

BI-ALGEBRAIC GEOMETRY AND THE ANDRÉ-OORT CONJECTURE

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1. INTRODUCTION

Shimura varieties are algebraic varieties of enormous interest. Introduced by Shimura and Deligne in order to generalize the modular curves, they play nowadays a central role in the theory of automorphic forms (Langlands program), the study of Galois representations and in Diophantine geometry. A Shimura variety is a moduli space of mixed Hodge structures of a restricted type. The main examples are the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension g and the universal abelian variety \mathfrak{A}_g above it. The geometry and arithmetic of a Shimura variety are governed by its special points (also called CM points) parametrizing the Hodge structures with complex multiplication, and more generally its special subvarieties parametrizing “non-generic” Hodge structures.

The André-Oort conjecture describes the distribution of special points on a Shimura variety S : any irreducible closed subvariety of S containing a Zariski-dense set of special points ought to be special. It is the analog in a Hodge-theoretic context of the Manin-Mumford conjecture (a theorem of Raynaud [Ray88]) stating that an irreducible subvariety of a complex abelian variety containing a Zariski-dense set of torsion points is the translate of an abelian subvariety by a torsion point. The André-Oort conjecture has been proven for the Shimura variety \mathcal{A}_g (and more generally for mixed Shimura varieties whose pure part is of abelian type) following a strategy proposed by Pila and Zannier and through the work of many authors (see Section 2.5 for details). One goal of this survey paper is to provide an overview of the André-Oort conjecture and the Pila-Zannier strategy for a general Shimura variety, particularizing to \mathcal{A}_g when needed.

A particularly interesting feature of the Pila-Zannier strategy is its understanding of the special subvarieties of a Shimura variety in terms of functional and arithmetic transcendence. Our second goal in this paper is to popularize this idea into a general format, baptized *bi-algebraic geometry*, which unifies many problems in Diophantine geometry but also suggests interesting new questions. In a few words: given S an irreducible algebraic variety over \mathbb{C} one tries to define an algebraic structure (in a sense made precise in Section 4) on the universal cover $\widetilde{S}^{\text{an}}$ of its associated analytic space S^{an} and to study the transcendence properties of the complex analytic uniformization morphism $\pi : \widetilde{S}^{\text{an}} \rightarrow S^{\text{an}}$. On the geometric side one defines the bi-algebraic subvarieties of S by a functional transcendence constraint: these are the irreducible algebraic subvarieties of S that are images of algebraic subvarieties of $\widetilde{S}^{\text{an}}$ (in the sense of Definition 4.3). In many cases of interest there are few positive dimensional bi-algebraic subvarieties, encoding a lot of the geometry of S . If the bi-algebraic structure on S can be defined over the field of algebraic numbers $\overline{\mathbb{Q}}$, this format can be arithmetically enriched by restricting our attention to the $\overline{\mathbb{Q}}$ -bi-algebraic subvarieties. Shimura varieties can be seen as an instance of this format in a Hodge theoretic context. The universal cover $\widetilde{S}^{\text{an}}$ of a connected Shimura variety S is canonically realized as an open subset of a flag variety over \mathbb{Q} parametrizing periods, hence admits a natural $\overline{\mathbb{Q}}$ -bi-algebraic structure. The

$\overline{\mathbb{Q}}$ -bi-algebraic subvarieties of S , defined in terms of transcendence properties of periods, coincide with its special subvarieties, defined in terms of Hodge theory.

This text is organized as follows.

Section 2 introduces the André-Oort conjecture. After presenting the Hodge-theoretic background of the conjecture, we describe its simplest instance when the Shimura variety is \mathbb{C}^2 , introduce the formalism of Shimura varieties using Deligne’s language of Hodge theory (for simplicity we restrict ourselves to the *pure Shimura varieties*) and formulate the general conjecture. We then describe the history and results on the conjecture, and summarize the main steps in the Pila-Zannier approach.

Section 3 describes a general format where a reasonable Manin-Mumford-André-Oort type problem can be formulated: the notion of a *special structure* on a complex algebraic variety S , which axiomatizes the properties of the collection of special subvarieties on a Shimura variety or an abelian variety. We also notice that in all the cases we consider, special structures are related to Kähler geometry through the notion of *weakly special subvarieties*: in the case of semi-abelian varieties or pure Shimura varieties, weakly special subvarieties are exactly the totally geodesic subvarieties for the canonical Kähler metric on S . The special subvarieties of S are precisely the weakly special ones (a purely geometric notion) containing a special point (an arithmetic notion).

Section 4 develops the idea of bi-algebraic geometry, both over \mathbb{C} and $\overline{\mathbb{Q}}$. This idea is illustrated in the case of abelian and Shimura varieties. All the special structures we consider are of bi-algebraic origin. All the special structures we consider are of bi-algebraic origin (see **Section 4.3**), and bi-algebraic subvarieties and weakly special subvarieties coincide. Hence special subvarieties are exactly the bi-algebraic subvarieties containing a smooth special point. In the best cases, the bi-algebraic structure can be enriched over $\overline{\mathbb{Q}}$ (see **Section 4.2**) and the special points are exactly the arithmetic bi-algebraic points (see **Definition 4.12**).

The geometry of non-trivial bi-algebraic structures is governed by a natural heuristic in functional transcendence: given a connected algebraic variety S endowed with a bi-algebraic structure, the Ax-Lindemann principle predicts that the Zariski-closure $\pi(Y)$ of any algebraic subvariety Y of \tilde{S} should be bi-algebraic. In the case of Shimura varieties this conjecture is the main *geometric* step in the Pila-Zannier strategy.

In **Section 5** we turn to the techniques at our disposal for attacking the Ax-Lindemann and the Manin-Mumford-André-Oort problems in the general context of a bi-algebraic structure. Let S be an algebraic variety endowed with a bi-algebraic structure. Whether or not this bi-algebraic structure underlies a special structure on S seems to depend on the existence of a common geometric framework for S and \tilde{S} , more flexible than (semi-)algebraic geometry as the map $\pi : \tilde{S} \rightarrow S$ is far from algebraic, but topologically more constraining than analytic geometry in order to explain the special structure. Such a common framework is reminiscent of Grothendieck’s idea of “tame topology” [**Gro84**, section 5], and is described in model theoretic language as o-minimal geometry. **Section 5** presents a minimal recollection of o-minimal geometry, and state a deep diophantine criterion due to Pila and Wilkie for detecting (positive dimensional) semi-algebraic subsets

of \mathbb{R}^n among subsets definable in an o-minimal structure: if such a subset contains polynomially many (with respect to the height) points of \mathbb{Q}^n then it contains a non-trivial positive dimensional semi-algebraic subset (see [Theorem 5.10](#)).

The next three sections describes the results towards the André-Oort [Conjecture 2.2](#) following the Pila-Zannier strategy.

[Section 6](#) deals with first ingredient: the definability in an o-minimal structure of the uniformization map of a connected Shimura variety (restricted to a suitable fundamental domain), see [Theorem 6.2](#).

Using this result and the Pila-Wilkie theorem, [Section 7](#) sketches the proof of the second ingredient: the Ax-Lindemann [Theorem 4.28](#). While it is known for any Shimura variety, for simplicity we restrict ourselves to pure Shimura varieties.

[Section 8](#) explains the two main results who lead to the proof of the André-Oort conjecture for \mathcal{A}_g . The first one, which is geometric in nature, holds for any Shimura variety and is a consequence of the Ax-Lindemann [Theorem 4.28](#). Let W be a Hodge generic subvariety of a Shimura variety S . Under a mild assumption on W , one shows that the union of *positive dimensional* special subvarieties of S contained in W is not Zariski-dense in W (see [Theorem 8.1](#)). The second one is arithmetic in nature and is known for \mathcal{A}_g . It states that if a subvariety W of \mathcal{A}_g contains a special point of sufficient arithmetic complexity then W contains a positive dimensional special subvariety of \mathcal{A}_g . The proof uses the Ax-Lindemann [Theorem 4.28](#), the Pila-Wilkie counting theorem [Theorem 5.10](#) and a suitable lower bound for the size of Galois orbits of special points.

[Section 9](#) describes the results on the lower bounds for the size of Galois orbits of special points of \mathcal{A}_g .

In the extra [Section 10](#), we present the work of Orr [[Orr15](#)] in the direction of the André-Pink conjecture.

This text is largely inspired by the course on the André-Oort conjecture given by E. Ullmo at IHES in Spring 2016. For other surveys on the André-Oort conjecture following the Pila-Zannier method, we refer to [[Daw16](#)] for a more elementary introduction, to [[Sca12](#)] and [[Sca16](#)] for the description of the method in the geometrically easier case of $S = \mathbb{C}^n \times \mathbf{G}_m^k$ but with an expanded treatment of the o-minimal background.

Notations: In this paper, an algebraic variety is a separated reduced scheme of finite type over \mathbb{C} . Algebraic subvarieties are assumed to be closed, unless otherwise stated.

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of \mathbb{Q} in \mathbb{C} .

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2. THE ANDRÉ-OORT CONJECTURE

2.1. The Hodge theoretic motivation. Let us start by explaining the algebro-geometric problem underlying the André-Oort conjecture. Let $f : \mathcal{X} \rightarrow S$ be a smooth

family of algebraic varieties over a quasi-projective smooth base S . Can we describe the locus of points $s \in S$ where the fiber \mathcal{X}_s (and its Cartesian powers) contain more algebraic cycles than the very general fiber (and its Cartesian powers)? We work over \mathbb{C} and consider the Hodge incarnation of this problem. Let $\mathbb{V} \rightarrow S$ be an admissible variation of mixed \mathbb{Z} -Hodge structures on the complex quasi-projective smooth base S (cf. [PS08, Def. 14.49]). In particular \mathbb{V} is a \mathbb{Z} -local system on S such that each fiber \mathbb{V}_s , $s \in S$, carries a graded-polarized mixed Hodge structure. This is an abstraction of the geometric case corresponding to $\mathbb{V} = (R^p f_* \mathbb{Z})_{\text{prim}}$ (for some $p > 0$) for f as above. One wants to understand the Hodge locus $\text{HL}(S, \mathbb{V}) \subset S$, namely the subset of points s in S for which exceptional Hodge classes of type $(0, 0)$ do occur in some $\mathbb{V}_{\mathbb{Q}, s}^a \otimes (\mathbb{V}_{\mathbb{Q}, s}^{\vee})^b$, where $\mathbb{V}_{\mathbb{Q}, s}^{\vee}$ denotes the \mathbb{Q} -Hodge structure dual to $\mathbb{V}_{\mathbb{Q}, s}$.

The Tannakian formalism available for Hodge structures is particularly useful for describing $\text{HL}(S, \mathbb{V})$. Recall that for every $s \in S$, the Mumford-Tate group \mathbf{MT}_s of the Hodge structure $\mathbb{V}_{\mathbb{Q}, s}$ is the Tannakian group of the subcategory $\langle \mathbb{V}_{\mathbb{Q}, s}^{\otimes} \rangle$ of the Tannakian category of pure Hodge structures tensorially generated by $\mathbb{V}_{\mathbb{Q}, s}$ and $\mathbb{V}_{\mathbb{Q}, s}^{\vee}$. Equivalently, the group \mathbf{MT}_s is the stabiliser of the Hodge classes of type $(0, 0)$ in the rational Hodge structures tensorially generated by $\mathbb{V}_{\mathbb{Q}, s}$ and its dual. A point $s \in S$ is said to be Hodge generic if \mathbf{MT}_s is maximal when s varies in its connected component. If S is connected, two Hodge generic points of S have the same Mumford-Tate group, called the generic Mumford-Tate group $\mathbf{MT}_{S, \text{gen}}$ of (S, \mathbb{V}) . The Hodge locus $\text{HL}(S, \mathbb{V})$ is the subset of points of S which are not Hodge generic.

A fundamental result of Cattani-Deligne-Kaplan [CDK95] states that $\text{HL}(S, \mathbb{V})$ is a countable union of closed irreducible algebraic subvarieties of S , each not contained in the union of the others. The irreducible components of the intersections of these algebraic subvarieties are called *special subvarieties* of (S, \mathbb{V}) . Hodge subvarieties of dimension zero are called *special points* of (S, \mathbb{V}) . We would like to understand the distribution of special points in S .

2.2. The André-Oort conjecture for \mathbb{C}^2 . The André-Oort conjecture answers this question when S is a Shimura variety. We start with its most explicit incarnation.

The simplest Shimura variety is the classical modular curve $Y(1)$. As a complex analytic space it is the quotient $Y(1) := \mathbf{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$, where $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ is the Poincaré upper-half plane and the group $\mathbf{SL}_2(\mathbb{Z})$ acts on \mathcal{H} by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

The space $Y(1)$ can also be interpreted as the set of complex elliptic curves up to isomorphism:

$$\mathbf{SL}_2(\mathbb{Z}) \backslash \mathcal{H} \longrightarrow \{E/\mathbb{C}\} / \cong, \quad \tau \mapsto [E_{\tau} := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})].$$

As complex elliptic curves up to isomorphism are classified by their j -invariant, the quotient map $\pi : \mathcal{H} \longrightarrow Y(1)$ identifies with the holomorphic j -map $j : \mathcal{H} \longrightarrow \mathbb{C}$ given by

$$\tau \mapsto j(E_{\tau}) = q^{-1} + 744 + 196884q + \cdots, \quad q = e^{2\pi i \tau}.$$

Hence the quotient $Y(1) \stackrel{j}{\simeq} \mathbb{C}$ is the coarse moduli space of complex elliptic curves associated to the Deligne-Mumford stack $\mathcal{M}_{1,1}$ of elliptic curves. As such it is an algebraic variety naturally defined over \mathbb{Q} .

The universal family of elliptic curves over $\mathcal{M}_{1,1}$ defines a Hodge locus in $Y(1)$, i.e. special points. For $\tau \in \mathcal{H}$, $\text{End}(E_\tau) = \{z \in \mathbb{C} : z \cdot (\mathbb{Z}\tau + \mathbb{Z}) \subset \mathbb{Z}\tau + \mathbb{Z}\}$. Hence $\text{End}(E_\tau) = \mathbb{Z}$ if $\dim_{\mathbb{Q}} \mathbb{Q}(\tau) \neq 2$ and $\text{End}(E_\tau)$ is an order in $\mathbb{Q}(\tau)$ if $\dim_{\mathbb{Q}} \mathbb{Q}(\tau) = 2$, in which case E_τ is a CM-elliptic curve. It follows easily that the Mumford-Tate group at $j(\tau)$ is $\mathbf{GL}(2, \mathbb{Q})$ in the first case, while it is $\text{Res}_{\mathbb{Q}(\tau)/\mathbb{Q}} \mathbf{G}_m$ in the second. Hence special points (also called CM-points) in \mathbb{C} correspond to imaginary quadratic τ 's in \mathcal{H} , in particular they are dense (even for the analytic topology) in \mathbb{C} .

Let us now consider $Y(1)^2 \simeq \mathbb{C}^2$ as the moduli space of pairs of elliptic curves. Once more the Hodge locus for this family can be explicitly described:

- a point $x = (x_1, x_2) \in \mathbb{C}^2$ is special if both $x_1 \in \mathbb{C}$ and $x_2 \in \mathbb{C}$ are special.
- a special curve is either a line $\{x_1\} \times \mathbb{C}$ with x_1 special, a line $\mathbb{C} \times \{x_2\}$ with x_2 special, or the image T_n in \mathbb{C}^2 of the modular curve $Y_0(n)$ parametrizing isogenies $\mathbb{Z}/n\mathbb{Z} \hookrightarrow E_1 \rightarrow E_2$ between two elliptic curves. The curve T_n is obtained from $Y_0(n)$ by forgetting the isogeny (an equivalent definition of T_n is given below).

Each of these special curves contains a dense set of special points. Conversely André [An89] conjectured:

Conjecture 2.1. *Let $\Sigma \subset \mathbb{C}^2$ be a set of special points, and let Z be an irreducible component of the its Zariski-closure $\overline{\Sigma}^{\text{Zar}}$. Then Z is one of the following:*

- (1) a special point,
- (2) $\{x_1\} \times \mathbb{C}$ with x_1 special,
- (3) $\mathbb{C} \times \{x_2\}$ with x_2 special,
- (4) the image T_n (a Hecke correspondence) of

$$t_n: \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}^2, \quad \tau \mapsto (\tau, n\tau) \mapsto (j(\tau), j(n\tau))$$

for some $n \in \mathbb{Z}_{\geq 1}$,

- (5) \mathbb{C}^2 itself.

Conjecture 2.1 was proven by Edixhoven [Ed98] under the Generalized Riemann Hypothesis (GRH) and by André [An98] unconditionally.

2.3. The Conjecture. We turn to the general case. Informally, a pure Shimura variety S (resp. a mixed Shimura variety) is a complex quasi-projective moduli space of pure polarized (resp. mixed graded-polarized) Hodge structures with additional data, such that the universal family above S defines an admissible variation \mathbb{V} of (mixed) Hodge structure over S . As explained by Deligne [De79] this restricts severely the possible types of Hodge structures we can consider. The prototype of a pure Shimura variety is the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension g , the variation \mathbb{V} over \mathcal{A}_g is the Hodge incarnation $R^1 f_* \mathbb{Z}$ of the universal abelian variety $f: \mathfrak{A}_g \rightarrow \mathcal{A}_g$. An example of mixed Shimura variety to keep in mind is \mathfrak{A}_g , the variation \mathbb{V} over \mathfrak{A}_g is the Hodge incarnation of the universal semi-abelian variety over \mathfrak{A}_g .

As in **Section 2.1** the variation \mathbb{V} over S defines special subvarieties in S . A special point of \mathcal{A}_g , also called a CM-point, corresponds to an abelian variety with complex multiplication (CM). A special point of \mathfrak{A}_g is a torsion point on a CM-abelian variety.

A crucial feature of Shimura varieties is their purely group-theoretic description: any Shimura variety S is defined thanks to a Shimura datum (\mathbf{G}, X) , where \mathbf{G} is a connected linear algebraic group over \mathbb{Q} and X is a certain homogeneous space under a subgroup of $\mathbf{G}(\mathbb{C})$. Special subvarieties of S also have a purely group theoretic description: they are precisely the images of the natural morphisms between Shimura varieties. In the next subsection we review this formalism for pure Shimura varieties.

It enables to show first that any Shimura variety S contains one special point, then that any special subvariety of S contains a dense (even for the Archimedean topology) set of special points (see [Lemma 2.5](#)). The André-Oort conjecture is the converse statement:

Conjecture 2.2 (André-Oort). *Let Z be an irreducible subvariety of a mixed Shimura variety S . If Z contains a Zariski-dense set of special points then Z is a special subvariety of S .*

2.4. Pure Shimura varieties and their special subvarieties. This section provides the precise definitions we need for pure Shimura varieties. More detailed references are [\[De71\]](#), [\[De79\]](#), [\[Mi05\]](#). The interested reader will find an introduction to mixed Shimura varieties in [\[Pink05\]](#) and the full theory in [\[Pink89\]](#).

Recall that a pure \mathbb{Q} -Hodge structure on a \mathbb{Q} -vector space V is a linear decomposition $V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$. Equivalently it is a morphism of real algebraic groups $h: \mathbf{S} \rightarrow \mathbf{GL}(V_{\mathbb{R}})$, where $\mathbf{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_{\mathbf{m},\mathbb{C}}$ denotes the Deligne's torus (hence $\mathbf{S}(\mathbb{R}) = \mathbb{C}^*$). The Mumford-Tate group $\mathbf{MT}(h)$ we defined in [Section 2.1](#) is equivalently the smallest algebraic \mathbb{Q} -subgroup \mathbf{H} of $\mathbf{GL}(V)$ such that h factors through $\mathbf{H}_{\mathbb{R}}$. It is a reductive group if V is assumed to be polarized.

A *Shimura datum* is a pair (\mathbf{G}, X) , with \mathbf{G} a linear connected reductive group over \mathbb{Q} and X a $\mathbf{G}(\mathbb{R})$ -conjugacy class of a morphism of real algebraic groups $h \in \text{Hom}(\mathbf{S}, \mathbf{G}_{\mathbb{R}})$, satisfying the “Deligne’s conditions” [\[De79, 1.1.13\]](#):

- (D1) The Hodge structure on the Lie algebra \mathfrak{g} defined by $\text{Ad} \circ h$ has Hodge types $(-1, 1)$, $(0, 0)$ and $(1, -1)$ only.
- (D2) The conjugation by $h(i)$ defines a Cartan involution of the group of real points $\mathbf{G}^{\text{ad}}(\mathbb{R})$ of the adjoint group \mathbf{G}^{ad} : the subgroup $\{g \in \mathbf{G}^{\text{ad}}(\mathbb{C}), \quad h(i)^{-1} \bar{g} h(i) = g\}$ of $\mathbf{G}^{\text{ad}}(\mathbb{C})$ is compact.
- (D3) for every simple factor \mathbf{H} of \mathbf{G} , the composition of $h: \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ with the projection $\mathbf{G}_{\mathbb{R}} \rightarrow \mathbf{H}_{\mathbb{R}}$ is non-trivial.

These conditions imply, in particular, that the connected components of X are Hermitian symmetric domains. Any Hermitian symmetric domain can be obtained in this way. A morphism of Shimura data from (\mathbf{G}_1, X_1) to (\mathbf{G}_2, X_2) is a \mathbb{Q} -morphism $f: \mathbf{G}_1 \rightarrow \mathbf{G}_2$ mapping X_1 to X_2 .

Definition 2.3. *Let (\mathbf{G}, X) be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ (where \mathbb{A}_f denotes the ring of finite adèles of \mathbb{Q}). The Shimura variety $\text{Sh}_K(\mathbf{G}, X)$ is the complex analytic space $\mathbf{G}(\mathbb{Q}) \backslash (X \times \mathbf{G}(\mathbb{A}_f)/K)$, where $\mathbf{G}(\mathbb{Q})$ acts diagonally on $X \times \mathbf{G}(\mathbb{A}_f)/K$.*

Proposition 2.4. *Let $\mathbf{G}(\mathbb{R})_+$ be the stabilizer in $\mathbf{G}(\mathbb{R})$ of a connected component X^+ of X and $\mathbf{G}(\mathbb{Q})_+ := \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q})$. The class group $\mathcal{C} := \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}_f)/K$ is finite*

and one has the decomposition

$$(2.1) \quad \mathrm{Sh}_K(\mathbf{G}, X) = \coprod_{g \in \mathcal{C}} \Gamma_g \backslash X^+ ,$$

where Γ_g denotes the congruence arithmetic lattice $gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ of $\mathbf{G}(\mathbb{R})_+$.

Each $\Gamma_g \backslash X^+$ has finite volume for the natural (up to a non-zero multiple scalar) $\mathbf{G}(\mathbb{R})_+$ -invariant measure on the Hermitian symmetric space X^+ . It follows from results of Baily and Borel [BB66] that each $\Gamma_g \backslash X^+$ has a natural structure of complex quasi-projective variety, hence also $\mathrm{Sh}_K(G, X)$. Moreover the natural analytic morphism $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1) \rightarrow \mathrm{Sh}_{K_2}(\mathbf{G}_2, X_2)$ deduced from a morphism of Shimura data $f : (\mathbf{G}_1, X_1) \rightarrow (\mathbf{G}_2, X_2)$ mapping a compact open subgroup $K_1 \subset \mathbf{G}_1(\mathbb{A}_f)$ into $K_2 \subset \mathbf{G}_2(\mathbb{A}_f)$ is naturally algebraic.

If Γ_g has no torsion then the algebraic variety $\Gamma_g \backslash X^+$ is smooth. Usually we work with a stronger notion of *neat compact open subgroup* $K \subset \mathbf{G}(\mathbb{A}_f)$, in which case $\mathrm{Sh}_K(\mathbf{G}, X)$ is smooth.

The quotient $S = \Gamma_e \backslash X^+$ is called the connected Shimura variety associated to the Shimura datum (\mathbf{G}, X) , the connected component X^+ of X and the compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$.

The projective limit $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} = \lim_K \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a \mathbb{C} -scheme on which $\mathbf{G}(\mathbb{A}_f)$ acts continuously by multiplication on the right. The multiplication by $g \in \mathbf{G}(\mathbb{A}_f)$ on $\mathrm{Sh}(\mathbf{G}, X)$ induces an algebraic correspondence T_g on $\mathrm{Sh}_K(G, X)$, called a Hecke correspondence.

Let $\rho : \mathbf{G} \rightarrow \mathbf{GL}(V)$ be a rational representation of \mathbf{G} . Choose a \mathbb{Z} -structure $V_{\mathbb{Z}}$ on V such that $\rho(K) \subset \mathbf{GL}(V_{\mathbb{Z}})$. Every point $x \in X$ defines a polarized \mathbb{Z} -Hodge structure

$$\rho \circ x : \mathbf{S} \xrightarrow{x} \mathbf{G}_{\mathbb{R}} \xrightarrow{\rho} \mathbf{GL}(V_{\mathbb{R}})$$

on $V_{\mathbb{Z}}$. These $\rho \circ x$, $x \in X$, aggregate to form a polarized variation of \mathbb{Z} -Hodge structure \mathbb{V}_{ρ} on $\mathrm{Sh}_K(\mathbf{G}, X)$. The collection of special subvarieties on $\mathrm{Sh}_K(\mathbf{G}, X)$ associated with \mathbb{V}_{ρ} is shown to be independent of the choice of the faithful representation ρ and has a purely group-theoretic description: a subvariety $V \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is *special* if and only if there is a Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$, a morphism of Shimura data $f : (\mathbf{H}, X_{\mathbf{H}}) \rightarrow (\mathbf{G}, X)$ and an element $g \in \mathbf{G}(\mathbb{A}_f)$ such that V is an irreducible component of the image of the Hecke correspondence $\mathrm{Sh}(\mathbf{H}, X_{\mathbf{H}}) \xrightarrow{\mathrm{Sh}(f)} \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{-g} \mathrm{Sh}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$.

It can also be shown that the Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$ can be chosen in such a way that $\mathbf{H} \subset \mathbf{G}$ is the generic Mumford-Tate group on $X_{\mathbf{H}}$. A *special* point is a special subvariety of dimension zero. One sees that a point $[x, gK] \in \mathrm{Sh}_K(\mathbf{G}, X)$ (where $x \in X$ and $g \in \mathbf{G}(\mathbb{A}_f)$) is *special* if and only if the group $\mathbf{MT}(x)$ is commutative (in which case $\mathbf{MT}(x)$ is a torus).

Lemma 2.5. *Given a special subvariety S of $\mathrm{Sh}_K(\mathbf{G}, X)$, the set of special points of $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$ contained in S is dense in V for the strong (and in particular for the Zariski) topology.*

Idea of proof. As Hecke correspondences map special point to special points, it is equivalent to proving that any Shimura variety contains a dense set of special points. One first shows that every connected component of $\mathrm{Sh}_K(\mathbf{G}, X)$ contains *one* special point (we

follow [Mi05, Lemma 13.3]). Let $[x, gK]$ be a point of $\text{Sh}_K(\mathbf{G}, X)$, where $x : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ is a point of X . Let $\mathbf{T}_{\mathbb{R}} \subset \mathbf{G}_{\mathbb{R}}$ be a maximal torus containing $x(\mathbf{S})$. Then $\mathbf{T}_{\mathbb{R}}$ is the centralizer of any regular element λ of the Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of $\mathbf{T}_{\mathbb{R}}$. If $\lambda_0 \in \mathbf{G}(\mathbb{Q})$ is chosen sufficiently close to λ , it is still regular hence its centralizer \mathbf{T}_0 in \mathbf{G} is a maximal torus in \mathbf{G} . As there are only finitely many conjugacy classes of maximal real torus in $\mathbf{G}_{\mathbb{R}}$, one can moreover choose λ_0 so that $\mathbf{T}_{0, \mathbb{R}}$ and $\mathbf{T}_{\mathbb{R}}$ are conjugate in $\mathbf{G}_{\mathbb{R}}$: there exists $h \in \mathbf{G}(\mathbb{R})$ close to the identity such that $\mathbf{T}_{0, \mathbb{R}} = h\mathbf{T}_{\mathbb{R}}h^{-1}$. Now $hx := hxh^{-1} : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ has image contained in $\mathbf{T}_{0, \mathbb{R}}$ hence $\mathbf{MT}(hx)$ is commutative and $[hx, gK]$ is special. \square

Example 2.6 (The Siegel modular variety). Let us illustrate the definitions above in the case of \mathcal{A}_g . We refer to [Mi05, section 6] for more details on this example and to [Mi05, section 8] for the more general definition of a Shimura variety of abelian type.

Let g be a positive integer. Let V_{2g} be the \mathbb{Q} -vector space of dimension $2g$ and let $\psi : V_{2g} \otimes V_{2g} \rightarrow \mathbb{Q}$ be a non-degenerate alternating form. Define the reductive \mathbb{Q} -algebraic group

$$\mathbf{GSp}_{2g} := \{h \in \mathbf{GL}(V_{2g}) \mid \psi(hv, hv') = \nu(h)\psi(v, v') \text{ for some } \nu(h) \in \mathbf{G}_m\},$$

and let \mathcal{H}_g be the set of all homomorphisms $h : \mathbf{S} \rightarrow \mathbf{GSp}_{2g, \mathbb{R}}$ which induce a pure Hodge structure of type $\{(1, 0); (0, 1)\}$ on V_{2g} and for which either ψ or $-\psi$ is a polarisation. Let $\mathcal{H}_g^+ \subset \mathcal{H}_g$ be the set of all such homomorphism such that ψ defines a polarization. It has a natural structure of complex bounded symmetric domain: the Siegel upper half space.

The pair $(\mathbf{GSp}_{2g}, \mathcal{H}_g)$ is a pure Shimura datum. The Shimura variety $\text{Sh}(\mathbf{GSp}_{2g}, \mathcal{H}_g)$ is usually called the Siegel modular variety attached to (V_{2g}, ψ) . For $K \subset \mathbf{GSp}_{2g}(\mathbb{A}_f)$ a compact open subgroup, the variety $\text{Sh}_K(\mathbf{GSp}_{2g}, \text{cb}\mathcal{H}_g)$ is a moduli space for g -dimensional complex principally polarized abelian varieties with a level K -structure. Let us fix $V_{2g, \mathbb{Z}}$ a \mathbb{Z} -lattice in V_{2g} and assume that ψ is defined over \mathbb{Z} : $\psi : V_{2g, \mathbb{Z}} \otimes V_{2g, \mathbb{Z}} \rightarrow \mathbb{Z}$. For $K_1 = \mathbf{GSp}(V \otimes \hat{\mathbb{Z}}, \psi)$ one obtains a natural isomorphism between $\text{Sh}_{K_1}(\mathbf{GSp}_{2g}, \mathcal{H}_g)(\mathbb{C})$ and $\mathcal{A}_g(\mathbb{C})$.

In the Siegel modular variety \mathcal{A}_g the special points are precisely the CM points, i.e. the points corresponding to principally polarized abelian varieties A of CM type (see [Mum69, paragraph 2]).

2.5. History and results. André [An89, p.215, Problem 1] formulated **Conjecture 2.2** for a curve Z contained in a pure Shimura variety, apparently motivated by transcendence questions about periods of Shimura varieties. Oort [Oort94] was interested in the study of Jacobians with complex multiplication and proposed **Conjecture 2.2** for $S = \mathcal{A}_g$. Hence the name of the conjecture.

Both André and Oort were aware of the analogy with the Manin-Mumford conjecture. This analogy has inspired all the strategies for proving **Conjecture 2.2**.

(a) The p -adic methods of Raynaud's proof [Ray88] of the Manin-Mumford conjecture inspired works on **Conjecture 2.2** when S is a pure Shimura variety and Z is the Zariski-closure of a set of special points having good reduction properties at one fixed place p [Moo98, II], [Ya05].

(b) Edixhoven developed an approach to [Conjecture 2.2](#), based on Galois techniques and intersection theory, retrospectively close in spirit to Hindry’s approach to the Manin-Mumford conjecture [[Hin88](#)]. This method uses in a crucial way effective Chebotarev type results, known only under the Generalized Riemann Hypothesis (GRH). In [[Ed98](#)] Edixhoven proves [Conjecture 2.2](#) under GRH for S a product of two modular curves ; in [[EdYa03](#)] Edixhoven and Yafaev obtain the result under GRH for Z a curve in an arbitrary pure Shimura variety S ; and in [[Ed05](#)] Edixhoven proves [Conjecture 2.2](#) under GRH for Z an arbitrary subvariety of a product of modular curves. This approach, allied with ideas *à la Margulis-Ratner* from ergodic theory on homogeneous spaces ([[CloU105](#)], [[U07](#)]), culminated in the following result [[UY14a](#)], [[KY14](#)] (announced in 2006 and published in 2014):

Theorem 2.7. *The André-Oort [Conjecture 2.2](#) for pure Shimura varieties is true under the Generalized Riemann Hypothesis. It is also true unconditionally if Z is the Zariski-closure of a set of special points contained in a Hecke orbit.*

The proof was made purely algebraic by Daw [[Daw16](#)], who replaced the ergodic arguments by a systematic use of Prasad’s formula for the covolume of a congruence group [[Prasad89](#)].

This text will say nothing about Edixhoven’s approach, for which many surveys are available. We refer for instance to [[Ya07](#)] or [[Panorama](#)] and the references therein.

(c) Pila and Zannier [[PiZa08](#)] developed a method based on o-minimal geometry for proving the Manin-Mumford conjecture. Pila adapted it to obtain an unconditional proof of [Conjecture 2.2](#) for S an arbitrary product $\mathbb{C}^n \times \mathbf{G}_m^k$ [[Pil11](#)] (as we already mentioned, André obtained an unconditional proof for S the product of two modular curves but his method using Puiseux expansion did not generalize). The combination of the work of many authors (whose contributions are detailed below) then lead to the following:

Theorem 2.8. *The André-Oort [Conjecture 2.2](#) is true for \mathcal{A}_g and more generally for any mixed Shimura variety whose pure part is of abelian type.*

The goal of this text is to present the ideas around [Conjecture 2.2](#) and sketch the proof of [Theorem 2.8](#) following the Pila-Zannier strategy. Following [[U14](#)], [Conjecture 2.2](#) for a general connected mixed Shimura variety S uniformized by $\pi : X^+ \rightarrow S := \Gamma \backslash X^+$ follows from three main ingredients (two of which are known in full generality while the third one is known only under GRH or unconditionally for mixed Shimura varieties whose pure part is of abelian type):

The first ingredient is the *definability in some o-minimal structure (in our case $\mathbb{R}_{\text{an,exp}}$) of the restriction of π to a semi-algebraic fundamental set \mathcal{F} for the action of Γ on X^+* : see [Theorem 6.2](#). This result is obtained by Peterzil-Starchenko [[PetStar13](#)] for $S = \mathcal{A}_g$, by Klingler-Ullmo-Yafaev [[KUY16](#)] for an arbitrary pure Shimura variety and extended by Gao [[Gao16b](#)] to any mixed Shimura variety.

The second ingredient is the *Ax-Lindemann conjecture for Shimura varieties*, see [Theorem 4.28](#), which says that the Zariski-closure $\pi(Y)$ of any algebraic subvariety Y of X^+ (in the sense of [Example 4.8](#)) should be weakly special (in the sense of [Section 3.3](#)). This is the main geometric ingredient in the Pila-Zannier strategy for solving the Manin-Mumford-André-Oort problem for Shimura varieties.

Theorem 4.28 is proven by Pila [Pil11] when S is a product $Y(1)^n \times (\mathbb{C}^*)^k$, by Ullmo-Yafaev [UY14b] for projective Shimura varieties, by Pila-Tsimerman [PT14] for \mathcal{A}_g , by Klingler-Ullmo-Yafaev [KUY16] for any pure Shimura variety and extended by Gao [Gao16b] to any mixed Shimura variety. All these proofs use o-minimal geometry as a tool. Mok has an entirely complex-analytic approach to the Ax-Lindemann conjecture in the pure case. We refer to [Mok10], [Mok12] for partial results.

The third ingredient is *a good lower bound for the size of Galois orbits of special points of S* . This ingredient is already crucial in the Edixhoven’s approach. We refer to [U14, conj.2.7] for the description of the expected lower bound for an arbitrary pure Shimura variety. These expected lower bounds are known under GRH for any pure Shimura variety following results of Tsimerman [Tsi12] and Ullmo-Yafaev [UY15]. They are known unconditionally only for mixed Shimura varieties whose pure part is of abelian type. For simplicity we restrict ourselves to the case $S = \mathcal{A}_g$.

Given a point $x \in \mathcal{A}_g$ let A_x be the principally polarized abelian variety parametrized by x and d_x the absolute value of the discriminant of the center of the ring of endomorphisms of A_x . When x is special its field of definition $k(x)$ is a number field. In 2001, motivated by applications to the André-Oort conjecture, Edixhoven conjectured in [EMO, Problem 14] that there should exist real positive numbers $c_2 = c_2(g)$ and $\beta = \beta(g)$ such that for any special point $x \in \mathcal{A}_g$ one has:

$$(2.2) \quad |\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \quad (= [k(x) : \mathbb{Q}]) > c_2 \cdot d_x^\beta .$$

In [Tsi] Tsimerman proves that the inequality (2.2) follows from the Masser-Wüstholz isogeny Theorem 9.3 [MaWü95] (which Orr [Orr15] already used for obtaining lower bounds for Galois orbits of special points, see Section 10) and an upper bound for the Faltings height $h_F(A_x)$ of the form

$$(2.3) \quad \forall \epsilon > 0, \quad h_F(A_x) \ll_\epsilon d_x^\epsilon .$$

He also shows that the upper-bound (2.3) follows from the so-called “Colmez Conjecture on average” and classical arguments from analytic number theory.

In [Col93] Colmez conjectured a closed formula for the Faltings height of an abelian variety with complex multiplication, depending only on its CM-type (E, Φ) . Fixing E and averaging on the 2^g possible CM-type Φ for E one obtains a simpler formula for the average of the Faltings height of abelian varieties with CM by the ring of integers \mathcal{O}_E of E . Two remarkable proofs of Colmez conjecture on average have been obtained independently by Andreatta-Goren-Howard-Madapusi Pera [AGHM] (studying CM-points on certain orthogonal Shimura varieties) and Yuan-Zhang [YuZh] (analyzing Heegner points on certain Shimura curves), see Theorem 9.5.

Daw and Orr [DawOrr15] show that the Pila-Zannier method gives a new proof of Conjecture 2.2 under GRH for an arbitrary pure Shimura variety.

Gao [Gao16a], [Gao16b] extends the Pila-Zannier method in the mixed setting, showing Conjecture 2.2 under GRH for any mixed Shimura variety and Conjecture 2.2 unconditionally for mixed Shimura varieties whose pure part is of abelian type.

3. SPECIAL STRUCTURES ON ALGEBRAIC VARIETIES

3.1. Special structures. In this section we introduce a general format in which a Manin-Mumford-André-Oort type problem can be formulated: the notion of a special structure on an algebraic variety. We refer to [U16] for more details and [Zil13] for a study of special subvarieties from the point of view of model theory.

Definition 3.1. (*special structure*) *Let S be a complex quasi-projective variety. A special structure on S is the datum of a countable set $\Sigma(S)$ of irreducible algebraic subvarieties of S , called special subvarieties of S , satisfying the following properties:*

- (i) $S \in \Sigma(S)$, i.e. S is special.
- (ii) An irreducible component of an intersection of special subvarieties of S is a special subvariety of S .
- (iii) For any $W \in \Sigma(S)$, special points of S are dense in W .
- (iv) The variety S admits an infinite countable set of finite algebraic correspondances mapping any special subvariety of S to a finite linear combination of special subvarieties.

It follows from the condition (ii) that for any irreducible algebraic subvariety Z of S , there exists a unique smallest special subvariety of S containing Z . One says that Z is Hodge generic if it not contained in any strict special subvariety of S .

The following are natural examples of complex algebraic varieties endowed with a special structure:

- (1) a complex semi-Abelian variety S extension of an Abelian variety A by a torus $T \simeq \mathbf{G}_m^n$. Its special points are torsion points. Its special subvarieties are the translate of an algebraic subgroup by a torsion point. The finite correspondances of the condition (iv) are the endomorphisms of A .
- (2) a Shimura variety S with its special subvarieties. The finite correspondances of the condition (iv) are the Hecke correspondances of S .

3.2. Manin-Mumford-André-Oort type problem for special structures. An abstract Manin-Mumford-André-Oort type problem can be formulated for any quasi-projective variety endowed with a special structure:

Problem 3.2. *Let S be a complex quasi-projective variety endowed with a special structure. Does it satisfy the following equivalent two statements ?*

- (1) *Let Z be an irreducible algebraic subvariety of S containing a Zariski-dense set of special points. Then Z is a special subvariety of S .*
- (2) *Let Z be an algebraic subvariety of S . The set of special subvarieties of S contained in Z and maximal for these properties is finite.*

Remark 3.3. The equivalence between (1) and (2) follows from the properties (ii) and (iii) of [Definition 3.1](#).

[Problem 3.2](#) for S a semi-abelian variety is the classical Manin-Mumford conjecture for S . [Problem 3.2](#) for S a Shimura variety is the André-Oort Conjecture [2.2](#).

Remark 3.4. Notice that any semi-Abelian variety can be realized as a subvariety of a mixed Shimura variety. However only the ones whose abelian part has complex multiplication can be realized a special subvarieties of a mixed Shimura variety. Hence the

André-Oort [Conjecture 2.2](#) implies the Manin-Mumford conjecture only for such semi-Abelian varieties. In [\[Zil02\]](#) and [\[Pink05\]](#), Zilber and Pink propose a general conjecture (now called the Zilber-Pink conjecture) about atypical intersections in mixed Shimura varieties, which implies both the Manin-Mumford and the André-Oort conjecture. We refer the reader to the volume [\[Panorama\]](#) for an exposition of the Zilber-Pink conjecture.

3.3. Weakly special subvarieties. This section relates special structures and Kähler geometry.

Notice first that any semi-abelian variety A is endowed with a canonical Kähler metric coming from the flat Euclidean metric on its uniformization \mathbb{C}^n . Define a *weakly special* subvariety of A as an irreducible algebraic subvariety whose smooth locus is totally geodesic in A . Equivalently, these are the translates of the algebraic subgroups of A . Thus special subvarieties are weakly special, and a weakly special subvariety is special if and only if it contains a special point.

Similarly, a connected pure Shimura variety S (assumed to be smooth) inherits an essentially canonical Kähler metric from its universal cover X^+ : any locally symmetric Kähler metric on the Hermitian symmetric space X^+ is invariant under Γ hence descends to $S = \Gamma \backslash X^+$. Notice that the locally symmetric Kähler metric on X^+ is unique (up to a scalar) if X^+ is irreducible as a symmetric space: it coincides with the Bergman metric of the bounded Harish-Chandra realization of X^+ .

Define once more a *weakly special* subvariety of S as an irreducible algebraic subvariety whose smooth locus is totally geodesic in S . Every special subvariety of S is easily seen to be weakly special. Similarly to the case of semi-abelian varieties, Moonen [\[Moo98,I\]](#) proved:

Theorem 3.5. *Let S be a pure connected Shimura variety. A weakly special subvariety of S is special if and only if it contains a special point.*

More precisely: let $(\mathbf{H}, X_{\mathbf{H}})$ be a sub-Shimura datum of the Shimura datum (\mathbf{G}, X) defining S . Assume that the adjoint Shimura datum $(\mathbf{H}^{\text{ad}}, X_{\mathbf{H}^{\text{ad}}})$ splits as a product:

$$(\mathbf{H}^{\text{ad}}, X_{\mathbf{H}^{\text{ad}}}) = (\mathbf{H}_1, X_1) \times (\mathbf{H}_2, X_2) .$$

Let x_2 be a point of X_2 and Z the image of $X_1^+ \times x_2$ in S . Then Z is weakly special, and Z is special if and only if x_2 is a special point of X_2 . Conversely any weakly special subvariety of S is obtained in this way.

When S is a general mixed Shimura variety, Pink [\[Pink05, def. 4.1\]](#) defines the weakly special subvarieties of S in terms of mixed Shimura data. Once more the special subvarieties are exactly the weakly special ones containing a special point.

4. BI-ALGEBRAIC GEOMETRY

4.1. Complex bi-algebraic geometry. Let X and S be (connected) complex algebraic varieties and suppose $\pi : X^{\text{an}} \rightarrow S^{\text{an}}$ is a complex analytic, *non-algebraic*, morphism between the associated complex analytic spaces. In this situation the image $\pi(Y)$ of a generic algebraic subvariety $Y \subset X$ is usually highly transcendental and the pairs $(Y \subset X, V \subset S)$ of irreducible algebraic subvarieties such that $\pi(Y) = V$ are rare and of particular geometric significance. We are especially interested in the case where

X is the universal cover \tilde{S} of S . In this case, however, the requirement that \tilde{S} is a complex algebraic variety is too restrictive for practical purposes. We relax it as follows:

Definition 4.1. *A bi-algebraic structure on a connected complex algebraic variety S is a pair*

$$(D : \tilde{S} \longrightarrow \hat{X}, \quad h : \pi_1(S) \longrightarrow \text{Aut}(\hat{X}))$$

where \tilde{S} denotes the universal cover of S , \hat{X} is a complex algebraic variety, $\text{Aut}(\hat{X})$ its group of algebraic automorphisms, $h : \pi_1(S) \longrightarrow \text{Aut}(\hat{X})$ is a group morphism and D is a non-constant, h -equivariant, holomorphic map.

Definition 4.2. *(Algebraic subvariety of \tilde{S}) Let S be a connected complex algebraic variety S endowed with a bi-algebraic structure (D, h) . A closed analytic subvariety $Y \subset \tilde{S}$ is said to be an irreducible algebraic subvariety of \tilde{S} if Y is an irreducible analytic component of $D^{-1}(\overline{D(Y)}^{\text{Zar}})$ (where $\overline{D(Y)}^{\text{Zar}}$ denotes the Zariski-closure of $D(Y)$ in \hat{X}).*

Definition 4.3. *(Bi-algebraic subvariety of S) Let S be a connected complex algebraic variety S endowed with a bi-algebraic structure (D, h) . An irreducible algebraic subvariety $Y \subset \tilde{S}$, resp. $W \subset S$, is said to be bi-algebraic if $\pi(Y)$ is an algebraic subvariety of S , resp. any (equivalently one) analytic irreducible component of $\pi^{-1}(W)$ is an irreducible algebraic subvariety of \tilde{S} .*

Remark 4.4. The bi-algebraic structures (D, h) we consider in this paper all have the property that the map D is an open embedding which realizes \tilde{S} as an analytic open subset of \hat{X} . However it is crucial for further applications to allow the generality we introduce here. We refer to [K16] for natural examples of bi-algebraic structures where D is not immersive.

Example 4.5. (Tori)

The simplest example of a bi-algebraic structure is provided by the multi-exponential

$$\pi := (\exp(2\pi i \cdot), \dots, \exp(2\pi i \cdot)) : \mathbb{C}^n \longrightarrow (\mathbb{C}^*)^n .$$

In this case $\tilde{S} = \hat{X} = \mathbb{C}^n$ and D is the identity morphism. An irreducible algebraic subvariety $Y \subset \mathbb{C}^n$ (resp. $W \subset (\mathbb{C}^*)^n$) is bi-algebraic if and only if Y is a translate of a rational linear subspace of $\mathbb{C}^n = \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{C}$ (resp. W is a translate of a subtorus of $(\mathbb{C}^*)^n$).

For the choice of the factor $2\pi i$ in the exponential, see [Section 4.2](#).

Example 4.6. (Abelian varieties)

Let $\pi : \text{Lie } A \simeq \mathbb{C}^n \longrightarrow A$ be the uniformizing map of a complex abelian variety A of dimension n . Once more $\tilde{S} = X = \mathbb{C}^n$ and D is the identity morphism. One checks that an irreducible algebraic subvariety $W \subset A$ is bi-algebraic if and only if W is the translate of an abelian subvariety of A (cf. [UY11, prop. 5.1] for example).

Example 4.7. (Semi-abelian varieties)

Any semi-abelian variety admits a bi-algebraic structure generalizing [Example 4.5](#) and [Example 4.6](#) (we leave the details to the reader).

Example 4.8. (Shimura varieties)

Let $S = \Gamma \backslash X^+$ be a connected pure Shimura variety associated to a Shimura datum (\mathbf{G}, X) (with the notations of [Section 2.4](#)). For simplicity we assume that Γ is torsion-free, equivalently that S is smooth (the meticulous reader will easily extend [Definition 4.1](#) and [Definition 4.3](#) to the orbifold case). Hence $\pi : X^+ \rightarrow S$ is the universal cover of S . Fix a faithful algebraic representation $\rho : \mathbf{G} \rightarrow \mathbf{GL}(V)$. As X is a $\mathbf{G}(\mathbb{R})$ -conjugacy class of morphisms from \mathbf{S} to $\mathbf{G}_{\mathbb{R}}$, any point $x \in X^+$ defines a morphism $\rho \circ x : \mathbf{S} \rightarrow \mathbf{GL}(V)_{\mathbb{R}}$, i.e. a Hodge structure V_x on V . Let F_x^\bullet be the corresponding Hodge filtration on $V_{\mathbb{C}}$. The Borel embedding $D : X^+ \rightarrow \hat{X}$ associates to a point $x \in X^+$ the filtration F_x in the complex algebraic flag variety \hat{X} parametrizing filtrations of $V_{\mathbb{C}}$ of a given type. This is an open holomorphic embedding of X^+ in its dual compact space. The flag variety \hat{X} is homogeneous under the algebraic action of $\mathbf{G}^{\text{ad}}(\mathbb{C})$ and the open embedding D is equivariant under the natural inclusion $h : \Gamma \hookrightarrow \mathbf{G}^{\text{ad}}(\mathbb{R})^+ \hookrightarrow \mathbf{G}^{\text{ad}}(\mathbb{C})$, hence (D, h) defines a bi-algebraic structure on S .

The identification of the bi-algebraic varieties for this bi-algebraic structure is due to Ullmo and Yafaev [[UY11](#)]:

Theorem 4.9. *Let S be a pure connected Shimura variety endowed with its canonical bi-algebraic structure. The bi-algebraic subvarieties of S are the weakly special ones.*

Sketch of proof. Let us sketch the proof of [Theorem 4.9](#), which illustrates typical reduction steps and monodromy arguments.

Let (\mathbf{G}, X) be the Shimura datum defining S (hence S is a connected component of the Shimura variety $\text{Sh}_K(\mathbf{G}, X)$, for some compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$).

Any weakly special subvariety W of S is an algebraic subvariety of S image under $\pi : X^+ \rightarrow S = \Gamma \backslash X^+$ of a totally geodesic Hermitian subdomain $X_{\mathbf{H}}^+ \subset X^+$. As $X_{\mathbf{H}}^+$ is the intersection of the algebraic subvariety $\hat{X}_{\mathbf{H}} \subset \hat{X}$ with X^+ , the weakly special W is bi-algebraic.

Conversely we want to show that any bi-algebraic subvariety of S is weakly special. Let $W \subset S$ be an algebraic subvariety. We perform first three reduction steps:

- Replacing if necessary S by its smallest special subvariety containing W , we can assume without loss of generality that W is Hodge generic in S .

- The morphism $\psi : \mathbf{G} \rightarrow \mathbf{G}^{\text{ad}}$ from \mathbf{G} to its adjoint group extends to a morphism of Shimura data $\psi : (\mathbf{G}, X) \rightarrow (\mathbf{G}^{\text{ad}}, X^{\text{ad}})$. Let $K^{\text{ad}} \subset \mathbf{G}^{\text{ad}}(\mathbb{A}_f)$ be a compact open subgroup containing the image of K . We thus have a morphism of Shimura varieties $\psi : \text{Sh}_K(\mathbf{G}, X) \rightarrow \text{Sh}_{K^{\text{ad}}}(\mathbf{G}^{\text{ad}}, X^{\text{ad}})$. In this situation one immediately checks that W is weakly special if and only if $\psi(W)$ is weakly special. Moreover as the connected components of X and X^{ad} coincide, W is bi-algebraic if and only if $\psi(W)$ is bi-algebraic. Hence we can assume that \mathbf{G} is adjoint.

- Changing the level if necessary we can also assume without loss of generality that K is sufficiently small so that S is smooth.

Fix a faithful rational representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(V)$ and an integral structure $V_{\mathbb{Z}} \subset V$ such that $\Gamma \subset \mathbf{GL}(V_{\mathbb{Z}})$. This defines a polarized \mathbb{Z} -variation of Hodge structures \mathbb{V} on S . Let $\rho : \pi_1(W^{\text{sm}}) \rightarrow \Gamma \subset \mathbf{GL}(V_{\mathbb{Z}})$ be the monodromy representation of the induced variation on the smooth locus W^{sm} of W and $\Gamma_W := \rho(\pi_1(W^{\text{sm}}))$. Let $\widetilde{W} \subset X^+$ be an analytic irreducible component of $\pi^{-1}(W)$. Hence the group Γ_W is exactly the stabilizer of \widetilde{W} in Γ .

Suppose from now on that W is bi-algebraic. Hence $\widetilde{W} \subset X^+$ is algebraic of the form $\widehat{W} \cap X^+$, where $\widehat{W} \subset \widehat{X}$ is the Zariski-closure of W in \widehat{X} . In particular \widehat{W} is stabilized by the algebraic monodromy group \mathbf{G}_1 , which is the connected component of the Zariski-closure of Γ_W in \mathbf{G} . Recall the following result of Deligne (generalized by André [An92] in the mixed case):

Theorem 4.10. *Let \mathbb{V} be an admissible variation of mixed Hodge structures on a smooth quasi-projective variety S with generic Mumford-Tate group \mathbf{G} .*

- (i) *The algebraic monodromy group $\mathbf{G}_1 \subset \mathbf{G}$ is a normal subgroup of the derived group \mathbf{G}^{der} .*
- (ii) *If moreover S contains a CM-point then $\mathbf{G}_1 = \mathbf{G}^{\text{der}}$.*

Applying (i) and as \mathbf{G} is adjoint, we obtain a decomposition of Shimura data

$$(\mathbf{G}, X) = (\mathbf{G}_1, X_1) \times (G_2, X_2)$$

and one checks that W is the π -image of $X_1^+ \times x_2$ for a Hodge generic point $x_2 \in X_2^+$. It follows from Moonen's [Theorem 3.5](#) that W is weakly special. \square

The construction of a natural bi-algebraic structure on a pure Shimura variety extends to mixed Shimura varieties, as well as the identification of bi-algebraic subvarieties with weakly special ones (see [Gao16b]).

4.2. $\overline{\mathbb{Q}}$ -bi-algebraic geometry. Let S be a complex algebraic variety with a bi-algebraic structure as in [Section 4.1](#). While positive dimensional bi-algebraic subvarieties are usually rare and of geometric significance, any point of S is bi-algebraic in the sense of [Definition 4.3](#). To obtain a more meaningful definition of bi-algebraic points we refine [Definition 4.1](#) as follows:

Definition 4.11. *A $\overline{\mathbb{Q}}$ -bi-algebraic structure on a complex algebraic variety S is a complex bi-algebraic structure $(D : \tilde{S} \rightarrow \widehat{X}, h : \pi_1(S) \rightarrow \text{Aut}(\widehat{X}))$ such that:*

- (1) *S is defined over $\overline{\mathbb{Q}}$.*
- (2) *$\widehat{X} = \widehat{X}_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ and the homomorphism h takes values in $\text{Aut}_{\overline{\mathbb{Q}}} \widehat{X}_{\overline{\mathbb{Q}}}$.*

Definition 4.12. *Let (D, h) be a $\overline{\mathbb{Q}}$ -bi-algebraic structure on S . A point $s \in S(\mathbb{C})$ is said to be an arithmetic bi-algebraic point if $s \in S(\overline{\mathbb{Q}})$ and any (equivalently one) π -pre-image $\tilde{s} \in \tilde{S}$ satisfies $D(\tilde{s}) \in \widehat{X}_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$.*

Let us emphasize that the choice of the $\overline{\mathbb{Q}}$ -structure on \widehat{X} and the normalization of the developing map D crucially determines the existence of a large supply of arithmetic bi-algebraic points.

Example 4.13. (Tori)

If we endow \mathbb{C}^n and $(\mathbb{C}^*)^n$ with their standard rational structure \mathbb{Q}^n and $(\mathbb{Q}^*)^n$, the arithmetic bi-algebraic points of $(\mathbb{C}^*)^n$ for the $\overline{\mathbb{Q}}$ -bi-algebraic structure defined in [Example 4.5](#) are exactly the torsion points. Indeed, without loss of generality we can assume $n = 1$. The Gelfond-Schneider theorem [Ge60] states that if α and β are complex numbers such that $\alpha \neq 0$ and e^α, β and $e^{\alpha\beta}$ are all in $\overline{\mathbb{Q}}$ then $\beta \in \mathbb{Q}$. Applying this to

$\alpha = 2\pi i$, we see that $x = \exp(2\pi i\beta) \in \mathbb{C}^*$ is bi-algebraic if and only if $\beta \in \mathbb{Q}$, i.e. x is a torsion point.

Notice that if we had chosen for the uniformization map the usual exponential $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ rather than $\exp(2\pi i \cdot) : \mathbb{C} \rightarrow \mathbb{C}^*$ (keeping the same rational structures $\mathbb{Q} \subset \mathbb{C}$ and $\mathbb{Q}^* \subset \mathbb{C}^*$), or if we had kept the same uniformization map but chosen the rational structure $\mathbb{Q}(1)$ of \mathbb{C} , the only arithmetic bi-algebraic point for \mathbb{C}^* would have been 1 by the Hermite-Lindemann theorem [Ge60].

Example 4.14. (Abelian varieties with CM)

In the setting of [Example 4.6](#), suppose from now on that A is an abelian variety over $\overline{\mathbb{Q}}$. If we define a $\overline{\mathbb{Q}}$ -bi-algebraic structure on $A_{\mathbb{C}}$ by choosing the standard $\overline{\mathbb{Q}}$ -model $\text{Lie}(A_{\overline{\mathbb{Q}}})$ of $\text{Lie}(A_{\mathbb{C}})$, the unique bi-algebraic point of $A_{\mathbb{C}}$ is the identity (see [Lang66, thm.3 p.28]).

When A is a complex abelian variety of dimension g with CM (hence A is in particular defined over $\overline{\mathbb{Q}}$) one can consider a better $\overline{\mathbb{Q}}$ -structure on $\text{Lie}(A_{\mathbb{C}})$: in this case the lattice of periods $\Gamma := \text{Ker } \pi \subset \text{Lie}(A)$ generates a $\overline{\mathbb{Q}}$ -vector space $V_{\overline{\mathbb{Q}}} \subset \text{Lie}(A)$ of dimension g , hence defines a $\overline{\mathbb{Q}}$ -structure on $\text{Lie}(A)$. In [Ma76] Masser proved:

Theorem 4.15. (Masser) *Let A be a complex abelian variety of dimension g with CM. Let $V_{\overline{\mathbb{Q}}} \subset \text{Lie}(A)$ be the $\overline{\mathbb{Q}}$ -vector space generated by the lattice of periods Γ . Arithmetic bi-algebraic points for this $\overline{\mathbb{Q}}$ -bi-algebraic structure on A are exactly the torsion points of A .*

Example 4.16. (Semi-abelian varieties whose abelian part has CM)

[Example 4.13](#) and [Example 4.14](#) can be combined to define a $\overline{\mathbb{Q}}$ -bi-algebraic structure on any semi-abelian variety whose abelian part has CM. Once more the arithmetic bi-algebraic points are the torsion points. We leave the details to the reader.

Example 4.17. (Shimura varieties)

Let (\mathbf{G}, X) be a pure Shimura datum and $K \subset \mathbf{G}(\mathbb{A}_f)$ a compact open subgroup. A fundamental result of the theory of Shimura varieties is that the complex quasi-projective variety $\text{Sh}_K(\mathbf{G}, X)$ is defined over a number field $E(\mathbf{G}, X)$ (called the reflex field) depending only on the Shimura datum (\mathbf{G}, X) . It follows that any pure connected Shimura variety $S = \Gamma \backslash X^+$, connected component of $\text{Sh}_K(\mathbf{G}, X)$, is defined over an abelian extension of $E(\mathbf{G}, X)$.

With the notations of [Section 2.4](#), the flag variety \hat{X} is naturally defined over \mathbb{Q} as V is. This defines a $\overline{\mathbb{Q}}$ -bi-algebraic structure on S . The arithmetic bi-algebraic points of S for this $\overline{\mathbb{Q}}$ -bi-algebraic structure on S are the points of $S(\overline{\mathbb{Q}})$ whose pre-images lie in $X^+ \cap \hat{X}(\overline{\mathbb{Q}})$. An easy argument given in [UY11, section 3.4] shows that special points are always arithmetic bi-algebraic points.

What about the converse? When $(\mathbf{G}, X) = (\mathbf{GL}_2, \mathcal{H}^{\pm})$ and S is the modular curve $Y(1) \simeq \mathbb{C}$, Schneider's theorem [Schn37] states that if $\tau \in \mathcal{H} \cap \overline{\mathbb{Q}}$ and $x = j(\tau) \in \overline{\mathbb{Q}}$ then τ is imaginary quadratic, i.e. x is a CM-point. Hence the bi-algebraic points are exactly the special points.

Cohen [Co96] and Shiga-Wolfart [ShWo95] generalize this result to \mathcal{A}_g . A formal argument generalize their result to Shimura varieties of abelian type:

Theorem 4.18. (Cohen, Shiga, Wolfart) *A point $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$ is an arithmetic bi-algebraic point if and only if it is special.*

More generally let (\mathbf{G}, X) be a Shimura datum of abelian type, $K \subset \mathbf{G}(\mathbb{A}_f)$ a compact open subgroup and S a connected component of $\mathrm{Sh}_K(\mathbf{G}, X)$ endowed with the $\overline{\mathbb{Q}}$ -bi-algebraic structure defined above. A point of S is bi-algebraic if and only if it is special.

Using [Example 4.13](#) and [Example 4.14](#), both the definition of a natural $\overline{\mathbb{Q}}$ -bi-algebraic structure and [Theorem 4.18](#) extend to mixed Shimura varieties whose pure part is of abelian type.

Remark 4.19. It is worth underlining that all *numerical* transcendence results used to define interesting $\overline{\mathbb{Q}}$ -bi-algebraic structures are subsumed in the fundamental analytic subgroup theorem of Wüstholz [[Wus89](#)]:

Theorem 4.20. Let \mathbf{G} be a commutative algebraic group over $\overline{\mathbb{Q}}$ with Lie algebra \mathfrak{g} and $\exp : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathbf{G}(\mathbb{C})$ its complex exponential map. Let $\mathfrak{b} \subset \mathfrak{g}$ be a $\overline{\mathbb{Q}}$ -vector subspace of positive dimension and $B := \exp(\mathfrak{b} \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$.

Then $B \cap \mathbf{G}(\overline{\mathbb{Q}}) \neq 0$ if and only if there exists a positive dimensional $\overline{\mathbb{Q}}$ -algebraic subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{H}(\mathbb{C}) \subset B$.

4.3. Special structures and bi-algebraic structures.

Definition 4.21. A special structure on a complex algebraic variety S is of bi-algebraic origin if S admits a bi-algebraic structure such that the special subvarieties of S are its bi-algebraic subvarieties containing a special point. Such a bi-algebraic structure is said to underlie the special structure.

A special structure on a complex algebraic variety S is said to be of $\overline{\mathbb{Q}}$ -bi-algebraic origin if it admits an underlying $\overline{\mathbb{Q}}$ -bi-algebraic structure whose arithmetic bi-algebraic points are the special points.

Thus the special structures we defined on semi-abelian varieties and mixed Shimura varieties are of bi-algebraic origin. If moreover the abelian part of the semi-abelian variety has CM or the pure part of the mixed Shimura variety is of abelian type, it follows from [Example 4.16](#) and [Example 4.17](#) that the special structure is of $\overline{\mathbb{Q}}$ -bi-algebraic origin.

4.4. The Ax-Lindemann principle. In the abstract context of bi-algebraic geometry, the Ax-Lindemann heuristic principle is the following functional transcendence statement:

Ax-Lindemann principle 4.22. Let S be an irreducible algebraic variety endowed with a bi-algebraic structure. For any irreducible algebraic subvariety $Y \subset \tilde{S}$, the Zariski-closure $\overline{\pi(Y)}^{\mathrm{Zar}}$ is a bi-algebraic subvariety of S .

Notice the following equivalent version of the Ax-Lindemann principle, which is the one we will work with:

Lemma 4.23. The [Ax-Lindemann principle 4.22](#) is equivalent to the statement that for any algebraic subvariety $V \subset S$, any irreducible algebraic subvariety Y of X contained in $\pi^{-1}(V)$ and maximal for this property is bi-algebraic.

Proof. Let us first assume that for any algebraic subvariety $V \subset S$, any irreducible algebraic subvariety Y of X contained in $\pi^{-1}(V)$ and maximal for this property is bi-algebraic. Let Y be an irreducible algebraic subvariety of X . Let W be the Zariski-closure of $\pi(Y)$. Let Z be an irreducible algebraic subvariety of $\pi^{-1}(W)$ containing Y , and maximal for these properties. By hypothesis, $\pi(Z)$ is weakly special, in particular $\pi(Z)$ is irreducible algebraic. As $\pi(Y) \subset \pi(Z) \subset W$, it follows that $\pi(Z) = W$, hence W is weakly special.

Conversely let us assume that for any irreducible algebraic subvariety $Y \subset \tilde{S}$, the Zariski-closure $\overline{\pi(Y)}^{\text{Zar}}$ is a bi-algebraic subvariety of S . Let W be an algebraic subvariety of S and Y an irreducible algebraic subvariety of $\pi^{-1}(W)$, maximal for these properties. By hypothesis the Zariski-closure W' of $\pi(Y)$ is weakly special. As $W' \subset W$, there exists an analytic irreducible component Y' of $\pi^{-1}(W')$ containing Y . As W' is weakly special, Y' is irreducible algebraic. By maximality of Y , one obtains $Y = Y'$ and $\pi(Y) = W'$ is weakly special. \square

Example 4.24. (semi-abelian varieties) Ax [Ax72] showed that the abstract Ax-Lindemann conjecture is true for any semi-abelian variety endowed with the bi-algebraic structure of [Example 4.7](#):

Theorem 4.25. (Ax) Let A be a semi-abelian variety endowed with the bi-algebraic structure of [Example 4.7](#). The [Ax-Lindemann principle 4.22](#) is true for A .

Remark 4.26. Notice that Ax's theorem for $\pi := (\exp(2\pi i \cdot), \dots, \exp(2\pi i \cdot)) : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$ is the functional analog of the classical Lindemann transcendence theorem stating that if $\alpha_1, \dots, \alpha_n$ are \mathbb{Q} -linearly independent algebraic numbers then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over \mathbb{Q} . This explain the terminology.

Example 4.27. (Shimura variety)

Theorem 4.28. (Ax-Lindemann for mixed Shimura varieties) Let $\pi : X \rightarrow S$ be the uniformization map of a connected mixed Shimura variety S . We endow S with the bi-algebraic structure of [Example 4.8](#). The [Ax-Lindemann principle 4.22](#) is true for S .

Let us repeat that [Theorem 4.28](#) is proven by Pila [Pil11] when S is a product $Y(1)^n \times (\mathbb{C}^*)^k$, by Ullmo-Yafaev [UY14b] for projective Shimura varieties, by Pila-Tsimerman [PT14] for \mathcal{A}_g , by Klingler-Ullmo-Yafaev [KUY16] for any pure Shimura variety and extended by Gao [Gao16b] to any mixed Shimura variety.

The proof of [Theorem 4.28](#) for pure Shimura varieties will be the topic of [Section 7](#).

5. O-MINIMAL GEOMETRY AND THE PILA-WILKIE'S THEOREM

5.1. O-minimal structures. For a more detailed treatment of o-minimality we refer to [vdD98], [PW06], [PetStar10], [Pil] and [Sca16].

Definition 5.1. A structure \mathbf{S} is a collection $\mathbf{S} = (S_n)_{n \in \mathbb{N}}$, where S_n is a set of subsets of \mathbb{R}^n , called the definable sets of the structure, such that for every $n \in \mathbb{N}$:

- (1) all algebraic subsets of \mathbb{R}^n are in S_n .
- (2) S_n is a boolean subalgebra of the power set of \mathbb{R}^n .
- (3) If $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$.

(4) Let $p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be a linear projection. If $A \in S_{n+1}$ the $p(A) \in S_n$.
A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be definable if its graph is.

A dual point of view starts from the functions, namely considers sets definable in a first-order structure

$$\langle \mathbb{R}, +, \times, <, (f_i)_{i \in I} \rangle$$

where I is a set and the $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$, $i \in I$, are functions. A subset $Z \subset \mathbb{R}^n$ is definable if it can be defined by a formula

$$Z := \{(x_1, \dots, x_n) \in \mathbb{R}^n / \phi(x_1, \dots, x_n) \text{ is true}\} ,$$

where ϕ is a first-order formula that can be written using only the quantifiers \forall and \exists applied to real variables, logical connectors, algebraic expressions written with the f_i 's, $<$ and fixed parameters $\lambda_i \in \mathbb{R}$. When the set I is empty the definable subsets are the semi-algebraic sets. Semi-algebraic subsets are thus always definable.

Remark 5.2. Our definition of “definable” is an abuse of notation: it coincides with what is usually called “definable with parameters”, see [vdD98, Chap.1, (5.3)].

The o-minimal axiom for a structure \mathbf{S} guarantees the possibility of doing geometry using definable sets as basic blocks. In particular it excludes infinite countable sets, like $\mathbb{Z} \subset \mathbb{R}$, to be definable.

Definition 5.3. A structure \mathbf{S} is said to be o-minimal if the definable subsets of \mathbb{R} are precisely the finite unions of points and intervals (i.e. the semi-algebraic subsets of \mathbb{R}).

Example 5.4. The structure $\mathbb{R}_{\sin} := \langle \mathbb{R}, +, \times, <, \sin \rangle$ is not o-minimal. Indeed the infinite union of points $\pi\mathbb{Z} = \{x \in \mathbb{R}, \sin x = 0\}$ is a definable subset of \mathbb{R} in this structure.

A deep theorem of Wilkie [Wil96] states:

Theorem 5.5. The structure $\mathbb{R}_{\exp} := \langle \mathbb{R}, +, \times, <, \exp \rangle$ is o-minimal.

Definition 5.6. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a restricted analytic function if it is zero outside $[0, 1]^n$ and if there exists a real analytic function g on a neighbourhood of $[0, 1]^n$ such that f and g are equal on $[0, 1]^n$.

One defines $\mathbb{R}_{\text{an}} := \langle \mathbb{R}, +, \times, <, \{f\} \text{ for } f \text{ restricted analytic function} \rangle$.

A theorem of Van den Dries based on Gabrielov’s results [Ga68] shows:

Theorem 5.7. The structure \mathbb{R}_{an} is o-minimal.

In diophantine geometry we will use the structure

$$\mathbb{R}_{\text{an}, \exp} := \langle \mathbb{R}, +, \times, <, \exp, \{f\} \text{ for } f \text{ restricted analytic function} \rangle$$

generated by \mathbb{R}_{an} and \mathbb{R}_{\exp} . The structure generated by two o-minimal structures is not o-minimal in general, but Van den Dries and Miller [vdDM85] prove in this case:

Theorem 5.8. The structure $\mathbb{R}_{\text{an}, \exp}$ is o-minimal.

5.2. Pila-Wilkie's counting theorem. In this section we fix one o-minimal expansion \mathbf{S} of \mathbb{R} .

Let H denote the standard multiplicative height function on $\overline{\mathbb{Q}}$. Thus if L is a number field, M_L its set of places, and $x \in L$ then

$$H(x) := \prod_{v \in M_L} \max(1, |x|_v) .$$

We also denote by H its extension to $\overline{\mathbb{Q}}^n$ defined by $H(x_1, \dots, x_n) := \max_i H(x_i)$. Given a subset $Z \subset \mathbb{R}^n$, a positive integer d and a real number T we define

$$(5.1) \quad \Theta_d(Z, T) := \{(x_1, \dots, x_n) \in Z / \max_i [\mathbb{Q}(x_i) : \mathbb{Q}] \leq d \text{ and } H(x_1, \dots, x_n) \leq T\} ,$$

and

$$(5.2) \quad N_d(Z, T) := |\Theta_d(Z, T)| .$$

Definition 5.9. Let $Z \subset \mathbb{R}^n$. We denote by Z^{alg} the union of all connected positive dimensional semi-algebraic subsets of \mathbb{R}^n contained in Z .

Theorem 5.10. ([PW06, Theor. 1.8]) Let $Z \subset \mathbb{R}^n$ be a subset definable in \mathbf{S} . Let d be a positive integer and ε a positive real number. There exists a constant $c = c(Z, d, \varepsilon)$ such that

$$(5.3) \quad \forall T > 0, \quad N_d(Z - Z^{\text{alg}}, T) \leq c \cdot T^\varepsilon .$$

In particular if there exists $\alpha > 0$ and $c' = c'(d, Z) > 0$ such that for any T sufficiently large we have $N_d(Z, T) \geq c' \cdot T^\alpha$ then Z^{alg} is non-empty.

Example 5.11. Let $Z \subset I^2$ be the intersection of a real analytic curve C defined in a neighborhood of I^2 with I^2 (where $I = [0, 1]$). Hence Z is definable in \mathbb{R}_{an} . Suppose that there exist a positive integer d , and real numbers $\alpha > 0$ and $c' = c'(d, Z) > 0$ such that for any T sufficiently large we have $N_d(Z, T) \geq c' \cdot T^\alpha$. Then the real analytic curve C is real algebraic.

Notice that in general the set Z^{alg} associated to a definable set Z is usually not definable. Consider for example [Sca16, Rem.4.5] the \mathbb{R}_{exp} -definable subset of \mathbb{R}^3 defined as

$$Z := \{(x, y, z) \in \mathbb{R}_+^3 ; z = x^y\}$$

whose algebraic part Z^{alg} is the union of triples $(x, y, z) \in Z$ such that $y \in \mathbb{Q}$. It will be crucial to pass from Z^{alg} to something more controllable: the semi-algebraic blocks.

Definition 5.12. A semi-algebraic block W in \mathbb{R}^n for \mathbf{S} is a connected infinite definable subset of \mathbb{R}^n such that there exists a connected semi-algebraic set $B \subset \mathbb{R}^n$ whose non-singular locus contains W and which coincides with W in the neighbourhood of every point of W . In particular a semi-algebraic block is covered by open semi-algebraic sets.

Example 5.13. Let $W := \{(x, y) \in \mathbb{R}^2, y < \exp(x)\}$. This is a semi-algebraic block of \mathbb{R}_{exp} with $B = \mathbb{R}^2$.

Using the notion of semi-algebraic blocks, [Theorem 5.10](#) can be refined in two directions:

Theorem 5.14. ([Pil11, Theor. 3.6]) *Let $Z \subset \mathbb{R}^n$ be a subset definable in \mathbf{S} . Let d be a positive integer and ε a positive real number. There exists a constant $c = c(Z, d, \varepsilon)$ such that $\Theta_d(Z, T)$ is contained in at most $c \cdot T^\varepsilon$ semi-algebraic blocks contained in Z .*

The second refinement deals with families.

Definition 5.15. *A definable family $Z := \{Z_b\}_{b \in B}$ of subsets of \mathbb{R}^n is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m$ whose projection on the second factor is $B \subset \mathbb{R}^m$.*

In this case every fiber $Z_b \subset \mathbb{R}^n$ for $b \in B$ is definable.

Theorem 5.16. ([PW06, Theor. 1.10]) *Let $Z := \{Z_b\}_{b \in B}$ be a definable family of subsets of \mathbb{R}^n in \mathbf{S} . Let ε be a positive real number. There exists a constant $c := c(\varepsilon, Z)$ and a definable family $Y := \{Y_b\}_{b \in B}$ of subsets of \mathbb{R}^n such that, for every $b \in B$, one has the inclusion $Y_b \subset Z_b^{\text{alg}}$ and*

$$(5.4) \quad N_d(Z_b - Y_b, T) \leq c \cdot T^\varepsilon .$$

Remarks 5.17. (a) The crux of this refinement is the uniformity (the constant c does not depend on $b \in B$).

(b) The definable family $\{Y_b\}_{b \in B}$ is needed as the sets $Z^{\text{alg}} \cap Z_b$ associated to the definable set Z are usually not definable.

The proof of [Theorem 5.10](#) and its refinements [Theorem 5.14](#) and [Theorem 5.16](#) relies on a reparametrization theorem generalizing a result of Yomdin [[Yo87a](#)], [[Yo87b](#)] and Gromov [[Gromov87](#)] for semi-algebraic sets:

Theorem 5.18. ([PW06, Theor.2.3]) *Let r be an integer. Let $Z \subset (0, 1)^n$ be a definable set in an o-minimal expansion of \mathbb{R} , of dimension m in the sense of [[vdD98](#), Chap. 4, ¶1]. There exists a finite set $I := I(Z, r)$, of uniformly bounded cardinality when Z varies in a definable family, such that*

$$Z = \bigcup_{i \in I} \phi_i((0, 1)^m)$$

where $\phi_i : (0, 1)^m \rightarrow (0, 1)^n$ is of class C^r and $|\partial_\alpha \phi_i| \leq 1$ for any multi-index α of length $|\alpha| \leq r$.

6. O-MINIMALITY AND SHIMURA VARIETIES

We will not pursue here how to use o-minimality in the general context of special structures of bi-algebraic origin. From now on we restrict ourselves to the context of (mixed) Shimura varieties.

Let $\pi : X^+ \rightarrow S := \Gamma \backslash X^+$ be the uniformization of a connected mixed Shimura variety S . The realization $X^+ \subset \hat{X}$ defines X^+ as a real semi-algebraic subset of \hat{X} . Of course the map π cannot be definable in any o-minimal structure as it is periodic under the countably infinite group Γ . We remove this difficulty by restricting π to a fundamental set of X^+ for the action of Γ :

Definition 6.1. *A fundamental set for the action of Γ on X^+ is a connected open subset \mathcal{F} of X^+ such that $\Gamma \overline{\mathcal{F}} = X^+$ and such that the set $\{\gamma \in \Gamma \mid \gamma \mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is finite.*

An essential step for using o-minimal geometry in the context of Shimura varieties is the following result:

Theorem 6.2. *There exists a semi-algebraic fundamental set \mathcal{F} for the action of Γ on X^+ such that the restriction $\pi|_{\mathcal{F}}: \mathcal{F} \rightarrow S$ is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.*

The special case of Theorem 6.2 when S is pure and compact is easy, see [UY14b, Prop.4.2]. In this case, the map $\pi|_{\mathcal{F}}$ is even definable in \mathbb{R}_{an} . Theorem 6.2 in the case where $X = \mathcal{H}_g$ is the Siegel upper half plane of genus g was proven by Peterzil and Starchenko (see [PetStar13] and [PetStar10]): in this case they use an explicit description for π in terms of θ -functions and delicate computations with these. Notice moreover that this particular case implies Theorem 6.2 for any special subvariety S of \mathcal{A}_g (see Proposition 2.5 of [U14]). On the other hand Peterzil and Starchenko's method does not generalize to general arithmetic varieties, where an explicit description of π is not available. The paper [KUY16] provides a completely geometric proof of Theorem 6.2 for any pure Shimura variety using the general theory of toroidal compactifications of arithmetic varieties (see [AMRT75]). Gao generalizes this result to mixed Shimura varieties in [Gao16b].

Let us give the proof of Theorem 6.2 in the baby-case of $S = Y(1)$ and $\pi = j : \mathcal{H} \rightarrow Y(1) = \mathbf{SL}(2, \mathbb{Z}) \backslash \mathcal{H} \simeq \mathbb{C}$. In this case we consider for \mathcal{F} the usual semi-algebraic fundamental set:

$$(6.1) \quad \mathcal{F} := \left\{ z = x + iy \in \mathcal{H}, -\frac{1}{2} < x < \frac{1}{2} \text{ and } y > \frac{\sqrt{3}}{2} \right\}.$$

Let us consider the diagram of holomorphic maps:

$$\mathcal{F} \subset \mathcal{H} \xrightarrow{z \mapsto e^{2\pi iz}} \Delta^* \xrightarrow{q} S = \mathbb{C},$$

where $\Delta^* := \{z \in \mathbb{C}^*, |z| < \exp(-\pi\sqrt{3})\}$. We claim that this composite is definable in $\mathbb{R}_{\text{an,exp}}$. It follows from the following observations:

- $\exp(2\pi iz) = \exp(-2\pi i \text{Im}(z)) \cdot \exp(2\pi i \text{Re}(z))$. The first factor is definable in \mathbb{R}_{exp} . On the other hand $\text{Re}(x)$ is bounded on \mathcal{F} , hence the second factor restricted to \mathcal{F} is definable in \mathbb{R}_{an} .

- The function $q : \Delta^* \rightarrow \mathbb{C}$ extends to $\Delta \rightarrow \mathbf{P}^1\mathbb{C}$ and hence is definable in \mathbb{R}_{an} .

For a general pure connected Shimura variety S associated with a Shimura datum (\mathbf{G}, X) , the fundamental set \mathcal{F} is a semi-algebraic Siegel set, whose construction we recall now (see [Bor69] for a general reference). Without loss of generality we can assume that \mathbf{G} is semi-simple of adjoint type. Let \mathbf{P} be a minimal \mathbb{Q} -parabolic subgroup of \mathbf{G} and $K_\infty \subset \mathbf{G}(\mathbb{R})$ a maximal compact subgroup such that $K_\infty \cap \mathbf{P}(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{P}(\mathbb{R})$. Let \mathbf{U} be the unipotent radical of \mathbf{P} and let \mathbf{A} be a maximal split torus of \mathbf{P} . We denote by \mathbf{S} a maximal split torus of $\mathbf{GL}(V)$ containing $\rho(\mathbf{A})$, by \mathbf{M} the maximal anisotropic subgroup of the connected centralizer $\mathbf{Z}(\mathbf{A})^0$ of \mathbf{A} in \mathbf{P} and by Δ the set of positive simple roots of \mathbf{G} with respect to \mathbf{A} and \mathbf{P} . We denote by $A \subset \mathbf{S}(\mathbb{R})$ the real torus $\mathbf{A}(\mathbb{R})$. For any real number $t > 0$ we let

$$A_t := \{a \in A \mid a^\alpha \geq t \text{ for any } \alpha \in \Delta\}.$$

A Siegel set of $\mathbf{G}(\mathbb{R})$ for the data $(K_\infty, \mathbf{P}, \mathbf{A})$ is a product:

$$\Sigma'_{t,\Omega} := \Omega \cdot A_t \cdot K_\infty \subset \mathbf{G}(\mathbb{R})$$

where Ω is a compact neighborhood of e in $\mathbf{M}^0(\mathbb{R}) \cdot \mathbf{U}(\mathbb{R})$.

The image

$$\Sigma_{t,\Omega} := \Omega \cdot A_t \cdot x_0 \subset X^+$$

of $\Sigma'_{t,\Omega}$ in X^+ (where x_0 is the point of $X^+ = \mathbf{G}(\mathbb{R})/K_\infty$ fixed under K_∞) is called a Siegel set in X^+ . The following is obtained in [KUY16]:

Theorem 6.3. *There exist a semi-algebraic Ω , a real number t and a finite subset J of $\mathbf{G}(\mathbb{Q})$ such that $\mathcal{F} := J \cdot \Sigma_{t_0,\Omega}$ is a fundamental set for the action of Γ on X^+ satisfying Theorem 6.2.*

7. THE HYPERBOLIC AX-LINDEMANN CONJECTURE

In this section we give some indications on the proof of the hyperbolic Ax-Lindemann Theorem 4.28 for a *pure* Shimura variety S . We follow closely [KUY16], to which we refer for more details.

7.1. Stabilizers of maximal algebraic subvarieties of $\pi^{-1}(W)$. Let $W \subset S$ be an irreducible algebraic subvariety and $Y \subset \pi^{-1}W$ an irreducible algebraic subvariety of X^+ , maximal for these properties. By Lemma 4.23 we have to show that $\pi(Y)$ is weakly special. The main intermediate step is the following:

Proposition 7.1. *There exists a connected \mathbb{Q} -algebraic subgroup \mathbf{H}_Y of \mathbf{G} , of positive dimension, such that $\mathbf{H}_Y(\mathbb{R})^+ \subset \text{Stab}_{\mathbf{G}(\mathbb{R})^+}(Y)$.*

Proof that Proposition 7.1 implies Theorem 4.28. The arguments are close to the ones used in the proof of Theorem 4.9. Let \mathbf{H}_Y be the largest connected \mathbb{Q} -algebraic subgroup of \mathbf{G} such that $\mathbf{H}_Y(\mathbb{R})^+ \subset \text{Stab}_{\mathbf{G}(\mathbb{R})^+}(Y)$. By Proposition 7.1 the group \mathbf{H}_Y is positive dimensional.

Let $W' \subset S$ be the Zariski-closure of $\pi(Y)$. Replacing W by W' we can assume that $\pi(Y)$ is Zariski-dense in W . Replacing S by the smallest special subvariety of S containing W , one can also assume that W is Hodge generic. In this situation it follows that $\pi(Y)$ is also Hodge-generic in the sense that $\pi(Y)$ is not contained in any strict special subvariety S' of S . Otherwise $\pi(Y) \subset S' \cap W \subsetneq W$ contradicting the Zariski-density of $\pi(Y)$ in W .

Lemma 7.2. *Let \widetilde{W} be an irreducible component of $\pi^{-1}(W)$ containing Y . Then $\mathbf{H}_Y(\mathbb{Q})$ stabilizes \widetilde{W} .*

Proof. Let $h \in \mathbf{H}_Y(\mathbb{Q})$. As $Y \subset \widetilde{W} \cap h\widetilde{W}$ is irreducible algebraic there exists an irreducible component Z of $\widetilde{W} \cap h\widetilde{W}$ containing Y . Notice that $\pi(Z)$ is an irreducible component of $W \cap T_h(W)$ containing $\pi(Y)$. As $\pi(Y)$ is Zariski-dense in W it follows that $\pi(Z) = W$. Hence $\widetilde{W} = h\widetilde{W}$. \square

Without loss of generality we can assume that \mathbf{G} is semi-simple of adjoint type. Indeed consider the morphism of Shimura data $\psi : (\mathbf{G}, X) \rightarrow (\mathbf{G}^{\text{ad}}, X^{\text{ad}})$. Let $K^{\text{ad}} \subset \mathbf{G}^{\text{ad}}(\mathbb{A}_f)$ be a compact open subgroup containing the image of K . We thus have a morphism of

Shimura varieties $\psi : \mathrm{Sh}_K(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, K^{\mathrm{ad}})$ and the conjectures for W and $\psi(W)$ are equivalent.

For simplicity let us first assume that \mathbf{G} is \mathbb{Q} -simple.

We choose a Hodge-generic point $z \in W^{\mathrm{sm}}$ and a point $\tilde{z} \in \widetilde{W}$ above z . Let $\rho : \pi_1(W^{\mathrm{sm}}, z) \longrightarrow \Gamma \subset \mathbf{GL}(V_{\mathbb{Z}})$ be the associated monodromy representation with image $\Gamma_W := \rho(\pi_1(W^{\mathrm{sm}}, z)) \subset \Gamma$. By Galois theory Γ_W is the subgroup of Γ stabilizing \widetilde{W} . In particular the group Γ_W contains

$$\mathbf{H}_Y(\mathbb{Z}) := \mathbf{H}_Y(\mathbb{Q}) \cap \Gamma = \mathbf{H}_Y(\mathbb{Q}) \cap \mathbf{G}(\mathbb{Z}).$$

Deligne's [Theorem 4.10](#) then states that the Zariski-closure $\overline{\Gamma_W}$ of Γ_W is normal in \mathbf{G} . As we assumed that \mathbf{G} is simple, it follows that $\overline{\Gamma_W} = \mathbf{G}$.

Lemma 7.3. *The group Γ_W normalizes \mathbf{H}_Y .*

Proof. Let $\gamma \in \Gamma_W$. Thus $\gamma \mathbf{H}_Y(\mathbb{R}) \gamma^{-1} \cdot \widetilde{W} = \widetilde{W}$. Hence

$$Y' := \gamma \mathbf{H}_Y(\mathbb{R}) \gamma^{-1} \cdot Y \subset \widetilde{W} .$$

But Y' is semi-algebraic and contains Y . In this situation Y' is contained in an irreducible algebraic subvariety of X^+ contained in \widetilde{W} and maximal for these properties. By our maximality assumption on Y it follows that $Y = Y'$. Hence $\gamma \mathbf{H}_Y(\mathbb{R}) \gamma^{-1}$ fixes Y and it follows that $\gamma \mathbf{H}_Y \gamma^{-1} = \mathbf{H}_Y$. \square

Assuming [Proposition 7.1](#), we finish the proof of [Theorem 4.28](#) for a pure Shimura variety by noticing that the normaliser of \mathbf{H}_Y is algebraic and contains Γ_W . Hence it contains $\overline{\Gamma_W} = \mathbf{G}$. As we supposed that \mathbf{G} is simple it follows that $\mathbf{G} = \mathbf{H}_Y$. Hence \mathbf{G} stabilizes \widetilde{W} and Y . Finally $Y = \widetilde{W} = X^+$ and $\pi(Y) = W = S$.

In general the adjoint group \mathbf{G} is a product of simple factors. One obtains a decomposition $(\mathbf{G}, X) = (\mathbf{G}_1, X_1) \times (\mathbf{G}_2, X_2)$ with \mathbf{G}_1 the Zariski-closure of the monodromy Γ_W . The same kind of arguments as in the simple case then show that

$$\pi(Y) = W = \pi(X_1^+ \times \{x_2\})$$

for some point $x_2 \in X_2$. \square

7.2. O-minimal arguments and hyperbolic geometry. Before proving [Proposition 7.1](#) we need, in addition to the notations of [Section 6](#), some precise notions of norm, distance and height.

We choose $\|\cdot\|_{\infty} : V_{\mathbb{R}} \longrightarrow \mathbb{R}$ a Euclidean norm which is K_{∞} -invariant. We still denote by $\|\cdot\|_{\infty} : \mathrm{End} V_{\mathbb{R}} \longrightarrow \mathbb{R}$ the operator norm associated to the norm $\|\cdot\|_{\infty}$ on $V_{\mathbb{R}}$. By restriction we also denote by $\|\cdot\|_{\infty} : \mathbf{G}(\mathbb{R}) \longrightarrow \mathbb{R}$ the function $\|\cdot\|_{\infty} \circ \rho$. As K_{∞} preserves the norm $\|\cdot\|_{\infty}$ on $V_{\mathbb{R}}$, the function $\|\cdot\|_{\infty} : \mathbf{G}(\mathbb{R}) \longrightarrow \mathbb{R}$ is K_{∞} -bi-invariant, in particular descends to a K_{∞} -invariant function $\|\cdot\|_{\infty} : X^+ \longrightarrow \mathbb{R}$.

Let $*$ be the adjunction on $\mathrm{End} V_{\mathbb{R}}$ associated to $\|\cdot\|_{\infty}$. The restriction to the Lie algebra $\mathrm{Lie}(\mathbf{G}(\mathbb{R}))$ of the bilinear form $(u, v) \mapsto \mathrm{tr}(u^*v)$ on $\mathrm{End} V_{\mathbb{R}}$ defines a $\mathbf{G}(\mathbb{R})$ -invariant Kähler metric g_{X^+} on X^+ . We denote by $d : X^+ \times X^+ \longrightarrow \mathbb{R}$ the associated distance.

We define the (multiplicative) height function $H : \mathbf{G}(\mathbb{Z}) \longrightarrow \mathbb{R}$ as the restriction of the height function

$$\forall \varphi \in \mathrm{End} V_{\mathbb{Z}}, \quad H(\varphi) = \max(1, \|\varphi\|_{\infty}) .$$

To prove [Proposition 7.1](#) we introduce the set

$$\Theta(Y) := \{g \in \mathbf{G}(\mathbb{R}) : \dim(gY \cap \pi^{-1}W \cap \mathcal{F}) = \dim(Y)\} ,$$

where \mathcal{F} is a fundamental set for the action of Γ on X^+ as in [Theorem 6.2](#).

[Theorem 6.2](#) implies that $\Theta(Y)$ is definable in $\mathbb{R}_{\text{an,exp}}$. This relies on the fact that the dimension function is a well-defined definable function in any o-minimal theory [[vdD98](#)].

The inclusion $gY \subset \pi^{-1}(W)$ holds for any $g \in \Theta(Y)$. This follows from the inclusion $gY \cap \mathcal{F} \subset \pi^{-1}(W)$ and analytic continuation.

Lemma 7.4.

$$\Theta(Y) \cap \Gamma = \{\gamma \in \Gamma / \gamma^{-1}\mathcal{F} \cap Y \neq \emptyset\} .$$

Moreover for any $\gamma \in \Theta(Y) \cap \Gamma$ the translate γY is a maximal irreducible algebraic subvariety of $\pi^{-1}(W)$

Proof. The Γ -invariance of $\pi^{-1}(W)$ implies:

$$\begin{aligned} \Theta(Y) \cap \Gamma &= \{\gamma \in \Gamma / \dim(\gamma Y \cap \pi^{-1}W \cap \mathcal{F}) = \dim(Y)\} \\ &= \{\gamma \in \Gamma / \dim(Y \cap \gamma^{-1}\mathcal{F}) = \dim(Y)\} . \end{aligned}$$

As \mathcal{F} is open in X^+ the conditions $\dim(Y \cap \gamma^{-1}\mathcal{F}) = \dim(Y)$ and $\gamma^{-1}\mathcal{F} \cap Y \neq \emptyset$ are the same. The first part of the lemma follows. The second part follows from the inclusion $\gamma Y \subset \pi^{-1}(W)$ obtained by analytic continuation as above and the maximality of Y among the irreducible algebraic subvarieties of X^+ contained in $\pi^{-1}(W)$. \square

The heart of the proof of [Proposition 7.1](#) is the following statement. For every positive real number T we define

$$N_Y(T) := \{\gamma \in \Gamma / Y \cap \gamma^{-1}\mathcal{F} \neq \emptyset \text{ and } H(\gamma) \leq T\} .$$

Theorem 7.5. *There exists positive real numbers a and $c(Y)$ such that for T large enough*

$$N_Y(T) \geq c(Y)T^a .$$

Indications on the proof of [Theorem 7.5](#) will be given in the next section. For now let us show how it implies [Proposition 7.1](#).

Proof that [Theorem 7.5](#) implies [Proposition 7.1](#). First notice that if B is a semi-algebraic block of $\Theta(Y)$ containing an element $\gamma \in \Theta(Y) \cap \Gamma$ then

$$B \subset \gamma \cdot \text{Stab}_G(Y) .$$

Indeed if U_γ is an open semi-algebraic subset of B containing γ then $U_\gamma \cdot Y$ is semi-algebraic contained in $\pi^{-1}(W)$ and contains the maximal algebraic γY of $\pi^{-1}(W)$. Hence $U_\gamma \cdot Y = \gamma Y$. For $b \in B$ one can construct a connected semi-algebraic set $U(\gamma, b)$ of B containing γ and b . The same argument shows that

$$\gamma Y = bY = B \cdot Y .$$

Applying the block version [Theorem 5.14](#) of Pila-Wilkie's counting theorem, we obtain positive real numbers b_1 and b_2 such that for T sufficiently large, there exists a block B in $\Theta(Y)$ such that

$$|\{\gamma \in B \cap \Gamma, H(\gamma) \leq T^{b_1}\}| \geq T^{b_2} .$$

If we fix $\gamma_0 \in B \cap \Gamma$ the previous discussion shows that the subset $\gamma_0^{-1} \cdot (B \cap \Gamma) \subset \text{Stab}_{\mathbf{G}}(Y)$ contains at least T^{b_2} elements. It follows that $\text{Stab}_{\mathbf{G}}(Y) \cap \Gamma$ is infinite. Hence the algebraic subgroup of \mathbf{G} generated by $\text{Stab}_{\mathbf{G}}(Y) \cap \Gamma$ is positive dimensional. This finishes the proof that [Theorem 7.5](#) implies [Proposition 7.1](#). \square

7.3. An algebraic curve of X^+ meets many fundamental sets.

Proof of [Theorem 7.5](#). [Theorem 7.5](#), which is the technical heart of the proof of [Theorem 4.28](#), is a statement in hyperbolic geometry. We have to show that an irreducible algebraic subvariety Y of X^+ cuts “many” Γ -translates of \mathcal{F} . Hence we can assume that Y is the intersection C of an irreducible algebraic curve \hat{C} of \hat{X} with X^+ .

The following comparisons between the norm and the distance on X^+ on the one hand, the norm and the height on the other hand, are crucial (if easy):

Lemma 7.6. (i) *For any $g \in \mathbf{G}(\mathbb{R})$ the following inequality holds:*

$$(7.1) \quad \log \|g\|_{\infty} \leq d(g \cdot x_0, x_0) .$$

(ii) *There exists a positive number B and a positive integer N such that:*

$$(7.2) \quad \forall \gamma \in \mathbf{G}(\mathbb{Z}), \quad \forall u \in \gamma \mathcal{F}, \quad H(\gamma) \leq B \cdot \|u\|_{\infty}^N .$$

We also have at our disposal a lower bound for the volume of complex-analytic subvariety of X^+ due to Hwang and To [[HwTo02](#)]. Let us denote by Vol_C the area on C for the restriction of the metric g_X to C . For a positive real number R we denote by $B(x_0, R)$ the geodesic ball of X^+ of center x_0 and radius R .

Theorem 7.7. *Let C be a complex analytic curve in X^+ . For any point $x_0 \in C$ there exist positive constants a, b such that for any positive real number R one has :*

$$(7.3) \quad \text{Vol}_C(C \cap B(x_0, R)) \geq a \exp(b \cdot R) .$$

The key lemma for the proof of [Theorem 7.5](#) is then the following upper-bound for the volume of an algebraic curve (the proof uses the full geometry of toroidal compactifications):

Lemma 7.8. *There exists a constant $A > 0$ such that for any algebraic curve $C \subset X^+$ of degree d we have the bound*

$$(7.4) \quad \text{Vol}_C(C \cap \mathcal{F}) \leq A \cdot d .$$

With all these ingredients we show [Theorem 7.5](#) as follows. Let T be a positive real number. Let us define

$$(7.5) \quad \begin{aligned} C(T) &:= \{u \in C \text{ and } \|u\|_{\infty} \leq T\} \\ &= \bigcup_{\substack{\gamma \in \Gamma \\ \gamma \mathcal{F} \cap C \neq \emptyset}} \{u \in \gamma \mathcal{F} \cap C \text{ and } \|u\|_{\infty} \leq T\} \end{aligned}$$

It follows from the [\(7.2\)](#) that

$$(7.6) \quad C(T) \subset \bigcup_{\substack{\gamma \in \Gamma, \gamma \mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T^N}} \{u \in \gamma \mathcal{F} \cap C\} .$$

Taking volumes:

$$(7.7) \quad \text{Vol}_C(C(T)) \leq \sum_{\substack{\gamma \in \Gamma, \gamma \mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T^N}} \text{Vol}_C(\mathcal{F} \cap \gamma^{-1}C) ,$$

hence

$$(7.8) \quad \text{Vol}_C(C(T)) \leq \sum_{\substack{\gamma \in \Gamma, \gamma \mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T^N}} \text{Vol}_C(\mathcal{F} \cap \gamma^{-1}C) .$$

Notice that all the curves $\gamma^{-1}C$, $\gamma \in \mathbf{G}(\mathbb{Z})$, have the same degree as algebraic curves. Hence it follows from (7.4) that

$$(7.9) \quad \text{Vol}_C(C(T)) \leq (A \cdot d) \cdot N_C(B \cdot T^N) .$$

Observe that Part (i) of [Lemma 7.6](#) implies that $C \cap B(x_0, \log T) \subset C(T)$. Thus:

$$(7.10) \quad \text{Vol}_C(C \cap B(x_0, \log T)) \subset \text{Vol}_C C(T) .$$

Using inequality (7.9) and [Theorem 7.7](#) it follows that

$$aT^b \leq A \cdot d \cdot N_C(B \cdot T^N) .$$

This finishes the proof of [Theorem 7.5](#). □

8. THE TWO MAIN STEPS IN THE PROOF OF THE ANDRÉ-OORT CONJECTURE.

The following two results are instrumental in the Pila-Zannier strategy for proving the André-Oort conjecture.

The first one, proven in [\[U14\]](#), is valid for any pure Shimura variety. It is geometric as it deals with positive dimensional special subvarieties.

Theorem 8.1. *Let W be a Hodge generic subvariety of a pure connected Shimura variety S . If $S = S_1 \times S_2$ is a product of connected Shimura varieties, we assume that W is not of the form $W = S_1 \times W_2$ for a subvariety W_2 of S_2 .*

Then the union of weakly special positive dimensional subvarieties contained in W is not Zariski-dense in W . In particular the union of positive dimensional special subvarieties contained in W is not Zariski-dense in W .

The second result is arithmetic in nature as it deals with special points. We restrict to the case $S = \mathcal{A}_g$. For $x \in \mathcal{A}_g$ we denote by A_x the principally polarized abelian variety parametrized by x and d_x the absolute value of the discriminant of the center of the ring of endomorphisms of A_x .

Theorem 8.2. *Let $W \subset \mathcal{A}_g$ be an algebraic subvariety. There exists a constant $C := C(g, W)$ with the following property. Let x be a special point of \mathcal{A}_g contained in W . If $d_x \geq C$ then there exists a positive dimensional special subvariety Z_x of \mathcal{A}_g contained in W and containing x .*

The proofs of [Theorem 8.1](#) and [Theorem 8.2](#) are sketched in the next sections. For now let us show how the André-Oort conjecture for \mathcal{A}_g follows from them.

Proof that [Theorem 8.1](#) and [Theorem 8.2](#) imply [Theorem 2.8](#). Let $W \subset \mathcal{A}_g$ be a closed irreducible subvariety containing a Zariski-dense set Σ of special points. We want to show that W is special. Let S be the smallest special subvariety of \mathcal{A}_g containing W . Hence W is Hodge generic in S .

For each point $x \in \Sigma$ let W_x be a special subvariety of S containing x , contained in W , and maximal for these properties. As there exist only finitely many special points x in \mathcal{A}_g with d_x smaller than a given constant, [Theorem 8.2](#) implies that for all but a finite number of points $x \in \Sigma$, the special subvariety W_x is positive dimensional. Hence the union of positive dimensional special subvarieties contained in W is Zariski-dense in W . Notice this finishes the proof if W is a curve.

By [Theorem 8.1](#), it follows that S is a product $S_1 \times S_2$ of Shimura varieties and W is of the form $S_1 \times W_2$, with $W_2 \subset S_2$ a closed subvariety. As special points are Zariski-dense in W , they are also Zariski-dense in W_2 .

Replacing W by W_2 , S by S_2 and arguing by induction on the dimension of W we are done. \square

8.1. Proof of [Theorem 8.1](#).

We follow [[U14](#)], to which we refer for details. Let $\mathcal{E}(W)$ be the set of weakly special subvarieties contained in W . For a positive integer r we denote by $\mathcal{E}_r(W)$ the subset of $\mathcal{E}(W)$ consisting of weakly special subvarieties of real dimension r . Let d be the biggest r such that $\mathcal{E}_r(W)$ is non-empty.

It follows from the description of weakly special subvarieties that there exist a semi-simple group $\mathbf{H}_{\mathbb{R}}$ of $\mathbf{G}_{\mathbb{R}}$ and $z_0 \in \mathcal{F}$ such that $\pi(\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ \cdot z_0)$ is a weakly special subvariety of W of dimension d . Without loss of generality we can assume that $\mathbf{H}_{\mathbb{R}}$ has no compact simple real factor: $\mathbf{H}_{\mathbb{R}} = \mathbf{H}_{\mathbb{R}}^{\text{nc}}$.

Let us define

$$B_{\mathbf{H}_{\mathbb{R}}} := \{(t, z) \in \mathbf{G}(\mathbb{R}) \times \mathcal{F}, \quad \pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z) \subset W\}.$$

By analytic continuation the set $B_{\mathbf{H}_{\mathbb{R}}}$ can also be described as:

$$B_{\mathbf{H}_{\mathbb{R}}} := \{(t, z) \in \mathbf{G}(\mathbb{R}) \times \mathcal{F}, \quad \pi|_{\mathcal{F}}(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z \cap \mathcal{F}) \subset W\}.$$

As $\pi|_{\mathcal{F}}$ is definable in $\mathbb{R}_{\text{an,exp}}$ (see [Theorem 6.2](#)) and W is algebraic, it follows that $B_{\mathbf{H}_{\mathbb{R}}}$ is a definable subset of $\mathbf{G}(\mathbb{R}) \times \mathcal{F}$.

Lemma 8.3. *Let $(t, z) \in B_{\mathbf{H}}$. Then $\pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z)$ is a weakly special subvariety of W .*

Proof. Let $(t, z) \in B_{\mathbf{H}_{\mathbb{R}}}$. It follows from the definition of $B_{\mathbf{H}_{\mathbb{R}}}$ that $t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z$ is a semi-algebraic subset of X^+ whose projection $\pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z)$ is contained in W . On the other hand the real dimension of $t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z$ is at least the dimension of $\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ \cdot z_0$, with equality if and only if $\text{Stab}_{\mathbf{G}(\mathbb{R})}(z) \cap t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1}$ is a maximal compact subgroup of $t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1}$.

Let Y be an irreducible algebraic subvariety of X^+ , containing $t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z$, such that $\pi(Y) \subset W$, and maximal for these properties. By the Ax-Lindemann [Theorem 4.28](#), $\pi(Y)$ is weakly special. It follows from the definition of d that

$$\dim(\pi(Y)) \leq d = \dim(\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ \cdot z_0) \leq \dim(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^+ t^{-1} \cdot z) \leq \dim(\pi(Y)) .$$

Hence $\pi(Y) = \pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z)$, and $\pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z)$ is weakly special. \square

Lemma 8.4. *The set $C(\mathbf{H}_{\mathbb{R}}, W)$ of conjugacy classes $t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+t^{-1}}$, $t \in \mathbf{G}(\mathbb{R})$, for which there exists $z \in \mathcal{F}$ satisfying $\pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z) \subset W$, is finite.*

Proof. Consider the map $\psi : B_{\mathbf{H}_{\mathbb{R}}} \rightarrow \mathbf{G}(\mathbb{R})/N_{\mathbf{G}(\mathbb{R})}(\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+})$ deduced from the projection on the first factor. Hence $C(\mathbf{H}_{\mathbb{R}}, W)$ is in bijection with $\psi(B_{\mathbf{H}_{\mathbb{R}}})$. As $B_{\mathbf{H}_{\mathbb{R}}}$ is definable and ψ is algebraic, the image $\psi(B_{\mathbf{H}_{\mathbb{R}}})$ is definable. Moreover if $(t, z) \in B_{\mathbf{H}_{\mathbb{R}}}$ then $\pi(t\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z)$ is weakly special by [Lemma 8.3](#). From the description of weakly special subvarieties there exists a \mathbb{Q} -algebraic subgroup $\mathbf{H}_t \subset \mathbf{G}$ such that $\mathbf{H}_{t, \mathbb{R}}^{\text{nc}} = t\mathbf{H}_{\mathbb{R}}t^{-1}$. As the set of \mathbb{Q} -algebraic subgroups of \mathbf{G} is countable, it follows that $C(\mathbf{H}_{\mathbb{R}}, W)$ is countable. Any countable set definable in some o-minimal structure is finite hence $C(\mathbf{H}_{\mathbb{R}}, W)$ is finite. \square

Lemma 8.5. *Under the hypotheses of [Theorem 8.1](#) the union $\bigcup_{V \in \mathcal{E}_d(W)} V$ of the weakly special subvarieties contained in W of maximal dimension d is not Zariski-dense in W .*

Proof. As $\mathbf{G}_{\mathbb{R}}$ has only finitely many conjugacy classes of semi-simple subgroups, there exists only finitely many (up to $\mathbf{G}(\mathbb{R})$ -conjugacy) subgroups $\mathbf{H}_{\mathbb{R}}$ of $\mathbf{G}_{\mathbb{R}}$ for which there exists $z_0 \in \mathcal{F}$ with $\pi(\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+} \cdot z_0) \in \mathcal{E}_d(W)$ and such that $\mathbf{H}_{\mathbb{R}} = \mathbf{H}_{\mathbb{R}}^{\text{nc}}$.

For such an $\mathbf{H}_{\mathbb{R}}$, there exists a semi-simple subgroup $\mathbf{H} \subset \mathbf{G}$ whose real base change is $\mathbf{H}_{\mathbb{R}}$ and the number of such \mathbf{H} is finite by [Lemma 8.4](#).

Let $\mathbf{H} \subset \mathbf{G}$ be such a subgroup.

If \mathbf{H} is a factor of \mathbf{G} then S decomposes as $S_1 \times S_2$ and any weakly special subvarieties of the form $\pi(\mathbf{H}(\mathbb{R})^{+} \cdot z)$ with $z \in \mathcal{F}$ is of the form $S_1 \times \{x_2\}$ for some $x_2 \in S_2$. The Zariski-closure of the union of weakly special subvarieties V of the form $\pi(\mathbf{H}(\mathbb{R})^{+} \cdot z)$ is $S_1 \times W'$, where W' denotes the Zariski-closure of the set of x_2 for which $S_1 \times \{x_2\} \subset W$. As W is not of the form $S_1 \times W'$, this union is not Zariski-dense in W .

If \mathbf{H} is not normal in \mathbf{G} , one shows the following:

Proposition 8.6. *Suppose \mathbf{H} is not normal in \mathbf{G} . Then the union of weakly special subvarieties of the form $\pi(\mathbf{H}_{\mathbb{R}}(\mathbb{R})^{+} \cdot z)$ is contained in a finite union $\bigcup_{1 \leq i \leq r} V_i$ of strict special subvarieties V_i of S .*

As W is Hodge generic, the intersection $W \cap \bigcup_{1 \leq i \leq r} V_i$ is not Zariski-dense in W .

This finishes the proof of [Lemma 8.5](#). \square

One concludes the proof of [Theorem 8.1](#) by induction on the dimension of the weakly special subvarieties of S contained in W . Let us indicate the argument.

Let $d_1 < d$ be the maximal dimension of a weakly special subvariety of W not contained in $\mathcal{E}_d(W)$. There exist a semi-simple subgroup $\mathbf{H}_{1, \mathbb{R}} = \mathbf{H}_{1, \mathbb{R}}^{\text{nc}}$ of $\mathbf{G}_{\mathbb{R}}$ and $z_1 \in \mathcal{F}$ such that $\pi(\mathbf{H}_{1, \mathbb{R}}(\mathbb{R})^{+} \cdot z_1) \subset W$ is of dimension d_1 and is not in $\mathcal{E}_d(W)$. Up to $\mathbf{G}(\mathbb{R})$ -conjugacy there are only finitely many possibilities for $\mathbf{H}_{1, \mathbb{R}}$. The proof of [Lemma 8.3](#) shows that if $(z, t) \in B_{\mathbf{H}_{1, \mathbb{R}}}$ and if $\pi(t\mathbf{H}_{1, \mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z)$ is not contained in $\mathcal{E}_d(W)$ then $\pi(t\mathbf{H}_{1, \mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z)$ is weakly special contained in W . The proof of [Lemma 8.4](#) shows that the set $C(\mathbf{H}_{1, \mathbb{R}}, W, \mathcal{E}_d(W))$ of conjugacy classes $t\mathbf{H}_{1, \mathbb{R}}(\mathbb{R})^{+t^{-1}}$, $t \in \mathbf{G}(\mathbb{R})$, such that there exists $z \in \mathcal{F}$ with $\pi(t\mathbf{H}_{1, \mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z) \subset W$ and $\pi(t\mathbf{H}_{1, \mathbb{R}}(\mathbb{R})^{+t^{-1}} \cdot z)$ does not belong to $\mathcal{E}_d(W)$, is finite. As in the proof of [Lemma 8.5](#) one concludes that the set of weakly special subvarieties of W of dimension at least d_1 is not Zariski-dense in W .

By decreasing induction on r one concludes that $\bigcup_{r \geq 0} \bigcup_{V \in \mathcal{E}_r(W)} V$ is not Zariski-dense in W . \square

8.2. Proof of [Theorem 8.2](#).

8.3. Heights of special points. In [Example 4.17](#) we define a $\overline{\mathbb{Q}}$ -bi-algebraic structure on any Shimura variety S whose pure part is of abelian type: special points are exactly the arithmetic bi-algebraic points. A crucial ingredient for applying the Pila-Wilkie's [Theorem 5.10](#) in this context consists in showing that for any special point $x \in S$, the fiber $\pi^{-1}(x)$ consists of algebraic points of X^+ defined over extensions of uniformly bounded degree over \mathbb{Q} ; moreover one controls the height of points of $\pi^{-1}(x) \cap \mathcal{F}$. For simplicity let us state the result for $S = \mathcal{A}_g$ (the first part is classical, the second is due to Pila and Tsimerman [[PT14](#)]):

Theorem 8.7. (1) *The uniformization $\pi : \mathcal{H}_g \rightarrow \mathcal{A}_g = \mathbf{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g$ can be normalized in such a way that the coordinates of the inverse images by π of CM-points of \mathcal{A}_g lie in algebraic extensions of uniformly bounded degree.*
 (2) *One can choose the fundamental set \mathcal{F} in [Theorem 6.2](#) for the action of $\mathbf{Sp}(2g, \mathbb{Z})$ on \mathcal{H}_g , and positive real numbers $\alpha = \alpha(g)$ and $c_1 = c_1(g)$ such that if $x \in \mathcal{A}_g$ is a CM-point parametrizing the abelian variety A_x and if $\tilde{x} \in \mathcal{F} \cap \pi^{-1}(x)$ then*

$$H(\tilde{x}) \leq c_1 \cdot d_x^\alpha ,$$

where H denotes the canonical multiplicative height on $M_g(\overline{\mathbb{Q}}) \cap \mathcal{H}_g \subset \overline{\mathbb{Q}}^{g^2}$ and d_x is the absolute value of the discriminant of the center of the ring of endomorphisms of A_x .

Let us write explicitly the case of $Y(1)$. Let $\tau \in \mathcal{F}$ where \mathcal{F} denotes the fundamental set defined in [Equation \(6.1\)](#). If the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ has complex multiplication then τ satisfies a reduced equation $aX^2 + bX + c = 0$ for integers a , b and c such that $|b| \leq a \leq c$. In particular the coordinates of

$$\tau = -\frac{b}{2a} + i \frac{\sqrt{4ac - b^2}}{2a}$$

lie in extensions of degree at most 2 of \mathbb{Q} . Moreover $\text{End}(E_\tau) = \mathbb{Z}[\tau]$ and the absolute value d_τ of the discriminant of $\text{End}(E_\tau)$ is $4ac - b^2$. With our conventions on the height:

$$H(\tau) = \max\left(H\left(\frac{b}{2a}\right), H\left(\frac{\sqrt{4ac - b^2}}{2a}\right)\right).$$

On the one hand $H\left(\frac{b}{2a}\right) = \max(|b|, 2|a|) = 2|a| \leq d_\tau$. On the other hand $\frac{\sqrt{4ac - b^2}}{2a}$ is a root of the integral polynomial $4a^2X^2 - d_\tau$ hence:

$$H\left(\frac{\sqrt{4ac - b^2}}{2a}\right) \leq \max(4a^2, d_\tau) \leq \frac{4}{3}d_\tau ,$$

where the last inequality follows by noticing that

$$3a^2 \leq 4ac - b^2 = d_\tau$$

in view of the inequalities satisfied by (a, b, c) .

Finally we obtain $H(\tau) \leq \frac{4}{3}d_\tau$.

The main ingredient in the proof of [Theorem 8.2](#) is the following result of Tsimerman [[Tsi](#)], and based on the results of Andreatta-Goren-Howard-Madapusi Pera [[AGHM](#)] and Yuan-Zhang [[YuZh](#)] on the Colmez conjecture:

Theorem 8.8. *Let g be a positive integer. There exist positive real numbers $\beta = \beta(g)$ and $c_2 = c_2(g)$ with the following property. For any special point x of \mathcal{A}_g one has:*

$$(8.1) \quad |\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| = [\mathbb{Q}(x) : \mathbb{Q}] \geq c_2 \cdot d_x^\beta.$$

We will sketch the proof of [Theorem 8.8](#) in [Section 9](#). For now let us show how [Theorem 8.8](#) and o-minimal techniques imply [Theorem 8.2](#).

Let $W \subset \mathcal{A}_g$ be as in [Theorem 8.2](#). Replacing if necessary W by the Zariski-closure of its set of special points, we can assume that the special points are Zariski-dense in W . In particular W is an algebraic subvariety of \mathcal{A}_g defined over $\overline{\mathbb{Q}}$. Replacing W by the union of its conjugate under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we can assume without loss of generality that W is defined over \mathbb{Q} .

Let $\mathcal{F} \subset \mathcal{H}_g$ be a semi-algebraic fundamental set for the action of $\mathbf{Sp}(2g, \mathbb{Z})$ on \mathcal{H}_g such that $\pi|_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{A}_g$ is definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$ (see [Theorem 6.2](#)). Hence the set $\widetilde{W}_{\mathcal{F}} := \pi^{-1}(W) \cap \mathcal{F}$ is definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$.

Let $x \in W$ be a special point. Notice that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $d_{\sigma \cdot x} = d_x$. It follows from [Theorem 8.7](#) that any point y in $\pi^{-1}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x) \cap \widetilde{W}_{\mathcal{F}}$ is defined in an extension of \mathbb{Q} of uniformly bounded degree and satisfies

$$(8.2) \quad H(y) \leq c_1 \cdot d_x^\alpha .$$

It follows from Pila-Wilkie [Theorem 5.10](#) and the inequalities [\(8.1\)](#) and [\(8.2\)](#) that if d_x is sufficiently large, there exists a semi-algebraic subset $Y \subset \widetilde{W}_{\mathcal{F}}$ of positive dimension, containing one point y in $\pi^{-1}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x)$. Let Z be an irreducible algebraic subvariety of \mathcal{H}_g contained in $\pi^{-1}(W)$ containing y , and maximal for these properties. Hence Z is positive dimensional. Moreover it follows from the Ax-Lindemann [Theorem 4.28](#) that $\pi(Z)$ is a special subvariety of \mathcal{A}_g contained in W and containing a Galois conjugate $\sigma \cdot x$ of x . As W is defined over \mathbb{Q} the positive dimensional special subvariety $\sigma^{-1}(\pi(Z))$ of \mathcal{A}_g is contained in W and contains x . \square

9. LOWER BOUNDS FOR GALOIS ORBITS OF CM-POINTS

9.1. Class groups for tori and reciprocity morphisms.

9.1.1. *Class groups for tori.* Let \mathbf{M} be an algebraic torus over \mathbb{Q} . We denote by $K_{\mathbf{M}}^m$ the unique maximal compact subgroup of $\mathbf{M}(\mathbb{A}_f)$.

Definition 9.1. *The absolute class group of \mathbf{M} is the finite group*

$$h_{\mathbf{M}} := \mathbf{M}(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{A}_f) / K_{\mathbf{M}}^m .$$

If $K_{\mathbf{M}} \subset \mathbf{M}(\mathbb{A}_f)$ is an arbitrary compact open subgroup we define the associated relative class group as the finite group

$$h_{\mathbf{M}, K_{\mathbf{M}}} := \mathbf{M}(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{A}_f) / K_{\mathbf{M}} ,$$

so that $h_{\mathbf{M}} = h_{\mathbf{M}, K_{\mathbf{M}}^m}$.

Notice that if F is a number field and $\mathbf{R}_F := \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ then $h_{\mathbf{R}_F}$ is equal to the classical class group h_F of the ring of integers \mathcal{O}_F of F .

9.1.2. Reciprocity morphisms. The notations are those of [Section 2.4](#). Let $x = [x, 1]$ be a CM-point of S . The Mumford-Tate group \mathbf{MT}_x is a \mathbb{Q} -torus \mathbf{T} and $(\mathbf{T}, \{x\})$ is a Shimura sub-datum of (\mathbf{G}, X) . Let $K_{\mathbf{T}} := K \cap \mathbf{T}(\mathbb{A}_f)$. Then

$$\text{Sh}_{K_{\mathbf{T}}}(\mathbf{T}, \{x\}) = \mathbf{T}(\mathbb{Q}) \backslash (\{x\} \times \mathbf{T}(\mathbb{A}_f) / K_{\mathbf{T}}) \subset \text{Sh}_K(\mathbf{G}, X)$$

is a zero-dimensional subvariety, of cardinality $h_{\mathbf{T}, K_{\mathbf{T}}}$, defined over the reflex field $E := E(\mathbf{T}, \{x\})$ of $(\mathbf{T}, \{x\})$.

The theory of Complex Multiplication gives a surjective morphism, called the *reciprocity morphism*

$$r := r(\mathbf{T}, \{x\}) : \mathbf{R}_E \longrightarrow \mathbf{T} .$$

9.2. Faltings height. Let K be a number field and A_K an abelian variety over K of dimension g . Let $p : \mathbf{A} \rightarrow \text{Spec}(\mathcal{O}_K)$ be its Néron model and $\epsilon : \text{Spec}(\mathcal{O}_K) \rightarrow \mathbf{A}$ its unit section. We denote by $\omega_{\mathbf{A}_K} := \epsilon^* \Omega_{\mathbf{A}/\text{Spec}(\mathcal{O}_K)}^g$.

Every field embedding $\sigma : K \rightarrow \mathbb{C}$ defines a Hermitian metric on

$$\omega_{\mathbf{A}_K, \sigma} := H^0 \left(A_{\sigma}(\mathbb{C}), \Omega_{A_{\sigma}(\mathbb{C})}^g \right)$$

given on any section $\alpha \in H^0 \left(A_{\sigma}(\mathbb{C}), \Omega_{A_{\sigma}(\mathbb{C})}^g \right)$ by

$$\|\alpha\|_{\sigma} := \left| \frac{1}{(2\pi)^g} \int_{A_{\sigma}(\mathbb{C})} \alpha \wedge \bar{\alpha} \right| .$$

We denote by $\overline{\omega_{\mathbf{A}_K}}$ the metrized line bundle $(\omega_{\mathbf{A}_K}, \|\cdot\|_{\sigma})$.

The Faltings height of A is defined as

$$h_F(A) := \frac{\text{deg}_{\text{Ar}}(\overline{\omega_{\mathbf{A}_K}})}{[K : \mathbb{Q}]} ,$$

where deg_{Ar} denotes the Arakelov degree (see for example [\[HS00, p.247\]](#)).

If A has semi-stable reduction over K the Faltings height $h_F(A)$ does not change under base change to a finite extension of K . If A has good reduction over K there exists a finite extension L of K such that $\omega_{\mathbf{A}_L} \simeq \mathcal{O}_L$. Choosing a Néron differential $\omega \in \Gamma(\mathbf{A}_L, \omega_{\mathbf{A}_L})$, one then obtains

$$h_F(A) = -\frac{1}{[L : \mathbb{Q}]} \sum_{\sigma: L \rightarrow \mathbb{C}} \log \|\omega\|_{\sigma} .$$

The Faltings height can be interpreted as a height on the set $\mathcal{A}_g(\overline{\mathbb{Q}})$ of algebraic points of \mathcal{A}_g . If $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$ parameterizes the abelian variety A_x one define $h_F(x) = h_F(A_x)$. Following [\[Fal83\]](#) this function satisfies the Northcott property: given d and T positive real integers the set

$$N_{d,T}(\mathcal{A}_g) := \{x \in \mathcal{A}_g(\overline{\mathbb{Q}}), [\mathbb{Q}(x) : \mathbb{Q}] \leq d \text{ and } h_F(x) \leq T\}$$

is finite. If the Faltings height h_F were uniformly bounded on CM-points of \mathcal{A}_g we would directly obtain that the fields of definition of these points have a degree tending to infinity. This type of argument is used in the proof of the Manin-Mumford conjecture

to obtain a lower bound on Galois-orbits of torsion points of an abelian variety, as these are the points of canonical height zero. For \mathcal{A}_g it is not true that the Faltings height is uniformly bounded but a direct consequence of the Colmez conjecture on average, (which we describe in the next section) is the following version of the inequality (2.3):

Theorem 9.2. *Let g be a positive integer and ϵ a positive real number. There exists a positive real number $c_3 = c_3(g, \epsilon)$ with the following property. Let E be a CM-field of degree $2g$ with discriminant d_E . Let A be a g -dimensional abelian variety with complex multiplication by the ring of integers \mathcal{O}_E of E . Then*

$$h_F(A) \leq c_3 |d_E|^\epsilon .$$

9.3. Lower bounds for Galois orbits. Tsimerman's main result in [Tsi] is the deduction of the lower bound for the size of Galois orbits of special points [Theorem 8.8](#) from [Theorem 9.2](#) and the deep Isogeny Theorem of Masser and Wüstholz [[MaWü95](#)] (which is also the crux of an alternative proof of Mordell's conjecture):

Theorem 9.3. (*Masser-Wüstholz*) *Let g be a positive integer. There exist positive real numbers $\mu = \mu(g)$ and $c_4 = c_4(g)$ with the following property. Let A and B be two abelian varieties defined over a number field k . We suppose that A and B are $\overline{\mathbb{Q}}$ -isogenous. Then there exists a $\overline{\mathbb{Q}}$ -isogeny from A to B of degree N with*

$$N \leq c_4 \max(h_F(A), [k : \mathbb{Q}])^\mu .$$

Let us now sketch Tsimerman's argument:

Proof of [Theorem 8.8](#) using [Theorem 9.2](#) and [Theorem 9.3](#): Let Σ be the locus in \mathcal{A}_g of abelian varieties with complex multiplication by \mathcal{O}_E and fixed CM-type Φ . For all $x, y \in \Sigma$ the abelian varieties A_x and A_y are $\overline{\mathbb{Q}}$ -isogenous. On the other hand the cardinal of Σ is the cardinality of the class group of \mathcal{O}_E . As E is CM the class number formula gives $|\Sigma| \gg d_E^\gamma$ for an absolute constant $\gamma > 0$ for d_E sufficiently large.

Let us fix $x_0 \in \Sigma$. Let N be a positive integer. There exists $\delta > 0$ such that the number of $\overline{\mathbb{Q}}$ -isogenies with source A_{x_0} of degree at most N is bounded above by N^δ for N sufficiently large. Let η be a positive real number such that $\eta < \frac{\gamma}{\delta}$. Taking $N = d_E^\eta$ and d_E large enough it follows that there exists $x \in \Sigma$ such that the minimal degree $d_{\min}(A_{x_0}, A_x)$ of a $\overline{\mathbb{Q}}$ -isogeny from A_{x_0} to A_x satisfies

$$d_{\min}(A_{x_0}, A_x) > d_E^\eta .$$

By the Masser-Wüstholz [Theorem 9.3](#) and the upper bound on the Faltings height given by [Theorem 9.2](#) it follows that:

$$d_E^\eta \leq c_4 \max(h_F(A_{x_0}), [k : \mathbb{Q}])^\mu \leq c_4 \max(c_3 d_E^\epsilon, [k : \mathbb{Q}])^\mu .$$

If we choose $\epsilon < \eta$ and d_E sufficiently large one obtains a constant c_2 depending only on g such that

$$[k : \mathbb{Q}] \geq c_2 d_E^{\eta/\mu} .$$

□

9.4. Colmez conjecture. The reference for this section is [Col93] and [Col98]. Let A be a simple abelian variety over \mathbb{C} , with complex multiplication and of dimension g . The field $E := \text{End}(A) \otimes \mathbb{Q}$ is CM with $[E : \mathbb{Q}] = 2g$. We suppose moreover that $\text{End } A = \mathcal{O}_E$. Let $\Phi \subset \text{Hom}(E, \mathbb{C})$ be the CM-type of A . Hence

$$\text{Lie}(A) = \bigoplus_{\sigma \in \Phi} \text{Lie}(A)_\sigma \quad ,$$

where $\text{Lie}(A)_\sigma$ is the subspace of $\text{Lie}(A)$ on which E acts through σ . Let K be a number field on which A is defined and has good reduction.

Let F be a number field. We denote by \mathcal{G}_F the Galois group $\text{Gal}(\overline{\mathbb{Q}}/F)$ and by $c \in \mathcal{G}_\mathbb{Q}$ the complex conjugation. Let $C(\mathcal{G}_\mathbb{Q}, \mathbb{C})$ be the complex vector space of locally constant complex functions on $\mathcal{G}_\mathbb{Q}$ and $C^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$ its subspace of central ones. Let $\mathbb{Q}^{\text{CM}} \subset \overline{\mathbb{Q}}$ be the extension of \mathbb{Q} generated by CM-fields. This is a Galois extension of \mathbb{Q} . We denote by $\text{CM}^0(\mathcal{G}_\mathbb{Q}, \mathbb{C}) \subset C^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$ the subspace of functions f such that $f(\sigma)$ depends only on the $\mathcal{G}_{\mathbb{Q}^{\text{CM}}}$ -conjugacy class of σ and such that $f(\sigma) + f(c\sigma)$ is independent of σ .

We define a Hermitian scalar product \langle, \rangle on $C(\mathcal{G}_\mathbb{Q}, \mathbb{C})$ by:

$$\forall \Theta_1, \Theta_2 \in C(\mathcal{G}_\mathbb{Q}, \mathbb{C}), \quad \langle \Theta_1, \Theta_2 \rangle := \frac{1}{|\mathcal{G}_\mathbb{Q}/\mathcal{G}_F|} \sum_{g \in \mathcal{G}_\mathbb{Q}/\mathcal{G}_F} \Theta_1(g) \overline{\Theta_2(g)} \quad ,$$

where F is any finite normal extension of \mathbb{Q} such that Θ_1 and Θ_2 depend only on residue classes modulo \mathcal{G}_F .

The set Art of Artin characters (i.e. characters of continuous finite dimensional complex representations of $\mathcal{G}_\mathbb{Q}$) is an orthonormal basis of $C^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$. Given any Artin character χ , we denote by $L(\chi, s)$ its L-function. One also checks that the set of Artin characters whose L-function does not vanish at 0 form an orthonormal basis of $\text{CM}^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$.

For $\Theta \in C(\mathcal{G}_\mathbb{Q}, \mathbb{C})$ we denote by Θ^0 its orthonormal projection

$$\Theta^0 = \sum_{\chi \in \text{Art}} \langle \Theta, \chi \rangle \chi$$

on $C^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$.

We also denote by $Z(\chi, s)$ the logarithmic derivative $L'(\chi, s)/L(\chi, s)$ and by $\mu_{\text{Art}}(\chi)$ the logarithm $\log f_\chi$ of the Artin conductor f_χ of χ . These functions admit local decompositions

$$\begin{aligned} \mu_{\text{Art}} &= \sum_{p \text{ prime}} \mu_{\text{Art}, p} \log p \quad , \\ \forall \text{Re}(s) > 1, \quad Z(\chi, s) &= - \sum_{p \text{ prime}} Z_p(\chi, s) \log p \quad . \end{aligned}$$

For any prime p , the local factor $Z_p(\chi, s)$ lies in $\mathbb{Q}(p^{-s})$. The function $Z(\chi, s)$ admits a holomorphic extension to \mathbb{C} and a functional equation.

Given a CM-type (E, Φ) we define the function $A_\Phi \in C(\mathcal{G}_\mathbb{Q}, \mathbb{C})$ by

$$A_\Phi(g) = \frac{|\Phi \cap g\Phi|}{[E : \mathbb{Q}]}$$

and denote by A_Φ^0 its projection on $C^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$. One checks that $A_\Phi^0 \in \text{CM}^0(\mathcal{G}_\mathbb{Q}, \mathbb{C})$. Colmez conjecture is the following:

Conjecture 9.4. *Let A be a complex abelian variety of CM-type (E, Φ) . Then:*

$$h_F(A) = Z(A_\Phi^0) - \frac{1}{2} \mu_{\text{Art}}(A_\Phi^0) = - \sum_{\chi \in \text{Art}} \langle A_\Phi, \chi \rangle \left(\frac{L'(\chi, 0)}{L(\chi, 0)} + \frac{1}{2} \mu_{\text{Art}}(\chi) \right) .$$

Notice that this conjecture implies that the height $h_F(A)$ depends on (E, Φ) only, which is proven in [Col93, Theor.0.3]. We will write $h_F(A) = h_F(\Phi)$ in the sequel.

Let F be the maximal totally real subfield of E , d_F its discriminant and $d_{E/F} := N_{E/F} d_E$ the relative discriminant of E over F . Let $\chi_{E/F}$ be the associated quadratic character of F . As noticed by Colmez, [Conjecture 9.4](#) simplifies if we average on the 2^g possible CM-types of E . It is this result which is proved by completely different methods by Andreatta-Goren-Howard-Madapusi Pera [AGHM] and Yuan-Zhang [YuZh] and which implies [Theorem 9.2](#):

Theorem 9.5. (*Colmez conjecture on average*)

$$\frac{1}{2^g} \sum_{\Phi} h_F(\Phi) = -\frac{1}{2} \frac{L'(\chi_{E/F}, 0)}{L(\chi_{E/F}, 0)} - \frac{1}{4} \log |d_{E/F} d_F| .$$

10. FURTHER DEVELOPMENTS: THE ANDRÉ-PINK CONJECTURE

In this section we briefly present a conjecture which is a special case of the Zilber-Pink conjecture and which is a family version of the Mordell-Lang conjecture in the context of (mixed) Shimura varieties. Instead of looking at the Zariski-closure of a set of special points, one looks at the Zariski closure of a subset of a (generalized) Hecke orbit in a (mixed) Shimura variety. The expectation is that components of this Zariski closure are weakly special ([An89], Problem 3 and [Pink05], Conjecture 1.6):

Conjecture 10.1 (André-Pink). *Let S be a mixed Shimura variety over \mathbb{C} and $\Lambda \subset S$ be the generalized Hecke orbit of a point x of S . Let Z be an irreducible subvariety of S such that $Z \cap \Lambda$ is Zariski-dense in Z . Then Z is a weakly special subvariety.*

The André-Pink [Conjecture 10.1](#), which implies the classic Mordell-Lang conjecture (see [Pink05, Theor.5.4]), is open in general. Pink obtained results on this conjecture under certain quite strong assumptions. We will not touch upon Pink's work and refer to Pink's (excellent) exposition in [Pink05].

In [Orr15] Orr has obtained a fairly general result when $S = \mathcal{A}_g$, using the techniques explained in this text. The primary aim of this section is to explain Orr's result and give an idea of its proof. In the case of \mathcal{A}_g the conjecture [Conjecture 10.1](#) becomes the following.

Conjecture 10.2. *Let Λ be the isogeny class of a point $x \in \mathcal{A}_g(\mathbb{C})$. Let Z be an irreducible closed subvariety of \mathcal{A}_g such that $Z \cap \Lambda$ is Zariski-dense in Z . Then Z is a weakly special subvariety of \mathcal{A}_g .*

In the case where x is Galois-generic, Pink in [Pink05], proves that [Conjecture 10.2](#) follows from results of Clozel, Oh and Ullmo ([COU01]) on equidistribution of Hecke orbits. In the case where x is a special point, [Conjecture 10.2](#) is a special case of the André-Oort conjecture, known for \mathcal{A}_g . In [Orr15] Orr proves the following.

Theorem 10.3 (Orr). *Let x , Λ and Z be as in [Conjecture 10.2](#). Then:*

- (1) *There exists a special subvariety $S \subset \mathcal{A}_g$, isomorphic to a product $S_1 \times S_2$ of connected Shimura varieties, such that $\dim(S_1) > 0$ and*

$$Z = S_1 \times Z_2 \subset S$$

where Z_2 is a closed subvariety of S_2 .

- (2) *If Z is a curve, then Z is weakly special.*

Sketch of proof: It is clear that (2) follows from (1).

The strategy of Orr's proof is again a combination of lower bounds for Galois orbits with Pila-Wilkie [Theorem 5.14](#), the Ax-Lindemann [Theorem 4.28](#) and Ullmo's [Theorem 8.1](#). Note that elaboration of suitable lower bounds for the Galois orbits makes essential use of the Masser-Wüstholz [Theorem 9.3](#).

Let x be a point of $\mathcal{A}_g(\mathbb{C})$, Λ its isogeny class and Z an irreducible subvariety of \mathcal{A}_g such that $Z \cap \Lambda$ is Zariski-dense. Let again $\pi: \mathcal{H}_g \rightarrow \mathcal{A}_g$ be the uniformization map and \mathcal{F}_g the classical Siegel fundamental domain. Let

$$\tilde{Z} = \mathcal{F}_g \cap \pi^{-1}Z \text{ and } \tilde{\Lambda} = \mathcal{F}_g \cap \pi^{-1}\Lambda.$$

Given a point x of $\mathcal{A}_g(\mathbb{C})$, we let A_x be the abelian variety associated to x . We define the complexity of t in Λ as the minimal degree of an isogeny between A_x and A_t . Similarly, we define the complexity of a point t in $\tilde{\Lambda}$. The height of a matrix in $M_n(\mathbb{Q})$ is defined as the maximum of heights of its entries.

Orr proves the following:

Proposition 10.4 ([\[Orr15\]](#), Proposition 3.2). *Let Z be a subvariety of \mathcal{A}_g and \tilde{x} a point in \mathcal{F}_g . Let $\epsilon > 0$. There exists a positive real number $c = c(Z, \tilde{x}, \epsilon)$ such that for every $n \geq 1$, there is a collection of at most cn^ϵ semi-algebraic blocks $W_i \subset \tilde{W}$ such that all points of $\tilde{Z} \cap \tilde{\Lambda}$ of complexity $\leq n$ are contained in $\bigcup_i W_i$.*

The idea of the proof is to construct a certain definable subset Y of $\mathbf{GL}_{2g}(\mathbb{R})$, show that it contains 'a lot' of points of $\mathbf{GL}_{2g}(\mathbb{Q})$ up to height n and then apply Pila-Wilkie [Theorem 5.14](#) to it.

The crucial lemma is the following which is of independent interest.

Lemma 10.5 ([\[Orr15\]](#), Lemma 3.3). *There exist constant c, k depending only on g and \tilde{x} such that: for any $\tilde{t} \in \tilde{Z} \cap \tilde{\Lambda}$ of complexity n , there is a rational matrix $\gamma \in Y$ such that $\gamma\tilde{x} = \tilde{t}$ and the height of γ is at most cn^k .*

On the other hand, Masser-Wüstholz theorem gives a polynomial (in the complexity) lower bound on the size of the Galois orbits of the points of Λ .

This implies, via Pila-Wilkie theorem and Ax-Lindemann, that positive dimensional weakly special subvarieties are dense in Z . Ullmo's [Theorem 8.1](#) then implies the conclusion of [Theorem 10.3](#). \square

[Conjecture 10.1](#) for the mixed Shimura variety \mathfrak{A}_g becomes (following Gao [\[Gao17\]](#)):

Conjecture 10.6. *Let B be an irreducible algebraic variety over \mathbb{C} and let $\pi: \mathfrak{A} \rightarrow B$ be an abelian scheme. Let $b \in B(\mathbb{C})$ and Σ be a finitely generated subgroup of \mathfrak{A}_b . Define*

$$\Lambda := \{t \in \mathfrak{A}(\mathbb{C}) : \exists n \in \mathbb{N} \text{ and an isogeny } f: \mathfrak{A}_b \rightarrow \mathfrak{A}_{\pi(t)} \text{ such that } nt \in f(\Sigma)\}.$$

If Z is an irreducible subvariety of \mathcal{A} dominating B such that $Z \cap \Lambda$ is Zariski dense in Z , then

- (1) Z is the translate of an abelian subscheme of \mathcal{A}/B by a torsion section and then by a constant section;
- (2) $i(B)$ is a weakly special subvariety of \mathcal{A}_g for the morphism $i : B \rightarrow \mathcal{A}_g$ induced by the abelian scheme \mathcal{A}/B .

Unlike for $S = \mathcal{A}_g$ in which case Orr has obtained a fairly complete result, there is still a lot to do on [Conjecture 10.6](#). The only cases known are for $\dim B = 0$ (this is the Mordell-Lang conjecture); $\dim B = 1$ and $\Sigma = \{0\}$, or $\dim B = 1$ and $\dim Z = 1$ (see [\[Gao17\]](#)).

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