalizes to our setting.

Let m, i_1, i_2 as in the statement of the proposition and denote by g the element $i_1 m i_2 \in \mathbf{G}(\mathbb{Q}_l)$. As $r(g) = \mathrm{ord}_{\mathbf{M}}(m)$ belongs to Λ^+ it follows from (8.4) that:

$$r(g^n) = n \cdot r(g) = n \cdot \mathrm{ord}_{\mathbf{M}}(m) \quad .$$

This implies that the chamber $\rho_{\mathcal{A},C}(g^n C) = n \cdot \mathrm{ord}_{\mathbf{M}}(m) + C$ leaves any compact of $\mathcal{A}$ as n tends to infinity. As a corollary the chamber $g^n C$ of $\mathcal{X}$ also leaves any compact of $\mathcal{X}$ when n tends to infinity. This proves that the group $g^{\mathbb{Z}}$ is not contained in a compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$. $\square$

8.1.3. *Lifting.* Recall that the notion of "good position" was defined in section 4.1.6. The following lemma controls uniformly the lifting to an Iwahori level and to the intersection of two Iwahori subgroups both contained in a given special maximal compact subgroup:

Lemma 8.1.6. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group.*

(*a*) *For any prime l, any $\mathbb{Q}_l$-split torus $\mathbf{T} \subset \mathbf{G}_{\mathbb{Q}_l}$ and any maximal compact subgroup $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}$, there exists an Iwahori subgroup $\mathcal{I}_l$ of K_l in good position with respect to $\mathbf{T}$.*

(*b*) *There exists an integer f such that for any reductive $\mathbb{Q}$-subgroup $\mathbf{H} \subset \mathbf{G}$ and any prime l such that $\mathbf{H}_{\mathbb{Q}_l}$ is $\mathbb{Q}_l$-isotropic the following holds :*
  - (i) *for any maximal compact subgroup K_l of $\mathbf{H}(\mathbb{Q}_l)$, any Iwahori subgroup $\mathcal{I}_l \subset K_l$ is of index $[K_l : \mathcal{I}_l]$ smaller than l^f.*
  - (ii) *for any maximal compact subgroup K_l of $\mathbf{H}(\mathbb{Q}_l)$, any Iwahori subgroup $\mathcal{I}_l^1$ of K_l and any Iwahori subgroup $\mathcal{I}_l^2$ of $\mathbf{H}(\mathbb{Q}_l)$ such that both $\mathcal{I}_l^1$ and $\mathcal{I}_l^2$ are contained in a common special maximal compact subgroup, the index $[K_l : \mathcal{I}_l^1 \cap \mathcal{I}_l^2]$ is smaller than l^f.*

Proof. To prove (a) let l, $\mathbf{T}$ and K_l be as in the statement. Choose a maximal split torus $\mathbf{A}$ of $\mathbf{G}_{\mathbb{Q}_l}$ containing $\mathbf{T}_{\mathbb{Q}_l}$, denote by $\mathbf{M}$ the centraliser of $\mathbf{A}$ in $\mathbf{G}_{\mathbb{Q}_l}$ and choose any minimal parabolic $\mathbf{P}$ of $\mathbf{G}_{\mathbb{Q}_l}$ with Levi $\mathbf{M}$. Let $\mathcal{A}$ be the apartment of the Bruhat-Tits building $\mathcal{X}$ of $\mathbf{G}_{\mathbb{Q}_l}$ associated to $\mathbf{A}$, let $x \in \mathcal{A}$ be the unique point of $\mathcal{X}$ fixed by the maximal compact subgroup K_l, and let $\mathcal{C} \subset \mathcal{A}$ be any Weyl chamber containing x whose stabiliser at infinity in $\mathbf{G}(\mathbb{Q}_l)$ is $\mathbf{P}(\mathbb{Q}_l)$. Let C be the unique chamber of $\mathcal{C}$ containing x in its closure. Then by construction the Iwahori subgroup $\mathcal{I}_l \subset K_l$ fixing C satisfies that $\mathcal{I}_l \cap \mathbf{A}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{A}(\mathbb{Q}_l)$. In particular $\mathcal{I}_l \cap \mathbf{T}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{T}(\mathbb{Q}_l)$.

To prove $(b)(i)$: first notice that among maximal compact subgroups of $\mathbf{H}(\mathbb{Q}_l)$ the hyperspecial ones have maximal volume, cf. [35, 3.8.2]. Thus one can assume that K_l is hyperspecial. In this case the index $[K_l : \mathcal{I}_l]$ coincides with $\sum_{w \in W_0} q_w$ where W_0 denotes the finite Weyl group of $\mathbf{H}_{\mathbb{Q}_l}$ and q_w denotes $[\mathcal{I}_l w \mathcal{I}_l : \mathcal{I}_l]$ for $w \in W_0$. With the notations of [35, section 3.3.1] for a reduced word $w = r_1 \cdots r_j \in W_0$ one has $q_w = l^d$ with $d = \sum_{i=1}^{j} d(\nu_i)$, where ν_i denotes the vertex of the local Dynkin diagram of $\mathbf{H}_{\mathbb{Q}_l}$ corresponding to the reflection r_i. As the cardinality of W_0 and its length function are bounded when $\mathbf{H}$ ranges through reductive $\mathbb{Q}$-subgroups of $\mathbf{G}$ and l ranges through prime numbers we are reduced to prove that for any positive integer r there exists a positive integer s such that $d(\nu_i) \leq s$ for any local Dynkin diagram of rank at most r. This follows from inspecting the tables in [35, section 4].

To prove $(b)(ii)$: notice that

$$[K_l : \mathcal{I}_l^1 \cap \mathcal{I}_l^2] = [K_l : \mathcal{I}_l^1] \cdot [\mathcal{I}_l^1 : \mathcal{I}_l^1 \cap \mathcal{I}_l^2] \ .$$

As $\mathcal{I}_l^1$ and $\mathcal{I}_l^2$ are both Iwahori subgroups of a special maximal compact subgroup K_l^{m} of $\mathbf{H}(\mathbb{Q}_l)$ the index $[\mathcal{I}_l^1 : \mathcal{I}_l^1 \cap \mathcal{I}_l^2]$ is bounded by $[K_l^{\mathrm{m}} : \mathcal{I}_l^2] = |W_0|$. As the cardinality of W_0 is bounded when $\mathbf{H}$ ranges through reductive $\mathbb{Q}$-subgroups of $\mathbf{G}$ and l ranges through prime numbers, statement $(b)(ii)$ follows from statement $(b)(i)$ (up to a change of the constant f). □

8.2. **A uniformity result.** The purpose of this section is to prove the following uniformity result:

Proposition 8.2.1. *Let $(\mathbf{G}', X')$ be a Shimura datum with $\mathbf{G}'$ semi-simple of adjoint type and X'^+ a connected component of X'. Let A be the positive integer defined in* [39], *proposition 2.9. Then the following holds.*

Let $(\mathbf{G}, X)$ be a Shimura subdatum of $(\mathbf{G}', X')$ and X^+ a connected component of X contained in X'^+. Let $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ be a neat open compact subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$.

Let V be a special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ which is not strongly special. Let $(\mathbf{H}_V, X_V)$ be a Shimura datum defining V, denote by $\mathbf{T}_V$ its connected centre. Let l be a prime splitting $\mathbf{T}_V$ and E_V the reflex field of $(\mathbf{H}_V, X_V)$. For any m in $\mathbf{T}_V(\mathbb{Q}_l)$ its power m^A satisfies the condition that for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ the following inclusion holds in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$:

$$\sigma(V) \subset T_{m^A}(V) \ .$$

Proof. Let V and m be as in the statement. For simplicity we write $\mathbf{H}$ for $\mathbf{H}_V$. We refer to section 2.1 of [39] for details and notations on reciprocity morphisms.

By Proposition 2.9 of [39] the image of m^A in $\overline{\pi_0}\pi(\mathbf{H})$ is of the form $r_{(\mathbf{H},X_{\mathbf{H}})}(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$.

The variety V is the image of $X_{\mathbf{H}}^+ \times \{1\}$ in $\mathrm{Sh}_K(\mathbf{G}, X)$. Let σ be the element of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ as above. By definition of the Galois action on the set of connected components of a Shimura variety, we get

$$\sigma(V) = \overline{X_{\mathbf{H}}^+ \times \{m^A\}} \subset T_{m^A} V$$

where $\overline{X_{\mathbf{H}}^+ \times \{m^A\}}$ stands for the image of $X_{\mathbf{H}}^+ \times \{m^A\}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. □

8.3. **Proof of theorem 8.1.** Let $\mathbf{G}'$, X', X'^+, K', ρ, $\mathbf{G}$, X, V and l be as in theorem 8.1.

8.3.1. *Definition of m.* As V is special but not strongly special, the torus $\mathbf{T}_V^{\mathrm{ad}} := \lambda(\mathbf{T}_V)$ is a non-trivial torus in $\mathbf{G}^{\mathrm{ad}}$, where $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ denotes the natural morphism.

As $K'_l = \mathbf{G}'(\mathbb{Z}_l)$ the compact subgroup $K_l = K'_l \cap \mathbf{G}(\mathbb{Q}_l)$ of $\mathbf{G}(\mathbb{Q}_l)$ contains $\mathbf{G}(\mathbb{Z}_l)$. In particular for any element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ one has the inequality:

$$[K_l : K_l \cap m K_l m^{-1}] \leq [K_l : K_l \cap m \mathbf{G}(\mathbb{Z}_l) m^{-1}] \quad . \tag{8.5}$$

By lemma 2.6 of [39] the coordinates of the characters of $\mathbf{T}_V$ intervening in the representation $\rho_{|\mathbf{T}_V} : \mathbf{T}_V \longrightarrow \mathbf{GL}_n$ with respect to a suitable $\mathbb{Z}$-basis of $X^*(\mathbf{T}_V)$ are bounded uniformly on V. By assumption the reduction $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a torus, hence $(\mathbf{T}_V)_{\mathbb{Z}_l}$ is also a torus by lemma 3.3.1 of [18]. Thus we can apply proposition 7.4.3 of [18] for $r = 1$, $q_1 = \lambda_{|\mathbf{T}_V} : \mathbf{T}_V \longrightarrow \mathbf{T}_V^{\mathrm{ad}}$ and $e = A$ (the positive integer given by proposition 8.2.1): there exists a constant k_1 depending only on $\mathbf{G}'$, X' and K', and an element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ such that $\lambda(m)$ does not lie in a compact subgroup of $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$ and satisfies

$$[K_l : K_l \cap m^A \mathbf{G}(\mathbb{Z}_l) m^{-A}] < l^{k_1} \quad . \tag{8.6}$$

8.3.2. *Definition of I_l.* As l splits $\mathbf{T}_V$ and $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a torus, the group $\mathbf{G}(\mathbb{Z}_l)$, and thus also K_l, is in good position with respect to $\mathbf{T}_V$ by lemma 4.1.5.

Let f be the constant defined in lemma 8.1.6, (b) (for the ambient group $\mathbf{G}'$). We claim that there exists an Iwahori subgroup $\mathcal{I}_l^1$ of $\mathbf{G}(\mathbb{Q}_l)$ such that $[K_l : K_l \cap \mathcal{I}_l^1] < l^f$. Indeed let K_l^1 be any maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$ containing K_l. As K_l is in good position with respect to $\mathbf{T}$ the group K_l^1 too. By lemma 8.1.6(b)(i) there exists an Iwahori subgroup $\mathcal{I}_l^1 \subset K_l^1$ in good position with respect to $\mathbf{T}_V$ and satisfying $[K_l^1 : \mathcal{I}_l^1] < l^f$. This implies $[K_l : K_l \cap \mathcal{I}_l^1] < l^f$ as required.

Let $\mathbf{A}$ be a maximal split torus of $\mathbf{G}_{\mathbb{Q}_l}$ containing $\mathbf{T}_{V,\mathbb{Q}_l}$ and such that $\mathcal{I}_l^1$ is in good position with respect to $\mathbf{A}$. Let $\mathbf{M}$ be its centralizer in $\mathbf{G}_{\mathbb{Q}_l}$. Choose K_l^{m} a special maximal compact subgroup containing $\mathcal{I}_l^1$. Let $\mathcal{X}$ be the Bruhat-Tits building of $\mathbf{G}_{\mathbb{Q}_l}$. Denote by $\mathcal{A} \subset \mathcal{X}$ the apartment fixed by $\mathbf{A}$, by $x \in \mathcal{A}$ the unique special vertex fixed by K_l^{m} and by C^1 the unique chamber of $\mathcal{A}$ fixed by $\mathcal{I}_l^1$. The vertex x lies in the closure of C^1. The vector $\mathrm{ord}_{\mathbf{M}}(m) \in \Lambda := \mathrm{ord}_{\mathbf{M}}(\mathbf{M}(\mathbb{Q}_l))$ is non-trivial. Let $\mathcal{C} \subset \mathcal{A}$ be a Weyl chamber of $\mathcal{A}$ with apex x such that $C^1 + \mathrm{ord}_{\mathbf{M}}(m) \subset \mathcal{C}$. In particular:

$$\mathrm{ord}_{\mathbf{M}}(m) \in \Lambda^+ \setminus \{0\} \quad , \tag{8.7}$$

where $\Lambda^+ \subset \Lambda$ denotes the positive cone associated to the Weyl chamber $\mathcal{C}$.

Finally let $\mathcal{I}_l^2$ be the Iwahori subgroup of K_l^{m} fixing the unique chamber of $\mathcal{C}$ with apex x. As $\mathcal{I}_l^2$ is the fixator of a chamber of $\mathcal{A}$ it is in good position with respect to $\mathbf{A}$, hence also with respect to $\mathbf{T}_V$.

Definition 8.3.1. *We define $I_l := \mathcal{I}_l^1 \cap \mathcal{I}_l^2 \cap K_l$.*

Remark 8.3.2. Lifting to the Iwahori level I_l chosen as above will enable us to apply proposition 8.1.4, as the Iwahori $\mathcal{I}_l^2$ is in the required position with respect to m. The definition of I_l is simpler in the case where K_l is hyperspecial. In this case necessarily $K_l^1 = K_l^{\mathrm{m}} = K_l$ and we can take $\mathcal{I}_l^1 = \mathcal{I}_l^2$. Moreover the choice of $\mathcal{I}_l^2$ is unique if m is regular.

8.3.3. *End of the proof.* Let us show that the uniform constants $k = (k_1 + f)$ and f, the open subgroup I_l and the element $m^A \in \mathbf{T}_V(\mathbb{Q}_l)$ satisfy the conclusions of the theorem 8.1.

As the groups K_l, $\mathcal{I}_l^1$ and $\mathcal{I}_l^2$ are in good position with respect to $\mathbf{T}_V$, the group I_l is also in good position with respect to $\mathbf{T}_V$. As $\mathcal{I}_l^1$ and $\mathcal{I}_l^2$ are both contained in the special maximal compact subgroup K_l^{m}, the lemma 8.1.6(b)(ii) implies the following inequality:

$$[K_l : I_l] = [K_l : K_l \cap \mathcal{I}_l^1 \cap \mathcal{I}_l^2] \leq [K_l^1 : \mathcal{I}_l^1 \cap \mathcal{I}_l^2] < l^f \quad . \tag{8.8}$$

This is condition (1) of theorem 8.1.

By proposition 8.2.1 there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ such that $\sigma(\widetilde{V}) \subset T_{m^A}\widetilde{V}$: this is condition (2) of theorem 8.1.

Let $\mathbf{A}^{\mathrm{ad}}$ be the maximal split torus $\lambda(\mathbf{A})$ of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$, denote by $\mathbf{M}^{\mathrm{ad}} := \lambda(\mathbf{M})$ its centralizer in $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$, let $\mathcal{I}_l$ be the Iwahori $\lambda(\mathcal{I}_l^2)$ of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$, let C^{ad} be the unique chamber of the Bruhat-Tits building $\mathcal{X}^{\mathrm{ad}}$ of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$ fixed by $\mathcal{I}_l$ and x^{ad} the vertex in the closure of C^{ad} fixed by $\lambda(K_l^{\mathrm{m}})$. Finally let $\mathcal{C}^{\mathrm{ad}} \subset \mathcal{A}^{\mathrm{ad}}$ be the unique Weyl chamber with apex x^{ad} and containing C^{ad} and $\Lambda^{\mathrm{ad},+} \subset \Lambda^{\mathrm{ad}} := \mathrm{ord}_{\mathbf{M}^{\mathrm{ad}}}(\mathbf{M}^{\mathrm{ad}}(\mathbb{Q}_l))$ the associated positive cone. It follows from (8.7) that $\mathrm{ord}_{\mathbf{M}^{\mathrm{ad}}}(\lambda(m))$ lies in $\Lambda^{\mathrm{ad},+}$; it is non-zero as $\lambda(m)$ does not lie in a compact subgroup of $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$ (hence of $\mathbf{A}^{\mathrm{ad}}(\mathbb{Q}_l)$). Hence also $\mathrm{ord}_{\mathbf{M}^{\mathrm{ad}}}(\lambda(m^A))$ belongs to $\Lambda^{\mathrm{ad},+} \setminus \{0\}$. It follows from the proposition 8.1.4 that for any k_1, k_2 in $\mathcal{I}_l^2$ (in particular for any k_1, k_2 in I_l) the image of $k_1 m^A k_2$ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$. This is condition (3) of theorem 8.1.

Finally from the inequalities (8.5), (8.6) and (8.8) one deduces:

$$\begin{aligned}[I_l : I_l \cap m^A I_l m^{-A}] &= [I_l : I_l \cap m^A K_l m^{-A}] \cdot [I_l \cap m^A K_l m^{-A} : I_l \cap m^A I_l m^{-A}] \\ &\leq [K_l : K_l \cap m^A K_l m^{-A}] \cdot [K_l : I_l] \\ &\leq [K_l : K_l \cap m^A \mathbf{G}(\mathbb{Z}_l) m^{-A}] \cdot [K_l : I_l] \leq l^{k_1+f} = l^k \quad .\end{aligned} \tag{8.9}$$

This is condition (4) of theorem 8.1.

This finishes the proof of theorem 8.1.

9. Conditions on the prime l.

In this section, we use theorem 2.4.4, theorem 7.2.1 and theorem 8.1 to show (under one of the assumptions of the theorem 3.1.1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a special subvariety V which is not strongly special to contain a special subvariety V' containing V properly.

9.1. **Situation.** We will consider the following set of data:

Let $(\mathbf{G}', X')$ be a Shimura datum with $\mathbf{G}'$ semi-simple of adjoint type and let X'^+ a connected component of X'. We fix R, as in definition 2.5.1 for $\mathbf{G}'$, X' and X'^+, a uniform bound on the degrees of the Galois closures of the fields $E(\mathbf{H}, X_{\mathbf{H}})$ with $(\mathbf{H}, X_{\mathbf{H}})$ ranging through the Shimura subdata of $(\mathbf{G}', X')$.

Let $K' = \prod_{p \text{ prime}} K'_p$ be a neat compact open subgroup of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$. We fix a faithful representation $\rho\colon \mathbf{G}' \hookrightarrow \mathbf{GL}_n$ such that K' is contained in $\mathbf{GL}_n(\hat{\mathbb{Z}})$. We suppose that with respect to ρ, the group K'_3 is contained in the principal congruence subgroup of level three of $\mathbf{GL}_n(\mathbb{Z}_3)$.

Fix N be a positive integer, let B and $C(N)$ be the constants from the theorem 2.4.4, k the positive integer defined in theorem 8.1 for the data $\mathbf{G}'$, X', X'^+ and K', and f the positive integer defined in theorem 8.1 for the data $\mathbf{G}'$, X' and X'^+.

Consider an infinite set Σ of special subvarieties of $S_{K'}(\mathbf{G}', X')_{\mathbb{C}}$. For each W in Σ, we let $(\mathbf{H}_W, X_W)$ be a Shimura subdatum of $(\mathbf{G}', X')$ defining W. Let $\mathbf{T}_W$ be the connected centre of $\mathbf{H}_W$ and α_W, β_W be as in definitions 2.4.1 and 2.5.1.

Remark 9.1.1. Let $(\mathbf{G}, X)$ be a Shimura subdatum of $(\mathbf{G}', X')$, define $K = K' \cap G(\mathbf{A}_{\mathrm{f}})$ and choose X^+ a connected component of X contained in X'^+. Let $p\colon \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')_{\mathbb{C}}$ be the natural morphism. If $V \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ is a special subvariety which is an irreducible component of $p^{-1}(W)$ for some $W \in \Sigma$, then V is still defined by the Shimura subdatum $(\mathbf{H}_V := \mathbf{H}_W, X_V := X_W)$ of $(\mathbf{G}, X)$. Accordingly we have $\mathbf{T}_V = \mathbf{T}_W$, $\alpha_V = \alpha_W$, $\beta_V = \beta_W$.

9.2. **The criterion.** We can now state the main result of this section:

Theorem 9.2.1. *Let $\mathbf{G}'$, X', X'^{+}, R, K', N, k, f and Σ as in the situation 9.1.*

We assume either the GRH or that the tori $\mathbf{T}_W$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as W ranges through Σ.

Let $(\mathbf{G}, X)$ be a Shimura subdatum of $(\mathbf{G}', X')$ with reflex field $F_{\mathbf{G}} := E(\mathbf{G}, X)$. Define $K = K' \cap G(\mathbf{A}_{\mathrm{f}})$ and choose X^+ a connected component of X contained in X'^{+}. Let $p\colon \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')_{\mathbb{C}}$ be the natural morphism.

Let $W \in \Sigma$, let $V \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ be an irreducible component of $p^{-1}(W)$ and let Z be a Hodge generic $F_{\mathbf{G}}$-irreducible subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing V.

Define $r := \dim Z - \dim V$ and suppose $r > 0$. Suppose moreover that V and Z satisfy the following conditions:

(1) *the variety V is special but not strongly special in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.*
(2) *there exists a prime l such that K'_l is a hyperspecial maximal compact open subgroup in $\mathbf{G}'(\mathbb{Q}_l)$ which coincides with $\mathbf{G}'(\mathbb{Z}_l)$, the prime l splits $\mathbf{T}_V$, the reduction $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a torus and the following inequality is satisfied:*

$$l^{(k+2f)\cdot 2^r} \cdot (\deg_{L_K} Z)^{2^r} < C(N)\alpha_V \beta_V^N \quad . \tag{9.1}$$

Then Z contains a special subvariety V' that contains V properly.

Proof. The proof of theorem 9.2.1 proceeds by induction on $r = \dim Z - \dim V > 0$. For simplicity we denote $d_Z := \deg_{L_K} Z$.

9.2.1. *Case $r = 1$.* Let $\mathbf{G}$, X, X^+, K, $F_{\mathbf{G}}$, W, V and Z as in theorem 9.2.1 with $\dim Z - \dim V = 1$. The inequality (9.1) for $r = 1$ gives us:

$$l^{2(k+2f)} \cdot d_Z^2 < C(N)\alpha_V \beta_V^N \quad . \tag{9.2}$$

Let $I_l \subset K_l$ and $m \in \mathbf{T}_V(\mathbb{Q}_l)$ satisfying the conclusion of theorem 8.1. Let $I \subset K$ be the neat compact open subgroup $K^l I_l$ of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and $\tau\colon \mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ the finite morphism of Shimura varieties deduced from the inclusion $I \subset K$. It follows from the condition (1) in theorem 8.1 that the degree of τ is bounded above by l^f.

Let $\widetilde{V} \subset S_I(\mathbf{G}, X)_{\mathbb{C}}$ be an irreducible component of the preimage $\tau^{-1}(V)$, this is a special but not strongly special subvariety of $S_I(\mathbf{G}, X)_{\mathbb{C}}$ still defined by the Shimura subdatum $(\mathbf{H}_V, X_V)$. Let $E_V = E(\mathbf{H}_V, X_V)$. Notice that $F_{\mathbf{G}} \subset E_V$. By the projection formula stated in section 5.1 and proposition 5.3.2(1), we have the inequality

$$\deg_{L_I}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot \widetilde{V}) \geq \deg_{L_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V) \quad .$$

By the corollary 5.3.10 the following inequality holds :

$$\deg_{L_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot V) \geq \deg_{L_{K_{\mathbf{H}_V}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot V) \quad .$$

On the other hand, as R satisfies the definition 2.5.1, theorem 2.4.4 applied to the special subvariety V of $S_K(\mathbf{G},X)_{\mathbb{C}}$ provides the following lower bound:

$$\deg_{L_{K_{\mathbf{H}_V}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot V) > C(N)\alpha_V\beta_V^N \quad .$$

We thus obtain:

$$\deg_{L_I}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot \widetilde{V}) > C(N)\alpha_V\beta_V^N \quad . \tag{9.3}$$

Let $\widetilde{Z}$ be an $F_{\mathbf{G}}$-irreducible component of $\tau^{-1}(Z)$ containing $\widetilde{V}$. In particular $\widetilde{Z}$ is Hodge generic in $\mathrm{Sh}_I(\mathbf{G},X)_{\mathbb{C}}$ and is the union of the $\mathrm{Gal}(\overline{F}/F_{\mathbf{G}})$-conjugates of a geometrically irreducible component of $\tau^{-1}(Z)$. The image of $\widetilde{Z}$ in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ is Z and as τ is of degree bounded above by l^f the following inequality follows from section 5.1:

$$\deg_{L_I}\widetilde{Z} \leq l^f\cdot d_Z \quad . \tag{9.4}$$

As the morphism $\tau\colon \mathrm{Sh}_I(\mathbf{G},X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ is finite and preserves the property of a subvariety of being special, exhibiting a special subvariety V' such that $V \subsetneq V' \subset Z$ is equivalent to exhibiting a special subvariety $\widetilde{V'}$ such that $\widetilde{V} \subsetneq \widetilde{V'} \subset \widetilde{Z}$.

By conclusion (2) of theorem 8.1 there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$ such that $\sigma\widetilde{V} \subset T_m\widetilde{V} \subset T_m\widetilde{Z}$. As $T_m\widetilde{Z}$ is defined over $F_{\mathbf{G}}$ hence over E_V we deduce that $\widetilde{V} \subset T_m\widetilde{Z}\cap\widetilde{Z}$ and thus $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot\widetilde{V} \subset \widetilde{Z}\cap T_m\widetilde{Z}$.

If $\widetilde{Z}$ and $T_m\widetilde{Z}$ have no common (geometric) irreducible component, then any $\sigma(\widetilde{V})$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$, is an irreducible component of $\widetilde{Z}\cap T_m\widetilde{Z}$ for dimension reasons. We get

$$\begin{aligned} C(N)\alpha_V\beta_V^N &\leq \deg_{L_I}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)\cdot\widetilde{V}) \leq \deg_{L_I}(\widetilde{Z}\cap T_m\widetilde{Z}) \\ &\leq (\deg_{L_I}\widetilde{Z})^2[I_l : I_l\cap mI_lm^{-1}] < l^{k+2f}\cdot d_Z^2 \quad , \end{aligned} \tag{9.5}$$

where the first inequality on the left comes from the inequality (9.3), the second from Bezout's theorem (as in [19], Example 8.4.6) and the last one from inequality (9.4) and the condition (4) on m from theorem 8.1. This contradicts the inequality (9.2). Therefore, the intersection $\widetilde{Z}\cap T_m\widetilde{Z}$ is not proper and, as both $\widetilde{Z}$ and $T_m\widetilde{Z}$ are defined over $F_{\mathbf{G}}$ and $\widetilde{Z}$ is $F_{\mathbf{G}}$-irreducible, we have $\widetilde{Z} \subset T_m\widetilde{Z}$.

As m also satisfies condition (3) of theorem 8.1, we can apply theorem 7.2.1 to this m: there exists $\widetilde{V'}$ special subvariety of $\widetilde{Z}$ containing $\widetilde{V}$ properly.

9.2.2. *Case* $r > 1$. Fix $r > 1$ an integer and suppose by induction that the conclusion of theorem 9.2.1 holds for all Shimura subdata $(\mathbf{G}, X)$ of $(\mathbf{G}', X')$, connected components X^+ of X contained in X'^+, compact open subgroups $K = K' \cap \mathbf{G}'(\mathbf{A}_{\mathrm{f}})$, varieties $W \in \Sigma$, and subvarieties V and Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ as in the statement of theorem 9.2.1, satisfying moreover $0 < \dim Z - \dim V < r$.

Now let $\mathbf{G}$, X, X^+, K, F, W, V and Z satisfying the assumptions of theorem 9.2.1 with $\dim Z = \dim V + r$. Let I, m, $\widetilde{V}$ and $\widetilde{Z}$ be constructed as in the case $r = 1$. In particular the inequalities (9.3) and (9.4) still hold.

Suppose that $\widetilde{Z} \subset T_m \widetilde{Z}$. In this case we can apply theorem 7.2.1 with this m: there exists $\widetilde{V'}$ special subvariety of $\widetilde{Z}$ containing $\widetilde{V}$ properly. This implies that there exists V' special subvariety of Z containing V properly.

Suppose now that the intersection $\widetilde{Z} \cap T_m \widetilde{Z}$ is proper. The same argument as in the case $r = 1$ shows that this is equivalent to $\widetilde{Z}$ not being contained in $T_m \widetilde{Z}$. As the intersection $\widetilde{Z} \cap T_m \widetilde{Z}$ contains $\widetilde{V}$, we choose an $F_{\mathbf{G}}$-irreducible component $\widetilde{Y} \subset \mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}}$ of $\widetilde{Z} \cap T_m \widetilde{Z}$ containing $\widetilde{V}$ and we denote by Y its image in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Thus Y is $F_{\mathbf{G}}$-irreducible and satisfies $r_Y := \dim Y - \dim V < r$. To show that $r_Y > 0$ we need to check that $\widetilde{V}$ is not a component of $\widetilde{Z} \cap T_m(\widetilde{Z})$. As $\widetilde{Z} \cap T_m(\widetilde{Z})$ is defined over $F_{\mathbf{G}}$ hence over E_V, we have $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot \widetilde{V} \subset \widetilde{Z} \cap T_m(\widetilde{Z})$. If $\widetilde{V}$ were a component of $\widetilde{Z} \cap T_m(\widetilde{Z})$ by taking degrees and arguing as in the proof of inequality (9.5) one still obtains:

$$C(N) \alpha_V \beta_V^N < l^{k+2f} d_Z^2 \quad .$$

This contradicts the condition (2). Hence $0 < r_Y < r$.

Let $(\mathbf{H}, X_{\mathbf{H}})$ be a Shimura subdatum of $(\mathbf{G}, X)$ defining the smallest special subvariety of $S_I(\mathbf{G}, X)_{\mathbb{C}}$ containing a geometrically irreducible component of $\widetilde{Y}$ containing $\widetilde{V}$, let $X_{\mathbf{H}}^+ \subset X^+$ be the corresponding connected component of $X_{\mathbf{H}}$. We define $K_{\mathbf{H}} := K \cap \mathbf{H}(\mathbf{A}_{\mathrm{f}})$ and $I_{\mathbf{H}} := I \cap \mathbf{H}(\mathbf{A}_{\mathrm{f}})$. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Sh}_{I_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} & \xrightarrow{q} & \mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}} \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\
\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} & \xrightarrow[q]{} & \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}
\end{array} \quad .$$

Let $F_{\mathbf{H}}$ be the reflex field $E(\mathbf{H}, X_{\mathbf{H}})$ and let $\widetilde{V_{\mathbf{H}}}$ be an irreducible component of $q^{-1}(\widetilde{V})$ contained in $S_{I_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$. We denote $V_{\mathbf{H}} := \tau(\widetilde{V_{\mathbf{H}}})$ its image in $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$. Hence $V_{\mathbf{H}}$ is also an irreducible component of $(p \circ q)^{-1}(W)$.

Let $\widetilde{Y_{\mathbf{H}}} \subset \mathrm{Sh}_{I_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$ be an $F_{\mathbf{H}}$-irreducible component of $q^{-1}(\widetilde{Y})$ containing $\widetilde{V_{\mathbf{H}}}$. In particular $\widetilde{Y_{\mathbf{H}}}$ is an $F_{\mathbf{H}}$-irreducible Hodge generic subvariety of $\mathrm{Sh}_{I_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$. We define $Y_{\mathbf{H}} := \tau(\widetilde{Y_{\mathbf{H}}})$, it is an $F_{\mathbf{H}}$-irreducible Hodge generic subvariety of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$.

Finally we have the commutative diagram of triples of varieties:

$$\begin{array}{ccc}
(\mathrm{Sh}_{I_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}, \widetilde{Y_{\mathbf{H}}}, \widetilde{V_{\mathbf{H}}}) & \xrightarrow{q} & (\mathrm{Sh}_{I}(\mathbf{G}, X)_{\mathbb{C}}, \widetilde{Y}, \widetilde{V}) \\
\Big\downarrow{\scriptstyle\tau} & & \Big\downarrow{\scriptstyle\tau} \\
(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}, Y_{\mathbf{H}}, V_{\mathbf{H}}) & \xrightarrow[q]{} & (\mathrm{Sh}_{K}(\mathbf{G}, X)_{\mathbb{C}}, Y, V)
\end{array} .$$

Notice that $Y_{\mathbf{H}}$ satisfies

$$(9.6)\quad \deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}} \leq \deg_{L_{I_{\mathbf{H}}}} \widetilde{Y_{\mathbf{H}}} \leq \deg_{L_I} q(\widetilde{Y_{\mathbf{H}}}) \leq \deg_{L_I} \widetilde{Y} \leq \deg_{L_I}(\widetilde{Z} \cap T_m \widetilde{Z}) < l^{k+2f} d_Z^2 \quad .$$

Indeed, the inequality $\deg_{L_{K_H}} Y_{\mathbf{H}} \leq \deg_{L_{I_{\mathbf{H}}}} \widetilde{Y_{\mathbf{H}}}$ comes from section 5.1, the inequality $\deg_{L_{I_{\mathbf{H}}}} \widetilde{Y_{\mathbf{H}}} \leq \deg_{L_I} q(\widetilde{Y_{\mathbf{H}}})$ from corollary 5.3.10, the inequality $\deg_{L_I} q(\widetilde{Y_{\mathbf{H}}}) \leq \deg_{L_I} \widetilde{Y}$ from the inclusion $q(\widetilde{Y_{\mathbf{H}}}) \subset \widetilde{Y}$, the inequality $\deg_{L_I} \widetilde{Y} \leq \deg_{L_I}(\widetilde{Z} \cap T_m \widetilde{Z})$ from the fact that $\widetilde{Y}$ is an $F_{\mathbf{G}}$-irreducible component of $\widetilde{Z} \cap T_m \widetilde{Z}$, and the last inequality on the right is proven as in (9.5).

Proposition 9.2.2. *The data* $\mathbf{H}$, $X_{\mathbf{H}}$, $X_{\mathbf{H}}^+$, $K_{\mathbf{H}}$, $F_{\mathbf{H}}$, W, $V_{\mathbf{H}}$ *and* $Y_{\mathbf{H}}$ *satisfy the conditions of theorem 9.2.1 (in place respectively of* $\mathbf{G}$, X, X^+, K, $F_{\mathbf{G}}$, W, V *and* Z*).*

Proof. Let $r_{\mathbf{H}} := \dim Y_{\mathbf{H}} - \dim V_{\mathbf{H}}$, thus $r_{\mathbf{H}} = r_Y > 0$.

We first check that $\mathbf{H}$, $X_{\mathbf{H}}$, $X_{\mathbf{H}}^+$, $K_{\mathbf{H}}$, $F_{\mathbf{H}}$, W, $V_{\mathbf{H}}$ and $Y_{\mathbf{H}}$ satisfy condition (2) of theorem 9.2.1, for the same prime l. From the inequality (9.6) we obtain:

$$l^{(k+2f)\cdot 2^{r_{\mathbf{H}}}} (\deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}})^{2^{r_{\mathbf{H}}}} \leq l^{(k+2f)\cdot 2^{r_{\mathbf{H}}+1}} d_Z^{2^{r_{\mathbf{H}}+1}}$$

and, as $r_{\mathbf{H}} + 1 \leq r$, we deduce from the inequality (9.1) that

$$(9.7)\qquad l^{(k+2f)\cdot 2^{r_{\mathbf{H}}}} (\deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}})^{2^{r_{\mathbf{H}}}} < C(N) \alpha_V \beta_V^N \quad .$$

This is condition (2) for $\mathbf{H}$, $X_{\mathbf{H}}$, $X_{\mathbf{H}}^+$, $K_{\mathbf{H}}$, $F_{\mathbf{H}}$, W, $V_{\mathbf{H}}$ and $Y_{\mathbf{H}}$.

Proposition 9.2.2 then follows from the following lemma proving that $\mathbf{H}$, $X_{\mathbf{H}}$, $X_{\mathbf{H}}^+$, $K_{\mathbf{H}}$, $F_{\mathbf{H}}$, W, $V_{\mathbf{H}}$ and $Y_{\mathbf{H}}$ satisfy the condition (1) of theorem 9.2.1. □

Lemma 9.2.3. *The special subvariety* $V_{\mathbf{H}}$ *is not strongly special in* $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$.

Proof. Suppose the contrary. Then $\mathbf{T}_{V_{\mathbf{H}}} (= \mathbf{T}_V)$ is contained in the connected centre $Z(\mathbf{H})^0$ of $\mathbf{H}$ and by the lemma 6.2, $\mathbf{T}_V = Z(\mathbf{H})^0$. Recall that $K_{\mathbf{T}_V}^{\mathrm{m}}$ denotes the maximal compact open subgroup of $\mathbf{T}_V(\mathbf{A}_{\mathrm{f}})$ and $K_{\mathbf{T}_V} = \mathbf{T}_V(\mathbf{A}_{\mathrm{f}}) \cap K_{\mathbf{H}}$. Let $K^{\mathrm{m}}_{\mathbf{H}} := K^{\mathrm{m}}_{\mathbf{T}_V} K_{\mathbf{H}}$ and let

$$\pi \colon \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K^{\mathrm{m}}_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$$

be the natural morphism. Notice that $K^{\mathrm{m}}_{\mathbf{H}}/K_{\mathbf{H}} = K^{\mathrm{m}}_{\mathbf{T}_V}/K_{\mathbf{T}_V}$ acts transitively on the fibres of π. Let A be the positive integer defined by [39, prop.2.9] for Shimura subdata of $(\mathbf{G}', X')$ (notice that the constant A already appeared in proposition 8.2.1). Let $\Theta_A \subset K^{\mathrm{m}}_{\mathbf{T}_V}/K_{\mathbf{T}_V}$ be the image of the map $x \mapsto x^A$ on $K^{\mathrm{m}}_{\mathbf{T}_V}/K_{\mathbf{T}_V}$.

Sublemma 9.2.4. *The orbit $\Theta_A V_{\mathbf{H}}$ is contained in* $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_{\mathbf{H}} \cap \pi^{-1}\pi(V_{\mathbf{H}})$.

Proof. Let $f \colon \mathrm{Sh}_{K_{V_{\mathbf{H}}}}(\mathbf{H}_V, X_V)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$ be the morphism defining $V_{\mathbf{H}}$, where $K_{V_{\mathbf{H}}} := K_{\mathbf{H}} \cap \mathbf{H}_V(\mathbf{A}_{\mathrm{f}})$. It is naturally $K^{\mathrm{m}}_{\mathbf{T}_V}/K_{\mathbf{T}_V}$-equivariant and defined over $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$.

Let $\mathcal{V}$ be the component of $\mathrm{Sh}_{K_{V_{\mathbf{H}}}}(\mathbf{H}_V, X_V)$ such that $V_{\mathbf{H}} = f(\mathcal{V})$. The $K^{\mathrm{m}}_{\mathbf{T}_V}/K_{\mathbf{T}_V}$-equivariance of f implies:

$$\forall\, \theta \in \Theta_A, \quad f(\theta \cdot \mathcal{V}) = \theta \cdot V_{\mathbf{H}} \ .$$

On the other hand, by the first claim of [39, lemma 2.15] applied to the Shimura datum $(\mathbf{H}_V, X_V)$, we see that

$$\theta \cdot \mathcal{V} = \sigma \mathcal{V}$$

for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. Hence

$$\theta \cdot V_{\mathbf{H}} = f(\theta \cdot \mathcal{V}) = f(\sigma \mathcal{V}) = \sigma f(\mathcal{V}) = \sigma V_{\mathbf{H}} \ .$$

Hence the result. □

Sublemma 9.2.5. *There exists a geometrically irreducible subvariety Y' of $Y_{\mathbf{H}}$ defined over $\overline{\mathbb{Q}}$ and containing $V_{\mathbf{H}}$ such that the following holds:*

(1) $\deg_{L_{K_{\mathbf{H}}}} \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y' \leq (\deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}})^{2^{r_{\mathbf{H}}}}$.

(2) The variety $\Theta_A Y'$ is contained in $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y'$.

Proof. Let Y_1 be a geometrically irreducible component of $Y_{\mathbf{H}}$ containing $V_{\mathbf{H}}$.

If $\Theta_A Y_1$ is contained in $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y_1$, then take $Y' = Y_1$. As $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y'$ is contained in $Y_{\mathbf{H}}$, the condition (1) is obviously satisfied.

Otherwise there exists a $\theta \in \Theta_A$ such that θY_1 is not a $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-conjugate of Y_1. Let $\overline{Y_1} = \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y_1$. Recall that the action of Θ_A commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$. In particular the intersection $\overline{Y_1} \cap \theta \overline{Y_1}$ is proper. Moreover, as $\theta V_{\mathbf{H}}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/E_V)$-conjugate of $V_{\mathbf{H}}$ by sublemma 9.2.4, we obtain that:

$$V_{\mathbf{H}} \subset \overline{Y_1} \cap \theta \cdot \overline{Y_1} \ .$$

Let Y_2 be a geometrically irreducible component of $\overline{Y_1} \cap \theta\overline{Y_1}$ containing $V_{\mathbf{H}}$ and let $\overline{Y_2} = \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y_2$. We have

$$\overline{Y_2} \subset \overline{Y_1} \cap \theta\overline{Y_1} \subset Y_{\mathbf{H}} \cap \theta Y_{\mathbf{H}} \quad .$$

It follows that

$$\deg_{L_{K_{\mathbf{H}}}} \overline{Y_2} \leq (\deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}})^2 \quad .$$

On the other hand:

$$\deg_{L_{K_{\mathbf{H}}}} \mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot V_{\mathbf{H}} > C(N)\alpha_V \beta_V^N > (\deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}})^{2^{r_{\mathbf{H}}}}$$

where the first left inequality follows from theorem [39, Theorem 2.19] applied with $Y = V_{\mathbf{H}}$ and the second one from (9.7). These inequalities show that $\dim Y_2 > \dim V_{\mathbf{H}}$.

We now iterate the process replacing Y_1 by Y_2. As $\dim V_{\mathbf{H}} < \dim Y_2 < \dim Y_1 = \dim Y_{\mathbf{H}}$ after at most $r_{\mathbf{H}} = \dim Y_{\mathbf{H}} - \dim V_{\mathbf{H}}$ iterations we construct the variety Y' satisfying the required conditions. □

We now finish the proof of lemma 9.2.3. Condition (2) of sublemma 9.2.5 enables us to apply theorem [39, theor.2.19]:

$$\deg_{L_{K_{\mathbf{H}}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y') \geq C(N)\alpha_V \beta_V^N \quad .$$

By sublemma 9.2.5 (1), and inequality (9.7) we have

$$\deg_{L_{K_{\mathbf{H}}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_V) \cdot Y') \leq (\deg_{L_{K_{\mathbf{H}}}} Y_{\mathbf{H}})^{2^{r_{\mathbf{H}}}} < C(N)\alpha_V \beta_V^N \quad .$$

These inequalities yield a contradiction. This finishes the proof of lemma 9.2.3. □

Let us now finish the proof of theorem 9.2.1. As $r_{\mathbf{H}} < r$ by induction hypothesis we can apply theorem 9.2.1 to $\mathbf{H}$, $X_{\mathbf{H}}$, $X_{\mathbf{H}}^+$, $K_{\mathbf{H}}$, $F_{\mathbf{H}}$, W, $V_{\mathbf{H}}$ and $Y_{\mathbf{H}}$. Thus $Y_{\mathbf{H}}$ contains a special subvariety $V'_{\mathbf{H}}$ which contains $V_{\mathbf{H}}$ properly. This implies that Z contains a special subvariety V' which contains V properly. This finishes the proof of theorem 9.2.1 by induction on r. □

10. The choice of a prime l.

In this section we complete the proof of the theorem 3.2.1, and thus also of the main theorem 1.2.2, using the theorem 9.2.1. The choice of a prime l satisfying the conditions of the theorem 9.2.1 will be made possible by the effective Chebotarev theorem, which we now recall.

10.1. **Effective Chebotarev.**

Definition 10.1.1. *Let L be a number field of degree n_L and absolute discriminant d_L. Let x be a positive real number. We denote by $\pi_L(x)$ the number of primes p such that p is split in L and $p \leq x$.*

Proposition 10.1.2. *Assume the Generalized Riemann Hypothesis (GRH). There exists a constant A such that the following holds. For any number field L Galois over $\mathbb{Q}$ and for any $x > \max(A, 2\log(d_L)^2(\log(\log(d_L)))^2)$ we have*

$$\pi_L(x) \geq \frac{x}{3n_L \log(x)} \quad .$$

Furthermore, if we consider number fields such that d_L is constant, then the assumption of the GRH can be dropped.

Proof. The first statement (assuming the GRH) is proved in the Appendix N of [17] and the second is a direct consequence of the classical Chebotarev theorem. □

10.2. **Proof of the theorem 3.2.1.**

Proof. Let $\mathbf{G}$, X, X^+, R, K and Z be as in theorem 3.2.1. Thus $(\mathbf{G}, X)$ is a Shimura datum with $\mathbf{G}$ semisimple of adjoint type, X^+ is a connected component of X, the positive integer R is as in definition 2.5.1, the group $K = \prod_{p \text{ prime}} K_p$ is a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$ and $Z \subset S_K(\mathbf{G}, X)_\mathbb{C}$ is a Hodge generic geometrically irreducible subvariety containing a Zariski dense set $\mathbf{\Sigma}$ of special subvarieties, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ such that for any modification Σ' of Σ the set $\{\alpha_V \beta_V,\ V \in \Sigma'\}$ is unbounded. We want to show, under each of the two assumptions (1) or (2) of theorem 3.2.1 separately, that for every V in Σ there exists a special subvariety V' such that $V \subsetneq V' \subset Z$ (possibly after replacing Σ by a modification).

From now on, we fix a faithful rational representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that K is contained in $\mathbf{GL}_n(\hat{\mathbb{Z}})$. In the case of the assumption (2) in theorem 3.2.1, we take for ρ the representation which has the property that the centres $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class (possibly replacing K by $K \cap \mathbf{GL}_n(\hat{\mathbb{Z}})$) as V ranges through Σ.

Lemma 10.2.1. *Without any loss of generality we can assume that:*

1. *The group K_3 is contained in the congruence subgroup of level three (with respect to the faithful representation ρ).*
2. *After possibly replacing Σ by a modification, Σ consists of special but not strongly special subvarieties.*

Proof. To fulfill the first condition, let $\widetilde{K} = \widetilde{K_3} \times \prod_{p \neq 3} K_p$ be a finite index subgroup of K with $\widetilde{K_3}$ contained in the congruence subgroup of level three (with respect to the faithful representation ρ). Let $\widetilde{Z}$ be an irreducible component of the preimage of $f^{-1}(Z)$, where $f : S_{\widetilde{K}}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow S_K(\mathbf{G}, X)_{\mathbb{C}}$ is the canonical finite morphism. Then $\widetilde{Z}$ contains a Zariski-dense set $\widetilde{\Sigma}$, which is a union of special subvarieties V, $V \in \widetilde{\Sigma}$, all of the same dimension $n(\Sigma)$: $\widetilde{\Sigma}$ is the set of all irreducible components $\widetilde{V}$ of $f^{-1}(V)$ contained in $\widetilde{Z}$ as V ranges through Σ. Notice that for any modification $\widetilde{\Sigma}'$ of $\widetilde{\Sigma}$ the set $\{\alpha_{V'}\beta_{V'},\ V' \in \widetilde{\Sigma}'\}$ is unbounded: $\beta_{V'} = \beta_{f(V')}$ and $\alpha_{V'}$ is equal to $\alpha_{f(V')}$ up to a factor independent of V'. Thus $\widetilde{Z}$ satisfies the assumptions of theorem 3.2.1. As a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is special if and only if some (equivalently any) irreducible component of its preimage by f is special, theorem 3.2.1 for $\widetilde{Z}$ implies theorem 3.2.1 for Z.

For the second condition : otherwise there is a modification Σ' of Σ consisting only of strongly special subvarieties. Contradiction with the assumption that the set $\{\alpha_V \beta_V, V \in \Sigma'\}$ is unbounded. □

Let B be the constant depending on $\mathbf{G}$, X and R given by the theorem 2.4.4. Fix N a positive integer and let $C(N)$ be the real number depending on R and N given by the theorem 2.4.4. Let k the constant depending on the data $\mathbf{G}$, X, X^+, K defined in the theorem 8.1. Let f be the constant depending on the data $\mathbf{G}$, X, X^+ defined in the theorem 8.1.

Let $F_{\mathbf{G}}$ be the reflex field $E(\mathbf{G}, X)$. As Z contains a Zariski dense set of special subvarieties, Z is defined over $\overline{\mathbb{Q}}$. We replace Z by the union of its conjugates under $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{\mathbf{G}})$. Thus Z is now an $F_{\mathbf{G}}$-irreducible $F_{\mathbf{G}}$-subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.

For all primes l larger than a constant C, the group K_l is a hyperspecial maximal compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$ (cf. [35, 3.9.1]) and furthermore $K_l = \mathbf{G}(\mathbb{Z}_l)$, where the $\mathbb{Z}$-structure on $\mathbf{G}$ is defined by taking the Zariski closure in $\mathbf{GL}_{n,\mathbb{Z}}$ via ρ.

Proposition 10.2.2. *To prove theorem 3.2.1 it is enough to show that for any V in Σ (up to a modification), there exists a prime $l > C$ satisfying the following conditions:*

(1) the prime l splits $\mathbf{T}_V$.
(2) $\mathbf{T}_{V,\mathbb{F}_l}$ is a torus.
(3) $l^{(k+2f)\cdot 2^r} \cdot (\deg_{L_K} Z)^{2^r} < C(N)\alpha_V \beta_V^N$, where $r = \dim Z - \dim V$.

Proof. Let V be an element of Σ.

Let us check that the conditions of the theorem 9.2.1 are satisfied for $\mathbf{G} = \mathbf{G}'$, X, X^+, K, $F_{\mathbf{G}}$, $W = V$ and Z:

- condition (1) of theorem 9.2.1 is automatically satisfied because $\mathbf{G} = \mathbf{G}'$ and Σ consists of special but non strongly special subvarieties of $\mathbf{G}$ by lemma 10.2.1(2).
- the conditions of proposition 10.2.2 immediately imply that the condition (2) of theorem 9.2.1 is satisfied.

As the set $\{\alpha_V\beta_V, V \in \Sigma\}$ is unbounded the difference $r := \dim Z - n(\Sigma)$ is necessarily positive. We now apply the theorem 9.2.1: for any V in Σ there exists a special subvariety V' of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ such that $V \subsetneq V' \subset Z$. □

Therefore, in order to prove theorem 3.2.1, it remains to check the existence of the prime l satisfying the conditions of proposition 10.2.2. We first prove the following.

Proposition 10.2.3. *For every $D > 0$, $\epsilon > 0$ and every integer $m \geq \max(\epsilon, 6)$, there exists an integer M such that (up to a modification of Σ): for every V in Σ with $\alpha_V\beta_V$ larger than M there exists a prime $l > C$ satisfying the following conditions:*

(1) the prime l splits $\mathbf{T}_V$.
(2) $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a torus.
(3) $l < D\alpha_V^{\epsilon}\beta_V^{m}$.

Proof. For V in Σ recall that n_V is the degree of the splitting field L_V of $\mathbf{C}_V = \mathbf{H}_V/\mathbf{H}_V^{\mathrm{der}}$ over $\mathbb{Q}$. By the proof of [39, lemma 2.5] the number n_V is bounded above by some positive integer n as V ranges through Σ.

Fix $D > 0$, $\epsilon > 0$ and $m \geq \max(\epsilon, 6)$. For V in Σ, let

$$x_V := D\alpha_V^{\epsilon}\beta_V^{m} \quad .$$

Lemma 10.2.4. *Up to a modification of Σ the following inequality holds for every V in Σ:*

$$\pi_{L_V}(x_V) \geq \frac{D^{\frac{1}{2}}}{3n} \cdot \alpha_V^{\frac{\epsilon}{2}} \cdot \beta_V^{\frac{m}{2}} \quad . \tag{10.1}$$

Proof. As we are assuming either the GRH, or that the connected centres $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class under ρ as V ranges through Σ, in which case d_{L_V} is independent of V, we can apply proposition 10.1.2:

$$\pi_{L_V}(x_V) \geq \frac{x_V}{3n\log(x_V)}$$

provided that x_V is larger than some absolute constant and β_V^3. Notice moreover that if $x_V \geq 4$ then $\sqrt{x_V} \geq \log(x_V)$.

It follows that

$$\pi_{L_V}(x_V) \geq \frac{\sqrt{x_V}}{3n} = \frac{D^{\frac{1}{2}}}{3n} \cdot \alpha_V^{\frac{\epsilon}{2}} \cdot \beta_V^{\frac{m}{2}} \quad ,$$

hence the result, provided that x_V is larger than some absolute constant and β_V^3 for all V in Σ.

It remains to show that (up to a modification of Σ) the quantity x_V is larger than any given constant and β_V^3 for all $V \in \Sigma$. As $\alpha_V\beta_V$ is unbounded as V ranges through any modification of Σ, we can assume up to a modification of Σ that β_V is non-zero for all V in Σ. As β_V is the logarithm of a positive integer there exists $b > 0$ such that $\beta_V > b$ for all V in Σ. Then up to a modification of Σ:

- either the inequality $\beta_V \leq 1$ holds for all $V \in \Sigma$. In this case the assumption that $\alpha_V\beta_V$ is unbounded as V ranges through any modification of Σ implies that α_V can be ensured to be larger than any given constant (hence also larger than any given constant and β_V^3) for all V in Σ and we are done.

- or the inequality $\beta_V > 1$ holds for all $V \in \Sigma$. On the one hand $m \geq \varepsilon$ hence we have $x_V = D(\alpha_V\beta_V)^{\varepsilon}\beta_V^{m-\varepsilon} \geq D(\alpha_V\beta_V)^{\varepsilon}$. On the other hand as $m \geq 6$ one has $x_V \geq D(\alpha_V\beta_V)^{\inf(\varepsilon,3)}\beta_V^3$. Up to a new modification of Σ we can assume that for all V in Σ the quantity $(\alpha_V\beta_V)^{\inf(\varepsilon,3)}$ is larger than any given constant and we are also done in this case.

This finishes the proof of lemma 10.2.4. $\square$

Let i_V be the number of primes p *unramified* in L_V such that $K^{\mathrm{m}}_{\mathbf{T}_V,p} \neq K_{\mathbf{T}_V,p}$. To prove the proposition 10.2.3 it is enough to show that $\pi_{L_V}(x_V) > \max(C, i_V)$ if $\alpha_V\beta_V$ is large enough. Indeed, this will yield a prime $l > C$ satisfying $l < D\alpha_V^{\epsilon}\beta_V^m$ and such that: l is split in L_V and $K_{\mathbf{T}_V,l} = K^{\mathrm{m}}_{\mathbf{T}_V,l}$. These last two conditions imply that $\mathbf{T}_{V,\mathbb{F}_l}$ is a torus (we refer to the proof of lemma 3.17 of [39] for the proof of this fact).

Lemma 10.2.5. *Let c be the uniform constant from the Proposition 4.3.9 of* [18]. *Then:*

$$\alpha_V \geq (Bc)^{i_V} \cdot i_V! \qquad .$$

Proof. Notice that

$$\alpha_V = \prod_{\substack{p\,\text{prime}\\ K^{\mathrm{m}}_{\mathbf{T}_V,p}\neq K_{\mathbf{T}_V,p}}} \max(1, B\cdot|K^{\mathrm{m}}_{\mathbf{T}_V,p}/K_{\mathbf{T}_V,p}|) \geq \prod_{\substack{p\,\text{prime}\\ p\,\text{unramified in}\,L_V\\ K^{\mathrm{m}}_{\mathbf{T}_V,p}\neq K_{\mathbf{T}_V,p}}} B\cdot|K^{\mathrm{m}}_{\mathbf{T}_V,p}/K_{\mathbf{T}_V,p}| \quad .$$

By proposition 3.15 of [39], for p unramified in L_V and such that $K^{\mathrm{m}}_{\mathbf{T}_V,p} \neq K_{\mathbf{T}_V,p}$ we have $|K^{\mathrm{m}}_{\mathbf{T}_V,p}/K_{\mathbf{T}_V,p}| \geq cp$. Thus

$$\alpha_V \geq (Bc)^{i_V}\cdot\Big(\prod_{\substack{p\,\text{prime}\\ p\,\text{unramified in}\,L_V\\ K^{\mathrm{m}}_{\mathbf{T}_V,p}\neq K_{\mathbf{T}_V,p}}} p\Big) \geq (Bc)^{i_V}\cdot i_V! \qquad ,$$

where we used in the last inequality that the pth prime in $\mathbb{N}$ is at least p.

□

Definition 10.2.6. *Given a positive real number t we denote by Σ_t the set of V in Σ with $i_V > t$.*

To finish the proof of proposition 10.2.3 we proceed by dichotomy:

- Suppose that for any t the set Σ_t is a modification of Σ. In particular the function i_V is unbounded as V ranges through Σ. For simplicity, we let $B' := Bc$. Recall the well-known inequality: for every integer $n > 1$,
$$e \cdot \left(\frac{n}{e}\right)^n < n! < e \cdot n \cdot \left(\frac{n}{e}\right)^n \quad .$$
The lower bound for α_V provided by lemma 10.2.5 gives:
$$\alpha_V > e\left(\frac{B' i_V}{e}\right)^{i_V} > \left(\frac{B' i_V}{e}\right)^{i_V} \quad .$$
Hence:
$$\alpha_V^{\frac{\epsilon}{2}} > \left(\frac{B' i_V}{e}\right)^{\frac{\epsilon i_V}{2}} \quad .$$
For $i_V > \frac{4}{\epsilon}$ we obtain:
$$\alpha_V^{\frac{\epsilon}{2}} > \left(\frac{B' i_V}{e}\right)^{2} \quad .$$
Using the lower bound (10.1) for $\pi_{L_V}(x_V)$, the trivial lower bound $\beta_V \geq 1$ and the fact that $m \geq 6$, we obtain that
$$\pi_{L_V}(x_V) \geq \frac{D^{\frac{1}{2}} B'^2}{3ne^2} \cdot i_V^2 \quad .$$
Hence, whenever
$$i_V > t = \max\left(\frac{3ne^2}{D^{\frac{1}{2}} B'^2}, \frac{4}{\epsilon}, C\right)$$
we get $\pi_{L_V}(x_V) > \max(i_V, C)$. As the set Σ_t is a modification of Σ we get the proposition 10.2.3.
- Otherwise there exists a positive number t such that $\Sigma \setminus \Sigma_t$ is a modification of Σ. Replacing Σ by $\Sigma \setminus \Sigma_t$ we can assume without loss of generality that the function i_V is bounded by t as V ranges through Σ.
We have for any V in Σ,
$$\pi_{L_V}(x_V) > \frac{D^{\frac{1}{2}}}{3n} \cdot \alpha_V^{\frac{\epsilon}{2}} \cdot \beta_V^{\frac{m}{2}} \geq \frac{D^{\frac{1}{2}}}{3n} (\alpha_V \beta_V)^{\frac{\epsilon}{2}} \quad .$$

Then $\pi_{L_V}(x_V) > \max(i_V, C)$ as soon as

$$\frac{D^{\frac{1}{2}}}{3n}(\alpha_V\beta_V)^{\frac{\epsilon}{2}} > \max(t, C) \quad .$$

Hence we can take $M = \left(\frac{3n\max(t,C)}{D^{1/2}}\right)^{2/\epsilon}$.

$\square$

Let us now finish the proof of theorem 3.2.1 by showing the existence of the prime l satisfying the conditions of proposition 10.2.2. Let $r := \dim Z - n(\Sigma)$. Let N be a positive integer, at least $6(k+2f)\cdot 2^r$ and such that $m := \frac{N}{(k+2f)\cdot 2^r}$ is an integer. Let $\varepsilon < \frac{1}{(k+2f)\cdot 2^r}$, $D = \{\frac{C(N)}{(\deg_{L_K} Z)^{2^r}}\}^{\frac{1}{(k+2f)\cdot 2^r}}$.

Let M be the integer obtained from proposition 10.2.3 applied to ϵ, m and D. Up to a modification of Σ we can assume that any V in Σ satisfies $\alpha_V\beta_V > M$. Thus by proposition 10.2.3 for every $V \in \Sigma$ we can choose a prime $l > C$ such that l splits $\mathbf{T}_V$, the reduction $\mathbf{T}_{V,\mathbb{F}_l}$ is a torus and $l < D\alpha_V^{\epsilon}\beta_V^{m}$: this is proposition 10.2.2. $\square$

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