LOCAL RIGIDITY FOR COMPLEX HYPERBOLIC LATTICES AND HODGE THEORY

B. KLINGLER

Contents

B.K. is supported by IAS Von Neumann fellowship NSF grant DMS 0635607.

1. Introduction

The main open question concerning lattices of Lie groups is certainly the study of complex hyperbolic lattices and their finite dimensional representations. Let $n > 1$ be an integer. Let V denote the $(n+1)$-dimensional $\mathbb{C}$-vector space and let h denote the Hermitian form $h(\mathbf{z}, \mathbf{w}) = z_0\overline{w_0} + \cdots + z_{n-1}\overline{w_{n-1}} - z_n\overline{w_n}$ on V. We denote by $\mathbf{SU}(n,1)$ the real algebraic group $\mathbf{SU}(V,h)$. Let $\Gamma \overset{i}{\hookrightarrow} \mathbf{SU}(n,1)(\mathbb{R}) = SU(n,1)$ be a lattice (discrete subgroup of finite co-volume). What are the finite dimensional Γ-modules ? What can we say about the algebraic structure of Γ ?

Recall that these questions are completely understood if we replace $SU(n,1)$ by a simple real Lie group L of real rank $r > 1$ (respectively by $Sp(n,1)$ or the exceptional group F_4^{-20}). Let Γ be a lattice in L and $\rho : \Gamma \longrightarrow G = \mathbf{G}(k)$ be an *unbounded* morphism of Γ into the group of k-points of a simple algebraic k-group $\mathbf{G}$, k local field. Then k is archimedean and ρ is *standard*, i.e. the restriction to Γ of a Lie morphism $\rho : L \longrightarrow G$ (Margulis's super-rigidity theorem, respectively Corlette [3] and Gromov-Schoen [11]). This implies that such a lattice is arithmetic, i.e. commensurable to $\mathbf{L}(\mathbb{Z})$ where $\mathbf{L}$ denotes a $\mathbb{Q}$-algebraic group such that $\mathbf{L}(\mathbb{R}) = L$ up to compact factors.

The remaining possible cases are the two families of real hyperbolic groups $SO(n,1)$, $n \geq 2$ and complex hyperbolic groups $SU(n,1)$, $n \geq 2$. Lattices in $SO(n,1)$ do not have many rigidity properties. Many of them are non-arithmetic (c.f. Makarov [17] and Vinberg [28] for small n, Gromov and Piatetski-Shapiro [10] for any $n \in \mathbb{N}$), thus admit unbounded representations not coming from $SO(n,1)$. Concerning $SU(n,1)$, Mostow [20] exhibited a striking counterexample to super-rigidity for $n = 2$: namely two co-compact (arithmetic) lattices Γ and Γ' in $SU(2,1)$ and a *surjective* morphism $\rho : \Gamma \longrightarrow \Gamma'$ with *infinite kernel*. A few examples of non-arithmetic lattices are known in $SU(2,1)$ and $SU(3,1)$ ([16], [20], [6]). In particular in [16] Livne exhibited a non-arithmetic lattice in $SU(2,1)$ surjecting onto a non-Abelian free group, thus admitting many non-standard representations. Nothing is known for $n > 3$.

In this paper we investigate the *local rigidity problem* for standard representations of complex hyperbolic lattices. Let $\Gamma \overset{i}{\hookrightarrow} SU(n,1)$ be a co-compact lattice and $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{G}$ be a representation of the complex hyperbolic group $\mathbf{SU}(n,1)$ in a simple real algebraic group $\mathbf{G}$. Does there exist any *non-trivial* deformation of $\rho_{|\Gamma} : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$, i.e. a continuous family of morphisms $\rho_t : \Gamma \longrightarrow G$, $t \in I = [0,1]$, with $\rho_0 = \rho$ not of the form $\rho_t = g_t \cdot \rho \cdot g_t^{-1}$ for some continuous family $g_t \in G$, $t \in I$?

Notice that for higher rank lattices (or lattices in $Sp(n,1)$ or F_4^{-20}) super-rigidity implies local rigidity. On the other hand lattices in $SO(n,1)$ are usually not locally rigid : one can deform many of them into $SO(n+1,1)$ or other groups [12].

1.1. **First order deformations.** Let $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R}) = (\mathrm{Hom}(\mathbf{\Gamma}, \mathbf{G})//\mathbf{G})(\mathbb{R})$ be the moduli space of representations of Γ in G up to conjugacy. The space of first-order deformations of ρ, i.e. the real Zariski tangent space at $[\rho]$ to $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$, naturally identifies with the first cohomology group $H^1(\Gamma, \mathrm{Ad}\,\rho)$, where $\mathrm{Ad}\,\rho : \Gamma \overset{\rho}{\hookrightarrow} G \overset{\mathrm{Ad}}{\to} \mathrm{Aut}(\mathfrak{g})$ is the natural representation deduced from ρ and the adjoint action of G on its Lie algebra $\mathfrak{g}$. Thus the non-vanishing of $H^1(\Gamma, \mathrm{Ad}\,\rho)$ is a necessary condition for $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ not being trivial at the point $[\rho]$.

1.1.1. *Raghunathan's theorem.* The following result of Raghunathan restricts the possible first-order non-rigid standard representations :

Theorem 1.1.1 (Raghunathan)**.** *Let* $\lambda : \mathbf{SU}(n,1) \longrightarrow \mathbf{GL}(W)$ *be a real finite dimensional irreducible representation of* $\mathbf{SU}(n,1)$*. Let* Γ *be a co-compact lattice in* $SU(n,1)$*. Then* $H^1(\Gamma, W) = 0$ *except possibly if* $W = S^j V_{\mathbb{R}}$ *for some* $j \geq 0$*, where* S^j *denotes the* $j-$*th symmetric power and* $V_{\mathbb{R}}$ *denotes* V *seen as a real representation of* $\mathbf{SU}(n,1)$ *(by convention* $S^0 V_{\mathbb{R}}$ *is the trivial representation* $1_{\mathbb{R}}$*).*

Remark 1.1.2. Notice that the Hermitian form h on V identifies the complex $\mathbf{SU}(n,1)$-modules V^* (contragredient of V) and $\overline{V}$ (complex conjugate of V). The real $\mathbf{SU}(n,1)$-modules $V_{\mathbb{R}}$ and $V_{\mathbb{R}}^*$ are thus isomorphic.

As a corollary to this theorem the point $[\rho] \in \mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ is isolated except possibly if the real $\mathbf{SU}(n,1)$-module $\mathfrak{g}$ under $\mathrm{Ad}\,\rho$ contains an $\mathbf{SU}(n,1)$-direct factor isomorphic to $S^j V_{\mathbb{R}}$ for some $j \geq 0$.

Example 1.1.3. Let $\rho = \mathrm{Id} : \mathbf{SU}(n,1) \longrightarrow \mathbf{SU}(n,1)$ be the identity morphism. As the irreducible $\mathbf{SU}(n,1)$-module $\mathfrak{su}(n,1)$ does not belong to Raghunathan's list any cocompact lattice Γ of $SU(n,1)$ is first-order (thus locally) rigid in $SU(n,1)$. This was already proven by Weil [29].

1.1.2. Notice that Raghunathan's result is essentially optimal : for any $j \geq 0$ there exist co-compact lattices $\Gamma \in SU(n,1)$ such that $H^1(\Gamma, S^j V_{\mathbb{R}}) \neq 0$, c.f. [13], [1].

1.1.3. *Trivial first-order deformations.* The simplest possible first-order deformations of $\rho : \Gamma \longrightarrow G$ are those belonging to the subspace $H^1(\Gamma, \mathfrak{g}_{1_{\mathbb{R}}}) \simeq H^1(\Gamma, \mathbb{R})^d$ of $H^1(\Gamma, \mathfrak{g})$, where $\mathfrak{g}_{1_{\mathbb{R}}} \simeq (1_{\mathbb{R}})^d$ denotes the trivial isotypic component of $\mathfrak{g}$ as an $\mathbf{SU}(n,1)$-module. As $\mathfrak{g}_{1_{\mathbb{R}}}$ is the Lie algebra of the centralizer $\mathbf{Z}_{\mathbf{G}}(\mathbf{SU}(n,1))$ of $\rho(\mathbf{SU}(n,1))$ in $\mathbf{G}$ such first-order deformations may integrate to deformations of $\rho : \Gamma \longrightarrow G$ of the form $\rho \cdot \chi_t$ (up to G-conjugacy), where $\chi_t : \Gamma \longrightarrow Z_G(SU(n,1))$ is a deformation of the trivial representation $\chi_0 = 1_{\mathbb{R}}$ of Γ in the centralizer $Z_G(SU(n,1))$ of $SU(n,1)$ in G.

Example 1.1.4. Let $\rho : \mathbf{SU}(n,1) = \mathbf{SU}(V,h) \hookrightarrow \mathbf{SO}(2n,2) = \mathbf{SO}(V_{\mathbb{R}}, \mathrm{Re}\, h)$ be the natural embedding. Notice that ρ factorizes as $\mathbf{SU}(n,1) \hookrightarrow \mathbf{U}(n,1) \hookrightarrow \mathbf{SO}(2n,2)$. One easily checks that the Lie algebra $\mathfrak{so}(2n,2)$ is isomorphic as an $\mathbf{SU}(n,1)$-module to the direct sum of irreducible modules $\mathbb{R} \oplus \mathfrak{su}(n,1) \oplus \Lambda^2 V_{\mathbb{R}}$, where $\mathbb{R}$ is the Lie algebra of the centralizer $\mathbf{Z}_{\mathbf{G}}(\mathbf{SU}(n,1)) \simeq \mathbf{U}(1)$ of $\mathbf{SU}(n,1)$ in $\mathbf{U}(n,1)$. Thus $H^1(\Gamma, \mathrm{Ad}\,\rho) = H^1(\Gamma, \mathbb{R})$ and any deformation of ρ in $SO(2n,2)$ is (up to conjugacy) of the form $\rho \cdot \chi_t$, where $\chi_t : \Gamma \longrightarrow \mathbf{Z}_{\mathbf{G}}(\mathbf{SU}(n,1))(\mathbb{R}) = S^1$ is a unitary character of Γ.

In this paper such deformations will be considered as trivial (even if the integrability problem for first order deformations in $H^1(\Gamma, \mathfrak{g}_{1_{\mathbb{R}}})$ is non-trivial when $\mathbf{Z}_{\mathbf{G}}(\mathbf{SU}(n,1))$ is non-Abelian, c.f. section 1.2).

Remark 1.1.5. A well-known conjecture asserts that any complex hyperbolic lattice admits a finite index subgroup Γ with $H^1(\Gamma, \mathbb{R}) \neq 0$.

1.1.4. *Non-trivial first-order deformation.* A first-order deformation $x \in H^1(\Gamma, \mathfrak{g})$ will be considered as non-trivial if $x \notin H^1(\Gamma, \mathfrak{g}_{1_{\mathbb{R}}})$.

Example 1.1.6. Let $\rho : \mathbf{SU}(n,1) = \mathbf{SU}(V,h) \longrightarrow \mathbf{SU}(n+1,1)$ be the natural embedding. Then for a lattice Γ of $SU(n,1)$,

$$H^1(\Gamma, \mathfrak{su}(n+1,1)) = H^1(\Gamma, \mathbb{R}) \oplus H^1(\Gamma, V_{\mathbb{R}}) \quad ,$$

where once more $H^1(\Gamma, \mathbb{R})$ denotes the tangent space to the deformations of Γ in the centralizer $\mathbf{Z}_{\mathbf{G}}(\mathbf{SU}(n,1)) \simeq \mathbf{U}(1)$ of $\mathbf{SU}(n,1)$ in $\mathbf{SU}(n+1,1)$. As $H^1(\Gamma, V_{\mathbb{R}})$ may be non-zero, there might be non-trivial deformations of Γ in $SU(n+1,1)$.

Example 1.1.7. Let $V \otimes_{\mathbb{C}} \mathbb{H}$ be the quaternionic right vector space of dimension $n+1$ (thus of real dimension $4n+4$) endowed with the quaternionic Hermitian form $h_{\mathbb{H}}$ of signature

$(n, 1)$ deduced from h. The complex Hermitian part H of $h_{\mathbb{H}}$ is a complex Hermitian form on $V \otimes_{\mathbb{C}} \mathbb{H} = V \oplus jV$ of signature $(2n, 2)$. Let $\mathbf{Sp}(n,1) = \mathbf{SU}(V \otimes_{\mathbb{C}} \mathbb{H}, h_{\mathbb{H}})$ be the special unitary algebraic $\mathbb{R}$-group of linear transformation of $(V_{\mathbb{H}}, h_{\mathbb{H}})$, $\mathbf{U}(2n, 2)$ the unitary $\mathbb{R}$-group of complex linear transformations of $(V \oplus jV, H)$ and $\mathbf{SO}(4n, 4)$ the special orthogonal group of linear transformation of $((V_{\mathbb{H}})_{\mathbb{R}}, \mathrm{Re}H)$. One obtains a natural sequence of embeddings

$$\mathbf{SU}(n,1) \overset{j_{\mathbf{U}(n,1)}}{\hookrightarrow} \mathbf{U}(n,1) \overset{j_{\mathbf{Sp}(n,1)}}{\hookrightarrow} \mathbf{Sp}(n,1) \overset{j_{\mathbf{U}(2n,2)}}{\hookrightarrow} \mathbf{U}(2n,2) \overset{j_{\mathbf{SO}(4n,4)}}{\hookrightarrow} \mathbf{SO}(4n,4) \ \ .$$

One easily checks that modulo $H^1(\Gamma, \mathbb{R})$ the 1-cohomology of Γ with coefficient in $\mathfrak{sp}(n, 1)$, resp. $\mathfrak{su}(2n, 2)$, resp. $\mathfrak{so}(4n, 4)$ identifies with $H^1(\Gamma, S^2 V_{\mathbb{R}})$, resp. $H^1(\Gamma, S^2 V_{\mathbb{R}})$, resp. $H^1(\Gamma, 2 \cdot S^2 V_{\mathbb{R}})$ which may be non-trivial.

1.2. **Integrability and local rigidity : some earlier results.** Basic obstruction theory shows that if a first-order deformation $x \in H^1(\Gamma, \mathfrak{g})$ is tangent to a one-parameter family $\rho_t : \Gamma \longrightarrow G$ (one says that x is integrable) then necessarily $[x, x] = 0 \in H^2(\Gamma, \mathfrak{g})$. By [7] this necessary condition for integrability is in fact sufficient (c.f. section 2.5).

I don't know of many examples in deformation theory where one is able to prove that a first order deformation is non-integrable. However in [7] Goldman and Millson showed that none of the non-trivial first-order deformation for the example 1.1.6 can be integrated. Thus any representation $\lambda : \Gamma \longrightarrow SU(n+1, 1)$ sufficiently close to the standard $\rho = i : \Gamma \hookrightarrow SU(n, 1)$ is conjugate to a representation of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow Z_{SU(n+1,1)}(SU(n, 1)) = S^1$ a unitary character of Γ.

In [14] we proved a similar result for the example 1.1.7. Thus let $\rho_{\mathbf{G}} : \mathbf{SU}(n, 1) \longrightarrow \mathbf{G} = \mathbf{Sp}(n, 1), \mathbf{U}(2n, 2)$ or $\mathbf{SO}(4n, 4)$ be one of the embedding of example 1.1.7 and $\Gamma \overset{i}{\hookrightarrow} SU(n, 1)$ a co-compact lattice. Then any morphism $\lambda : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ close enough to $\rho_{\mathbf{G}}$ is conjugate to a representation of the form $\rho_{\mathbf{G}} \cdot \chi$, where $\chi : \Gamma \longrightarrow Z_G(SU(n, 1))$ (thus $Z_{Sp(n,1)}(SU(n, 1)) = S^1$ and $Z_{U(2n,2)}(SU(n, 1)) = Z_{SO(4n,4)}(SU(n, 1)) = S^1 \times S^1$).

1.3. **Results.** In this paper we prove non-integrability results for first order deformations of co-compact complex hyperbolic lattices vastly generalizing [7] and [14]. Our main result is theorem 1.3.3. As its statement is a bit technical we give three explicit corollaries (theorem 1.3.4, 1.3.7, 1.3.8).

1.3.1. *Notations.* We denote by $\mathrm{Coh}^1_{\mathbb{R}}$ the set of isomorphism classes of cohomological real $\mathbf{SU}(n, 1)$-modules in degree 1 :

$$\mathrm{Coh}^1_{\mathbb{R}} = \{S^k V_{\mathbb{R}} \text{ for } k \geq 0\} \ \ .$$

Let $\mathbf{U}(n)$ be the maximal compact subgroup of $\mathbf{SU}(n, 1)$. As a $\mathbf{U}(n)$-module the standard $\mathbf{SU}(n, 1)$-module $V_{\mathbb{R}}$ splits as $V_{\mathbb{R}} = (\mathbb{C}^n \oplus \det^{-1})_{\mathbb{R}}$. For $\pi = S^k V_{\mathbb{R}} \in \mathrm{Coh}^1_{\mathbb{R}}$ one denotes by W_π the $(S^k \mathbb{C}^n \otimes \det^{-k})_{\mathbb{R}}$-isotypic $\mathbf{U}(n)$-component of the $\mathbf{U}(n)$-module π.

Definition 1.3.1. *Let $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{G}$ be a non-trivial representation of $\mathbf{SU}(n,1)$ into a real simple algebraic group $\mathbf{G}$ with Lie algebra $\mathfrak{g}$. For $\pi \in \mathrm{Coh}^1_{\mathbb{R}}$ let $\mathfrak{g}_\pi$ be the π-isotypic component of the $\mathbf{SU}(n,1)$-module $\mathfrak{g}$ under $\mathrm{Ad}\,\rho$. We denote by $\mathrm{Coh}^1_{\mathbb{R}}(\rho)$ the subset $\{\pi \in \mathrm{Coh}^1_{\mathbb{R}},\ \ \mathfrak{g}_\pi \neq 0\}$. For $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$ let $V_\pi \subset \mathfrak{g}_\pi$ be the W_π-isotypic $\mathbf{U}(n)$-component of $\mathfrak{g}_\pi$.*

Fix $\mathbf{K_G}$ a maximal compact subgroup of $\mathbf{G}$ containing $\mathbf{U}(n)$, with Cartan involution C_G, and Cartan decomposition $\mathfrak{g} = \mathfrak{k}_{\mathbf{G}} \oplus \mathfrak{p}_{\mathbf{G}}$. As this decomposition of $\mathfrak{g}$ is $\mathbf{U}(n)$-stable each $\mathbf{U}(n)$-module V_π decomposes into a “compact” and a “non-compact” part :

$$V_\pi = V_{\pi,c} \oplus V_{\pi,n} \quad ,$$

where $V_{\pi,c} = V_\pi \cap \mathfrak{k}_{\mathbf{G}}$ and $V_{\pi,n} = V_\pi \cap \mathfrak{p}_{\mathbf{G}}$. We say that V_π is of compact type if $V_\pi = V_{\pi,c}$. This decomposition of V_π induces the decomposition $\mathfrak{g}_\pi = \mathfrak{g}_{\pi,c} \oplus \mathfrak{g}_{\pi,n}$, where $\mathfrak{g}_{\pi,c}$ (resp. $\mathfrak{g}_{\pi,n}$) is the $\mathbf{SU}(n,1)$-module generated by $V_{\pi,c}$ (resp. $V_{\pi,n}$).

1.3.2. Our main result is as follows. Let $\mathbf{Z}(\rho)$ be the (connected) centralizer of $\mathbf{U}(n)$ in $\mathbf{G}$ and let $\mathbf{T}(\rho)$ be a maximal torus of $\mathbf{Z}(\rho) \cap \mathbf{K_G}$. As explained in section 3.4 these groups are always non-trivial. The $\mathfrak{t}(\rho) \times \mathfrak{u}(n)$-module V_π, $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$, decomposes as :

$$V_{\pi,c} = \bigoplus_{\chi \in \Phi_{\pi,c}} \chi^{d_\chi} \otimes W_\pi \quad \text{and} \quad V_{\pi,n} = \bigoplus_{\chi \in \Phi_{\pi,n}} \chi^{d_\chi} \otimes W_\pi \quad ,$$

with $\Phi_{\pi,c} \cup \Phi_{\pi,n} \subset (i\mathfrak{t}(\rho))^*$.

Definition 1.3.2. *We denote by $\Lambda(\rho) \subset i\mathfrak{t}(\rho)$ the open cone*

$$\Lambda(\rho) = \bigcap_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \setminus \{1\}} \Big(\bigcap_{\chi \in \Phi_{\pi,c}} (i\mathfrak{t}(\rho))^{\chi > 0} \cap \bigcap_{\chi \in \Phi_{\pi,n}} (i\mathfrak{t}(\rho))^{\chi < 0} \Big) \quad .$$

Theorem 1.3.3. *Let $i : \Gamma \hookrightarrow SU(n,1)$ be a co-compact complex hyperbolic lattice. Let $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{G}$ be a morphism of $\mathbf{SU}(n,1)$ into a real simple algebraic group $\mathbf{G}$. If the cone $\Lambda(\rho)$ is not empty then any morphism $\lambda : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ close enough to ρ is conjugate to a representation of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow Z_G(SU(n,1))$ is a deformation of the trivial representation in the centralizer $Z_G(SU(n,1))$ of $SU(n,1)$ in G.*

In other words under the assumption $\Lambda(\rho) \neq \emptyset$ any deformation of ρ is trivial.

1.3.3. The first corollary of theorem 1.3.3 is :

Theorem 1.3.4. *Let $i : \Gamma \hookrightarrow SU(n,1)$ be a co-compact complex hyperbolic lattice. Let $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{G}$ be a morphism of $\mathbf{SU}(n,1)$ into a real simple algebraic group $\mathbf{G}$. If the V_π's, $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \setminus \{1\}$, are all of compact type then any morphism $\lambda : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ close enough to ρ is conjugate to a representation of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow Z_G(SU(n,1)$ is a deformation of the trivial representation in the centralizer $Z_G(SU(n,1))$ of $SU(n,1)$ in G.*

Example 1.3.5. This theorem immediately implies the results of [7] and [14] : in each of these example all the V_π's are of compact type and the centralizer $Z_G(SU(n,1))$ is a torus.

1.3.4. An other corollary of theorem 1.3.3 is as follows. Let $\mathbf{Z}_0(\rho)$ be the (connected) centralizer of $\mathbf{SU}(n,1)$ in $\mathbf{G}$ and let $\mathbf{T}_0(\rho)$ be a maximal torus of $\mathbf{Z}_0(\rho) \cap \mathbf{K_G}$. The $\mathfrak{t}_o(\rho) \times \mathfrak{su}(n,1)$-modules $\mathfrak{g}_{\pi,c}$ and $\mathfrak{g}_{\pi,n}$ decompose as

$$\mathfrak{g}_{\pi,c} = \bigoplus_{\chi_0 \in \Phi_{\pi,c}} \chi_0^{d_{\chi_0}} \otimes \pi \quad \text{and} \quad \mathfrak{g}_{\pi,n} = \bigoplus_{\chi_0 \in \Phi_{\pi,n}} \chi_0^{d_{\chi_0}} \otimes \pi \ ,$$

with $\Phi_{\pi,c} \cup \Phi_{\pi,n} \subset (i\mathfrak{t}_0(\rho))^*$.

Definition 1.3.6. *We denote by $\Lambda_0(\rho) \subset i\mathfrak{t}_0(\rho)$ the open cone*

$$\Lambda_0(\rho) = \bigcap_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus\{1\}} \Big(\bigcap_{\chi_0 \in \Phi_{\pi,c}} (i\mathfrak{t}_0(\rho))^{\chi_0 > 0} \cap \bigcap_{\chi_0 \in \Phi_{\pi,n}} (i\mathfrak{t}_0(\rho)))^{\chi_0 < 0} \Big) \ .$$

Theorem 1.3.7. *Let $i : \Gamma \hookrightarrow SU(n,1)$ be a co-compact complex hyperbolic lattice. Let $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{G}$ be a morphism of $\mathbf{SU}(n,1)$ into a real simple algebraic group $\mathbf{G}$. If the cone $\Lambda_0(\rho)$ is not empty then any morphism $\lambda : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ close enough to ρ is conjugate to a representation of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow Z_G(SU(n,1)$ is a deformation of the trivial representation in the centralizer $Z_G(SU(n,1))$ of $SU(n,1)$ in G.*

1.3.5. An explicit example of theorem 1.3.7 is the following generalization of [7] and [14] :

Theorem 1.3.8. *Let $i : \Gamma \hookrightarrow SU(n,1)$ be a co-compact complex hyperbolic lattice. Let $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{SU}(n+p,1+q)$ (resp. $\rho : \mathbf{SU}(n,1) \hookrightarrow \mathbf{Sp}(n+p,1+q)$) be the standard embedding. Then any morphism $\lambda : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ close enough to ρ is conjugate to a representation of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow U(p,q)$ is a deformation of the trivial representation in the centralizer $U(p,q)$ of $SU(n,1)$ in $SU(n+p,1+q)$ (resp. $Sp(n+p,1+q)$).*

1.3.6. *Remarks.* Notice that the condition in the theorem 1.3.3 is really a property of the morphism $\rho : \mathbf{SU}(n,1) \longrightarrow \mathbf{G}$, not of the group $\mathbf{G}$ alone. However there is a class of group $\mathbf{G}$ for which the condition $\Lambda(\rho) \neq \emptyset$ is never satisfied : if $\mathbf{G}$ is not absolutely simple, in other words if $\mathbf{G}(\mathbb{R})$ is a *complex* Lie group (up to isogeny). The conceptual explanation is quite clear : as we will see the proof of theorem 1.3.3 is purely Hodge theoretic. When the group $\mathbf{G}(\mathbb{R})$ is complex real variations of Hodge structures are replaced by complex variations of Hodge structures and we loose a lot of information (like symmetry of the Hodge numbers).

1.4. **Strategy of the proof : Deligne's result and Eichler-Shimura isomorphism.** Both the results of [7] and [14] were obtained from two ingredients :

- Hodge theory for flat bundles as developed in [19]. Vectors of $H^1(\Gamma, \mathrm{Ad}\,\rho)$ are understood as harmonic one-forms with value in the flat bundle $\mathcal{E}_{\mathfrak{g}}$ on the ball quotient $X = \Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$.
- basic obstruction theory : if a first-order deformation $x \in H^1(\Gamma, \mathrm{Ad}\,\rho)$ is integrable then $[x,x] = 0 \in H^2(\Gamma, \mathrm{Ad}\,\rho)$.

In both cases a quite tricky and painful computation showed that any $x \in H^1(\Gamma, \mathrm{Ad}\,\rho)$ satisfying $[x,x] = 0 \in H^2(\Gamma, \mathrm{Ad}\,\rho)$ necessarily vanishes.

Our proof of the general theorem 1.3.3 on the other hand relies on more sophisticated Hodge theory. We use not only the existence of harmonic representatives for vectors of $H^1(\Gamma, W) = H^1(X, \mathcal{E}_W)$ for any real representation $\rho : \Gamma \longrightarrow GL(W)$ but the existence of a canonical polarized real Hodge structure on the cohomology $H^\bullet(X, \mathcal{E}_W)$ of the polarized real variation of Hodge structure $\mathcal{E}_W$ on the compact Kähler manifold X. This result of Deligne was explained (and generalized to the quasi-projective case) a long time ago by Zucker [30], [31] but (as far as I know) never systematically used in deformation theory.

The first step in our proof is a corollary of Deligne's result : the De Rham cohomology $H^\bullet(X, \mathcal{E}_W)$ can be completely computed in holomorphic terms (Dolbeault cohomology). This computation is particularly nice in the complex hyperbolic locally homogeneous case and gives rise to the following "Eichler-Shimura isomorphism" (c.f. section 6) :

Theorem 1.4.1. *Let $i : \Gamma \hookrightarrow SU(n, 1) = SU(V, h)$ be a co-compact complex hyperbolic lattice. Let $k > 0$ be a positive integer. Then*

$$H^1(\Gamma, S^k V) = H^1(X, S^k TX \otimes \mathcal{L}^{-k})$$
$$H^1(\Gamma, S^k V^*) = H^0(X, S^{k+1}\Omega^1_X \otimes \mathcal{L}^k)) \ ,$$

where TX denotes the holomorphic tangent bundle of X, Ω^1_X its sheaf of holomorphic one-forms and $\mathcal{L}^{-1}$ the natural n-th root of the canonical line bundle K_X.

The proof of this theorem is particularly pleasant. On the one hand all the algebraic computations reduces to manipulating truncated tautological Koszul complexes. On the other hand it relies on a purely Hodge theoretic vanishing theorem (c.f. proposition 6.5.1) for which I don't know any other argument.

The second step is an analysis of Hodge types for the equation $[x, x] = 0 \in H^2(X, \mathcal{E}_\mathfrak{g})$. Theorem 1.4.1 is crucial here : it says that the complex Hodge structures $H^1(\Gamma, S^k V)$ and $H^1(\Gamma, S^k V^*)$ have only one non-vanishing Hodge type. Together with the condition $\Lambda(\rho) \neq \emptyset$ and the existence of a polarization on $H^\bullet(X, \mathcal{E}_\mathfrak{g})$ suitably compatible with the Lie bracket, it forces x to belong to $H^1(\Gamma, \mathfrak{z}_\mathfrak{g}(\mathfrak{su}(n, 1))$.

1.5. **Organization of the paper.** Theorem 1.3.3 confirms two general ideas :

- complex hyperbolic lattices have an intermediate behavior between the essentially non-rigid real hyperbolic lattices and all the other super-rigid lattices.
- all the rigidity features of a co-compact complex hyperbolic lattice $\Gamma \overset{i}{\hookrightarrow} SU(n, 1)$ arise from its Kähler property : Γ is the fundamental group of a compact Kähler manifolds, namely the locally homogeneous Hermitian symmetric space $M = \Gamma \backslash \mathbf{H}^n_\mathbb{C}$ quotient of the complex hyperbolic n-space. Thus the natural context for studying representations of Γ is the study of finite dimensional representations of Kähler groups, i.e. non-Abelian Hodge theory as developed by Hitchin, Corlette and Simpson.

Even if the Hodge theoretical results we use in this paper strictly predate Simpson's non-Abelian Hodge theory, I decided to develop the most general approach for two reasons. On the one hand it clarifies the results we use as it generalizes them from the case of a complex variation of Hodge structures ($\mathbb{C}$VHS) and its monodromy to the case of any semi-stable $\mathbf{G}$-Higgs bundle on a Kähler manifold X and the corresponding reductive representation $\rho : \pi_1(X) \longrightarrow G$. On the other hand we will need the general case in a future paper studying *non-standard* representations of Γ and their deformations.

Section 2 gives a compact review of Simpson's correspondence. Section 3 develops the properties of $\mathbf{G}$-$\mathbb{C}$VHS for the group-minded reader. Section 4 clarifies the link between $\mathbf{G}$-$\mathbb{C}$VHS and usual variations of Hodge structures, in particular weight issues crucial for our proof. Section 5 studies the cohomology of $\mathbf{G}$-$\mathbb{C}$VHS. First one recovers Deligne's results as a particular case of the Higgs formalism. Second one studies in details the link between Lie bracket and polarization also crucial for our proof. Section 6 contains the proof of theorem 1.4.1 and section 7 the proof of the main theorem 1.3.3 and its corollaries.

1.6. **Notations.** Let X be a smooth complex analytic space. Any flat complex vector bundle $\mathcal{E}$ on X is endowed with a structure of holomorphic vector bundle on X, still denoted $\mathcal{E}$. We will denote by $\mathcal{O}(\mathcal{E})$ the associated sheaf of holomorphic sections. It will always be clear from the context if we refer to $\mathcal{E}$ as a flat or a holomorphic vector bundle. The notation $H^\bullet(X, \mathcal{E})$ will always refer to the de Rham cohomology of the flat bundle $\mathcal{E}$ (equivalently : to the Betti cohomology of the associated local system) while $H^\bullet(X, \mathcal{O}(\mathcal{E}))$ refers to the coherent cohomology.

2. Non-Abelian Hodge theory

2.1. **The moduli space $\mathbf{M}(\Gamma, \mathbf{G})$.** In this section, Γ is a finitely presented group, K is the field $\mathbb{R}$ or $\mathbb{C}$, $\mathbf{G}$ a reductive K-algebraic group. We refer to [21], [22] and [15, chap. 6] as nice references for geometric invariant theory.

Definition 2.1.1. *We denote by* $\mathbf{R}(\Gamma, \mathbf{G})$ *the representation scheme of* Γ *in* $\mathbf{G}$*, namely the* K*-affine scheme representing the functor*

$$\begin{array}{lcl} K-\text{algebras} & \longrightarrow & \mathbf{Set} \\ R & \longrightarrow & \mathrm{Hom}_{\mathbf{Group}}(\Gamma, \mathbf{G}(R)) \ . \end{array}$$

The group $\mathbf{G}$ acts (factorizing through the adjoint group $\mathbf{G}^{\mathrm{ad}}$) on $\mathbf{R}(\Gamma, \mathbf{G})$ by $\mathbf{G}$-conjugation on the target. The moduli space $\mathbf{M}(\Gamma, \mathbf{G})$ is the quotient $\mathbf{R}(\Gamma, \mathbf{G})//G$ in the GIT sense. Notice that we may *a priori* choose many different $\mathbf{G}$-linearizations on the line bundle $\mathbf{R}(\Gamma, \mathbf{G}) \times_{\mathrm{Spec}\, K} \mathbb{A}^1$ if $\mathbf{G}$ is not semi-simple, leading to different notions of *unstable points* and different quotients. In order to proceed canonically we will always consider $\mathbf{R}(\Gamma, \mathbf{G})$ as a $\mathbf{G}^{\mathrm{ad}}$-scheme. We choose $\mathbf{R}(\Gamma, \mathbf{G}) \times_{\mathrm{Spec}\, K} \mathbb{A}^1$ with the trivial $\mathbf{G}^{\mathrm{ad}}$-action on $\mathbb{A}^1$ as our $\mathbf{G}^{\mathrm{ad}}$-linearized line bundle on the affine $\mathbf{G}^{\mathrm{ad}}$-variety $\mathbf{R}(\Gamma, \mathbf{G})$.

Definition 2.1.2. *We denote by* $\mathbf{M}(\Gamma, \mathbf{G})$ *the affine* K*-scheme universal categorical quotient* $\mathbf{R}(\Gamma, \mathbf{G})//\mathbf{G}^{\mathrm{ad}}$ *, and by* $[\cdot] : \mathbf{R}(\Gamma, \mathbf{G}) \longrightarrow \mathbf{M}(\Gamma, \mathbf{G})$ *the canonical quotient map.*

Notice that our choice of $\mathbf{G}^{\mathrm{ad}}$-linearization ensures that all the points of $\mathbf{R}(\Gamma, \mathbf{G})$ are semi-stable : $\mathbf{R}(\Gamma, \mathbf{G}) = \mathbf{R}(\Gamma, \mathbf{G})^{ss}$, and the quotient map $[\cdot] : \mathbf{R}(\Gamma, \mathbf{G}) \longrightarrow \mathbf{M}(\Gamma, \mathbf{G})$ is a good quotient. Recall the

Definition 2.1.3. *A point* ρ *of* $\mathbf{R}(\Gamma, \mathbf{G})^{ss}(K) = \mathbf{R}(\Gamma, \mathbf{G})(K)$ *is said to be* stable *if the orbit map* $\phi_\rho : \mathbf{G}^{\mathrm{ad}} \longrightarrow \mathbf{R}(\Gamma, \mathbf{G})$ *defined by* $\phi_\rho(g) = \mathrm{Ad}\, g(\rho)$ *is proper.*

By [15, lemma 6.1.9] and [12, theor 1.1], one obtains the following characterization of stable points in $\mathbf{R}(\Gamma, \mathbf{G})$:

Lemma 2.1.4. *A point* ρ *of* $\mathbf{R}(\Gamma, \mathbf{G})$ *is stable if and only if one of the following equivalent assertion is satisfied :*

(1) *The orbit* $\mathbf{G}^{\mathrm{ad}} \cdot \rho = \phi_\rho(\mathbf{G})$ *is closed in* $\mathbf{R}(\Gamma, \mathbf{G})$ *and the stabilizer* $\mathbf{Z}(\rho)$ *of* ρ *in* $\mathbf{G}^{\mathrm{ad}}$ *is finite.*

(2) *The image of* ρ *is not contained in a proper* K*-parabolic subgroup of* $\mathbf{G}$*.*

2.2. **The moduli space** $\mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})$**.** In this section $\mathbf{G}$ is a reductive $\mathbb{C}$-algebraic group and $G = \mathbf{G}(\mathbb{C})$ its Lie groups of complex points. We fix once for all a maximal compact subgroup K of G. Let X be a smooth connected polarized complex projective variety with fundamental group $\Gamma = \pi_1(X)$ (the role of the base point will be unimportant in our discussion). We refer to [25] and [27] for the theory of $\mathbf{G}$-principal Higgs bundles and their moduli spaces and recall only the main definitions.

Definition 2.2.1. *A* G*-principal Higgs bundle on* X *is a pair* (P, θ)*, where*

- P *is a principal holomorphic* G*-bundle on* X
- $\theta \in \mathrm{Ad}\,\mathrm{P} \otimes \Omega^1_X$ *satisfies* $[\theta, \theta] = 0$ *(where* $\mathrm{Ad}\,\mathrm{P} := \mathrm{P} \times_G \mathfrak{g}$*).*

A G*-principal Higgs bundle* (P, θ) *on* X *is said to be of semi-harmonic type if its Chern classes vanish and for some irreducible* $\mathbf{G}$*-module* V *(and then for any) the Higgs vector-bundle* $\mathrm{P} \otimes_{\mathbf{G}} V$ *is Higgs semi-stable.*

Definition 2.2.2. *We denote by* $\mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})$ *the* $\mathbb{C}$*-scheme moduli space of* G*-principal Higgs bundles of semi-harmonic type on* X *constructed in* [27, section 9].

2.3. **Simpson's correspondence.** Let $\rho : \Gamma \longrightarrow G$ be a reductive representation and $\mathcal{P} = (P, D)$ the associated flat complex G-bundle on X. Here P denotes the principal G-bundle $\tilde{X} \times_\rho G$ with flat connection $D \in A^1(\mathrm{Ad}\, P)$. As X is polarized and ρ is reductive there exists an essentially unique ρ-equivariant harmonic map

$$f : \tilde{X} \longrightarrow G/K$$

(where G/K denotes the symmetric space of G), defining an *harmonic K-reduction* P_K of P. Decompose the flat connexion D as

$$D = \nabla + \alpha \ ,$$

where ∇ is the canonical connexion on the K-principal bundle P_K and $\alpha \in A^1(X, \mathrm{Ad}\, P)$. Decompose furthermore using types :

$$\nabla = \partial_K + \overline{\partial} \ ,$$
$$\alpha = \theta + \theta^* \ ,$$

where ∂_K is of type $(1,0)$, $\overline{\partial}$ is of type $(0,1)$, $\theta \in A^{1,0}(\mathrm{Ad}\, P)$ and $\theta^* = \tau(\theta) \in A^{0,1}(\mathrm{Ad}\, P)$ is the conjugate of θ with respect to the K-reduction. Define $D' = \partial_K + \theta^*$, $D'' = \overline{\partial} + \theta$, thus $D = D' + D''$. As D is flat and the K-reduction P_K is harmonic, $(D'')^2 = 0$, that is :

$$\overline{\partial}^2 = \overline{\partial}(\theta) = [\theta, \theta] = 0 \ .$$

Finally $(\mathrm{P} = (P, \overline{\partial}), \theta)$ is a $\mathbf{G}$-Higgs bundle (of semi-harmonic type). Notice that knowing (P, θ) is equivalent to knowing (P, D'').

Let $\mathbf{G} - dR$ be the differential graded-category of flat G-bundles on X : an object is a flat bundle $\mathcal{P} = (P, D)$ on X and

$$\mathbf{Hom}_{\mathbf{G}-dR}(\mathcal{P}, \mathcal{P}') = (A^\bullet(\mathrm{Hom}(\mathrm{Ad}\, P, \mathrm{Ad}\, P'), D_{\mathrm{Hom}(P,P')}) \ .$$

Let $\mathbf{G} - Dol$ be the differential graded-category of semi-harmonic G-Higgs bundles on X : an object is a semi-harmonic G-Higgs bundle $(\mathrm{P}, \theta_{\mathrm{P}})$ on X and

$$\mathbf{Hom}_{\mathbf{G}-Dol}(\mathrm{P}, \theta_{\mathrm{P}}), (\mathrm{P}', \theta_{\mathrm{P}'})) = (A^\bullet(\mathrm{Hom}(\mathrm{Ad}\, P, \mathrm{Ad}\, \mathcal{P}'), D''_{\mathrm{Hom}(P,P')}) \ .$$

Theorem 2.3.1 (Simpson)**.** *The functor $F : \mathbf{G} - dR \longrightarrow \mathbf{G} - Dol$ associating to the flat bundle $\mathcal{P}$ the Higgs bundle (P, θ) is a quasi-equivalence of differential graded categories.*

It implies the geometric (weaker) version [27, theor. 9.11 and lemma 9.14] :

Theorem 2.3.2 (Simpson)**.** *The functor F induces a real-analytic diffeomorphism*

$$\phi_{\mathbf{G}} : \mathbf{M}(\Gamma, \mathbf{G})(\mathbb{C}) \longrightarrow \mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})(\mathbb{C}) \ .$$

2.4. **Tangent spaces.** A direct corollary of theorem 2.3.1 is the following isomorphism of tangent spaces :

Corollary 2.4.1. *Suppose (P, D) is a reductive flat G-bundle with monodromy ρ and (P, θ) the corresponding G-Higgs semi-harmonic bundle. There are canonical quasi-isomorphisms of complexes of sheaves :*

$$(A^\bullet(\mathrm{Ad}\, P), D) \simeq (A^\bullet(\mathrm{Ad}\, P), D'') \simeq (\Omega^\bullet_X(\mathrm{Ad}\, \mathrm{P}), \theta) \ .$$

In particular the real-analytic diffeomorphism $\phi_G : \mathbf{M}(\Gamma, \mathbf{G})(\mathbb{C}) \longrightarrow \mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})(\mathbb{C})$ *induces a sequence of natural isomorphisms of tangent spaces :*

$$T_{[\rho]}\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{C}) = H^1(\Gamma, \mathrm{Ad}\,\rho) = \mathbb{H}^1((A^\bullet(\mathrm{Ad}\,P), D)) \\ \simeq \mathbb{H}^1((\Omega^\bullet_X(\mathrm{Ad}\,\mathrm{P}), \theta)) = T_{[(P,\theta)]}\mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})(\mathbb{C}) \ .$$

2.5. **Formality.** The previous analysis (and its generalization to real representations) implies that deforming locally a reductive representation of the fundamental group of a smooth complex projective variety is a formal problem : it reduces to studying second-order deformations.

Theorem 2.5.1 ([8], [25])**.** *Let* X *be a connected smooth complex projective variety with fundamental group* Γ*,* $\mathbf{G}$ *a real reductive algebraic group and* $\rho : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ *a reductive representation. Let* $C_\rho \subset H^1(\Gamma, \mathrm{Ad}\,\rho)$ *be the affine cone defined by*

$$C_\rho = \{u \in H^1(\Gamma, \mathrm{Ad}\,\rho) \ / [u, u] = 0 \in H^2(\Gamma, \mathrm{Ad}\,\rho)\} \ .$$

Then the formal completion of $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ *at* $[\rho]$ *is isomorphic to the formal completion of the good quotient* C_ρ/E_ρ*, where* E_ρ *denotes the centralizer of* $\rho(\Gamma)$ *in* G*.*

3. G-variations of Hodge structures

The moduli space $\mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})$ carries a natural $\mathbb{C}^*$-action : an element $t \in \mathbb{C}^*$ maps $[(\mathrm{P}, \theta)]$ to $[(\mathrm{P}, t \cdot \theta)]$ [27, p.62]. The fixed points of this action are of particular importance : they are systems of **G**-Hodge bundles [25, p.44] and correspond by Simpson's correspondence to (isomorphism classes of) **G**-complex variations of Hodge structure (**G**-$\mathbb{C}$VHS). We refer to the appendix A for notations in Hodge theory, to [4], [5], [25] for more details on **G**-VHS, and to [9] for a detailed study of period domains.

3.1. Hodge datum.

Definition 3.1.1. *A (pointed) Hodge datum is a pair* $(\mathbf{L}, u)$*, where* $\mathbf{L}$ *is a real reductive algebraic group and* $u : \mathbf{U}(1) \longrightarrow \mathbf{Aut}(\mathbf{L})^0 \subset \mathbf{L}^{\mathrm{ad}}$ *is a morphism of real algebraic groups such that* $C = u(-1)$ *is a Cartan involution of* $\mathbf{L}$ *(that is :* $C^2 = 1$ *and* $\tau := C\sigma = \sigma C$ *is the conjugation of* $\mathbf{L}_\mathbb{C}$ *with respect to a compact real form* $\mathbf{U}$*, where* σ *denotes the conjugation of* $\mathbf{L}_\mathbb{C}$ *with respect to* $\mathbf{L}$*).*

In particular the Cartan involution C is inner. One easily shows (c.f. [25, section 4.4]) that an algebraically connected real reductive group $\mathbf{L}$ admits a Hodge datum $(\mathbf{L}, u)$ (one says that $\mathbf{L}$ is of *Hodge type*) if and only if $\mathbf{L}$ contains an anisotropic maximal torus $\mathbf{T}$. In other words the reductive real Lie group $\mathbf{L}(\mathbb{R})$ has the same real rank than any of its maximal compact subgroups.

3.2. **Period domains.**

Definition 3.2.1. *Let* $(\mathbf{L}, u)$ *be a Hodge datum. We denote by :*

- $V = Z_L(u)$ *the centralizer of* u *in* $L = \mathbf{L}(\mathbb{R})$*. As* V *is invariant by* C*,* V *is contained in* $U = \mathbf{U}(\mathbb{R})$*, in particular* V *is compact.*
- K *the centralizer of* C *in* L*. Notice that* K *coïncide with the intersection* $U \cap L$*. Thus the group* K *is a maximal compact subgroup of* L*.*

Finally the Hodge datum $(\mathbf{L}, u)$ defines canonically the chain of inclusions of compact groups $V \subset K \subset U$.

Let $(\mathbf{L}, u)$ be a Hodge datum. Let $\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E)$ be a real (resp. complex) representation $\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E)$ of $\mathbf{L}$. If $\mathbf{L} = \mathbf{L}^{\mathrm{ad}}$ (or more generally if λ factorizes through $\mathbf{L}^{\mathrm{ad}}$) the composite $\lambda \circ u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}} \longrightarrow \mathbf{GL}(E)$ defines a weight 0 real (resp. complex) Hodge structure on E polarized by $\lambda \circ u(-1)$ (c.f. appendix 6.4). In particular the adjoint representation of $\mathbf{L}$ defines on the Lie algebra $\mathfrak{l}$ a weight 0 polarized real Hodge structure :

$$\mathfrak{l}_{\mathbb{C}} = \oplus_{i \in \mathbb{Z}} \mathfrak{l}_{\mathbb{C}}^{i} \ ,$$

where $u(z)$ acts on $\mathfrak{l}_{\mathbb{C}}^{i}$ via multiplication by z^{-i}. We will denote by $F^{\bullet}\mathfrak{l}_{\mathbb{C}}$ the corresponding decreasing Hodge filtration. The polarization is given by the Killing form $\beta_{\mathbf{L}}$.

Definition 3.2.2. *One denotes by* $\mathfrak{q} \subset \mathfrak{l}_{\mathbb{C}}$ *the Lie sub-algebra* $F^0\mathfrak{l}_{\mathbb{C}}$ *and* $Q \subset L_{\mathbb{C}}$ *the corresponding subgroup.*

One easily check that $\mathfrak{q}$ is a parabolic sub-algebra of $\mathfrak{l}_{\mathbb{C}}$, with Levi sub-algebra $\mathfrak{v}_{\mathbb{C}}$ the complexified Lie algebra of $\mathfrak{v} = \mathrm{Lie}(V)$.

Definition 3.2.3. *Let* $(\mathbf{L}, u)$ *be a Hodge datum. The period domain* D *associated to* $(\mathbf{L}, u)$ *is the* L*-conjugacy class of* u*.*

Thus D naturally identifies with L/V. Let $\check{D} = L_{\mathbb{C}}/Q$ be the flag manifold of $L_{\mathbb{C}} = \mathbf{L}(\mathbb{C})$ defined by Q, the natural morphism $D = L/V \hookrightarrow \check{D} = L_{\mathbb{C}}/Q$ is open and thus defines a natural L-invariant complex structure on D.

3.3. **Elliptic orbits.** Let $(\mathbf{L}, u)$ be a Hodge datum an $\rho : \mathbf{L} \hookrightarrow \mathbf{G}$ an embedding of real connected algebraic groups. The previous geometric picture generalizes as follows.

Let $u(U(1)) \subset V$ be the inclusion of compact subgroups of L canonically defined by the Hodge datum $(\mathbf{L}, u)$ (c.f. section 3.2) It generates the diagram :

$$\begin{tikzcd} u(U(1)) \arrow[r, equal] \arrow[d, hook] & V \arrow[d, hook] \\ \mathbf{Z}(\mathbf{L}(\rho)) \arrow[r, hook] & \mathbf{L}(\rho) \end{tikzcd} \ , \tag{3.1}$$

where $\mathbf{L}(\rho)$ denotes the centralizer of $\rho \circ u(\mathbf{U}(1))$ in $\mathbf{G}$ and $\mathbf{Z}(\mathbf{L}(\rho))$ its connected center.

Let us choose K_G a maximal compact subgroup of G containing K, C_G the corresponding Cartan involution and

$$\mathfrak{g} = \mathfrak{k}_G \oplus \mathfrak{p}_{\mathbf{G}}$$

the corresponding Cartan decomposition. We denote by $\sigma_{G_\mathbb{C}}$ (resp. $\tau_{G_\mathbb{C}}$) the complex conjugation of the Lie algebra $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$ (resp. to the compact form $\mathfrak{u}_{\mathfrak{g}_\mathbb{C}}$ of $\mathfrak{g}_\mathbb{C}$ defined by C_G).

Remark 3.3.1. In general C_G is not interior.

As the restriction to L of C_G is C the diagram 3.1 is C_G-stable. Let $\mathbf{T}$ be a maximal torus of K_G containing $\rho \circ u(U(1))$ and H the centralizer of T in G. This is a Cartan subgroup of G, with Cartan decomposition $H = TA$ where $\mathfrak{a}$ is the centralizer of T in $\mathfrak{p}_{\mathbf{G}}$.

Definition 3.3.2. *We write $\Phi_c \subset i\mathfrak{t}$ the set of roots of T in $(\mathfrak{k}_G)_\mathbb{C}$, $\Phi_n \subset i\mathfrak{t}$ the set of non-zero weights of T in $(\mathfrak{p}_{\mathbf{G}})_\mathbb{C}$ and $\Phi = \Phi_c \cup \Phi_n$.*

One can regard an element of Φ as a character of the group generated by T and C_G, thus remembering whether an element of Φ comes from Φ_c or Φ_n : C_G acts by $+1$ on Φ_c and by -1 on Φ_n.

Definition 3.3.3. *One denotes by $E_\rho \in i\mathfrak{t}$ the vector $i(\rho \circ u)_*(\frac{\partial}{\partial t})$.*

Definition 3.3.4. *One defines the C_G-stable parabolic sub-algebra $\mathfrak{q}(\rho)$ of $\mathfrak{g}_\mathbb{C}$ associated to ρ as*

$$\mathfrak{q}(\rho) = \mathfrak{h} \oplus \bigoplus_{\substack{\gamma \in \Phi \\ \gamma(E_\rho) \geq 0}} \mathfrak{g}_\gamma \quad .$$

Its Levi sub-algebra is :

$$\mathfrak{l}(\rho)_\mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\substack{\gamma \in \Phi \\ \gamma(E_\rho) = 0}} \mathfrak{g}_\gamma \quad .$$

It follows immediately that :

- as a vector space $\mathfrak{q}(\rho)$ is nothing else than the 0-th Hodge filtration $F^0\mathfrak{g}_\mathbb{C}$ defined in section 4.2.
- $\sigma_G(\mathfrak{q}(\rho)) \cap \mathfrak{q}(\rho) = \mathfrak{l}(\rho)_\mathbb{C}$.
- the complex Levi sub-algebra $\mathfrak{l}(\rho)_\mathbb{C}$ is defined over $\mathbb{R}$, the corresponding Levi subgroup $\mathbf{L}(\rho) \subset \mathbf{G}$ being the centralizer of E_ρ in $\mathbf{G}$.

In other words, the Hodge datum $(\mathbf{L}, u)$ and the representation $\rho : \mathbf{L} \longrightarrow \mathbf{G}$ defines an holomorphic embedding of elliptic (co)adjoint orbits :

$$D = L/V \hookrightarrow G/L(\rho) \quad .$$

In general the adjoint elliptic orbit $G/L(\rho)$ is not a period domain as $L(\rho)$ is not necessarily compact.

Notice that the compact part of the torus $\mathbf{Z}(\mathbf{L}(\rho)$ center of $\mathbf{L}(\rho)$ is non-trivial as it contains $u(\mathbf{U}(1))$.

3.4. **More centralizers.**

Definition 3.4.1. *We denote by $\mathbf{Z}(\rho)$ the centralizer of V in $\mathbf{G}$.*

Thus one has the sequence of C_G-stable inclusions :

$$\mathbf{Z}(\mathbf{L}(\rho)) \subset \mathbf{Z}(\rho) \subset L(\rho) \ \ .$$

Definition 3.4.2. *We denote by $\mathbf{T}(\rho)$ the C_G-stable compact maximal torus $\mathbf{T} \cap \mathbf{Z}(\rho)$ of $\mathbf{Z}(\rho)$.*

Once more $\mathbf{T}(\rho)$ contains $u(\mathbf{U}(1))$, thus is non-trivial.

3.5. **G-ℂVHS.**

3.5.1. *Horizontality.* The holomorphic tangent bundle TD naturally identifies with the L-equivariant bundle $(L_{\mathbb{C}} \times_Q \mathfrak{l}_{\mathbb{C}}/\mathfrak{q})_{|D}$.

Definition 3.5.1. *The horizontal tangent bundle T_hD is the holomorphic sub-bundle $(L_{\mathbb{C}} \times_Q F^{-1}\mathfrak{l}_{\mathbb{C}}/\mathfrak{q})_{|D}$ of TD.*

Definition 3.5.2. *Let $\mathcal{C}$ be the category whose objects are pairs (Y, R_Y), where Y is a complex smooth analytic space, $R_Y \subset TY$ a holomorphic distribution, and a morphism $f : (Y, R_Y) \longrightarrow (X, R_X)$ in $\mathcal{C}$ is a holomorphic* horizontal *map $f : X \longrightarrow Y$: one requires that $df(R_Y) \subset R_X$. We will look at the category of smooth analytic spaces as a subcategory of $\mathcal{C}$, the distribution being the full tangent space.*

3.5.2. With all these definitions we can define the main actors in Simpson's theory :

Definition 3.5.3. *Let X be a complex analytic manifold with fundamental group Γ and universal cover $\tilde{X}$. Let $\mathbf{G}$ be a complex reductive algebraic group. A $\mathbf{G}$-complex variation of Hodge structure ($\mathbf{G}$-ℂVHS) is a Hodge datum $(\mathbf{L}, u)$ with period domain D, an injection $i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}$, a representation $\rho : \Gamma \longrightarrow L = \mathbf{L}(\mathbb{R}) \subset G$ (called the monodromy of the variation) and a holomorphic* horizontal *ρ-equivariant map $f : \tilde{X} \longrightarrow D$ (called* period map*).*

4. G-ℂVHS and μ-ℂVHS

4.1. **G-ℂVHS and G-bundles.**

Definition 4.1.1. *Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}, u, \rho, f : \tilde{X} \longrightarrow D)$ be a $\mathbf{G}$-ℂVHS on X. Let $\tau : Q \longrightarrow V_{\mathbb{C}} \overset{i}{\hookrightarrow} G$ be the reduction of i to the Levi $V_{\mathbb{C}}$ of Q. One associates to it the following principal G-bundles on X :*

- *the flat G-bundle $\mathcal{P} = \tilde{X} \times_{\Gamma,\rho} G$, which is naturally a holomorphic bundle. Notice that this holomorphic structure is compatible with the identification $\mathcal{P} := f^*((L_{\mathbb{C}} \times_{Q,i} G)_{|D})$ (descent to X of the) pull-back via f of the holomorphic G-bundle $L_{\mathbb{C}} \times_{Q,i} G$ on $\check{D}$.*
- *the holomorphic G-bundle $\mathrm{P} = f^*((L_{\mathbb{C}} \times_{Q,\tau} G)_{|D}$ (descent to X of the) pull-back via f of the holomorphic G-bundle $L_{\mathbb{C}} \times_{Q,\tau} G$ on $\check{D}$.*

4.2. **μ-$\mathbb{C}$VHS attached to a G-$\mathbb{C}$VHS and a G-module.** As noted above if $(\mathbf{L}, u)$ is a Hodge datum and $\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E)$ is a complex representation *factorizing through* $\mathbf{L}^{\mathrm{ad}}$ then the composite $\lambda \circ u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}} \longrightarrow \mathbf{GL}(E)$ defines a weight 0 complex Hodge structure on E polarized by $\lambda \circ u(-1)$. Thus if $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{L}_{\mathbb{C}}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ an $\mathbf{L}_{\mathbb{C}}$-$\mathbb{C}$VHS on X, such a representation defines canonically a $\mathbb{C}$VHS on the bundle $\mathcal{E}_\lambda := \mathcal{P} \times_{L_{\mathbb{C}},\lambda} E$. If moreover $E_{\mathbb{R}}$ is a real form of E and λ is defined over $\mathbb{R}$ then $\mathcal{E}_\lambda$ is a weight 0 polarized $\mathbb{R}$VHS.

What if λ does not factorize through $\mathbf{L}^{\mathrm{ad}}$?

4.2.1. *Index.*

Definition 4.2.1. *Let $(\mathbf{L}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}})$ be a Hodge datum. Its index μ is the smallest positive integer n such that the co-character $\tilde{u} := u^n \in X_*(\mathbf{L}^{\mathrm{ad}})$ belongs to the finite index subgroup $X_*(\mathbf{L}^{\mathrm{der}})$ of $X_*(\mathbf{L}^{\mathrm{ad}})$.*

Thus the diagram

$$\begin{tikzcd} \mathbf{U}(1) \arrow[r, "\tilde{u}"] \arrow[d, "\mu"'] & \mathbf{L}^{\mathrm{der}} \arrow[d, "\mathrm{Ad}"] \\ \mathbf{U}(1) \arrow[r, "u^{\mathrm{ad}}"'] & \mathbf{L}^{\mathrm{ad}} \end{tikzcd}$$

is commutative and μ is the smallest positive integer for which such a diagram exists.

4.2.2.

Definition 4.2.2. *Let $(\mathbf{L}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}})$ be a Hodge datum and $\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E)$ a complex representation. We define a weight 0 complex Hodge structure on E :*

$$E < p >= \{v \in E \ / \ \forall z \in \mathbb{C}^*, \ \ \tilde{u}(z) \cdot v = z^{-p} \cdot v\}$$

so that $E = \bigoplus_{p \in \mathbb{Z}} E < p >$. This complex Hodge structure on E is polarizable by $\lambda(c)$ for any choice of $c \in \tilde{u}(U(1)) \subset L = \mathbf{L}(\mathbb{R})$ a lifting of C (thus $C = \mathrm{Int}(c)$).

Remark 4.2.3. For example $\mathfrak{l}_{\mathbb{C}}^i = \mathfrak{l}_{\mathbb{C}} < \mu \cdot i >$

Assume that λ is irreducible. We still denote by $\lambda : \mathfrak{l}_{\mathbb{C}} \longrightarrow \mathrm{End}(E)$ the induced Lie algebra morphism. The following properties follow from the definitions :

- $\lambda(\mathfrak{l}_{\mathbb{C}}^{-1}) \cdot E < n > \subset E < n - \mu >$.

- $\{n \,/\, E < n > \neq 0\} = \{n_0, n_0+\mu, n_0+2\mu, \cdots, n_0+k\mu\}$.
- If $E_{\mathbb{R}}$ is a real form of E and λ is defined over $\mathbb{R}$ then $E < -n > = \overline{E < n >}$.

The appearance of the index μ in the action of $\mathfrak{l}_{\mathbb{C}}^{-1} = \mathfrak{l}_{\mathbb{C}} < -\mu >$ on E suggests the following definition of a μ-$\mathbb{C}$VHS (essentially the same object as a usual $\mathbb{C}$VHS up to some index convention in the transversality condition ; in particular a $\mathbb{C}$VHS is a 1-$\mathbb{C}$VHS) :

Definition 4.2.4. *Let X be a complex analytic space. A μ-complex variation of Hodge structures of weight 0 on X is a flat complex vector bundle $(\mathcal{E}, D)$ on X and a $\mathcal{C}^\infty$ decomposition $\mathcal{E} = \bigoplus_{p\in\mathbb{Z}} \mathcal{E}^{p,-p}$ of $\mathcal{C}^\infty$-vector bundles such that :*

1. *For any $x \in X$ the induced decomposition of the x-fiber $\mathcal{E}_x = \bigoplus_{p\in\mathbb{Z}} \mathcal{E}_x^{p,-p}$ is a weight 0 complex Hodge structure on $\mathcal{E}_x$.*
2. *For any $p \in \mathbb{Z}$ the fiber bundle $F^p\mathcal{E} = \bigoplus_{r\geq p} \mathcal{E}^{r,-r}$ is a holomorphic sub-bundle of $\mathcal{E}$ and the fiber bundle $\overline{F}^p\mathcal{E} := \bigoplus_{r\leq -p} \mathcal{E}^{r,-r}$ is an anti-holomorphic sub-bundle of $\mathcal{E}$.*
3. *If D is the flat connection on $\mathcal{E}$ then $D(F^p\mathcal{E}) \subset F^{p-\mu}\mathcal{E} \otimes \Omega^1_X$ and $D(\overline{F}^p\mathcal{E}) \subset \overline{F}^{p-\mu}\mathcal{E} \otimes \overline{\Omega^1_X}$.*

Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{L}_{\mathbb{C}}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ be an $\mathbf{L}_{\mathbb{C}}$-$\mathbb{C}$VHS on X. Let $\lambda : \mathbf{L}_{\mathbb{C}} \longrightarrow \mathbf{GL}(E)$ be a finite dimensional representation. Notice that the Hodge filtration $F^\bullet E$ on the complex Hodge structure $E = \bigoplus_{p\in\mathbb{Z}} E < p >$ is naturally Q-invariant. Thus :

Definition 4.2.5. *Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{L}_{\mathbb{C}}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ an $\mathbf{L}_{\mathbb{C}}$-$\mathbb{C}$VHS on X. Let $\lambda : \mathbf{L}_{\mathbb{C}} \longrightarrow \mathbf{GL}(E)$ be a finite dimensional representation. It defines canonically a weight 0 μ-complex variation of Hodge structures $\mathcal{E}_\lambda$ on X (with fiber E) :*

$$\mathcal{E}_\lambda := \mathcal{P} \times_{L_{\mathbb{C}},\lambda} E = \Gamma\backslash f^*(L_{\mathbb{C}} \times_{Q,\lambda} E)\ , \quad F^\bullet\mathcal{E}_\lambda := P_f \times_{G,\lambda} F^\bullet E = \Gamma\backslash f^*(L_{\mathbb{C}} \times_{Q,\lambda} F^\bullet E)\ .$$

The horizontality of f guaranties property (3) of definition 4.2.4 (μ-Griffiths's transversality). If $E_{\mathbb{R}}$ is a real form of E and λ is defined over $\mathbb{R}$ then $\mathcal{E}_\lambda$ is a μ-$\mathbb{R}$VHS.

As explained in the appendix B any μ-$\mathbb{C}$VHS can be considered as a usual $\mathbb{C}$VHS by relabeling the Hodge types. However this process is non-canonical, in particular usually not compatible with other algebraic structures on the μ-$\mathbb{C}$VHS. We will deal with such an example in section 5.2 : for $\rho : \mathbf{L} \longrightarrow \mathbf{G}$ a morphism and $\lambda = \mathrm{Ad}\,\rho : \mathbf{L}_{\mathbb{C}} \longrightarrow \mathbf{Aut}(\mathfrak{g}_{\mathbb{C}})$, the Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ induces a Lie bracket on $\mathcal{E}_{\mathrm{Ad}\,\rho}$, compatible with its natural μ-$\mathbb{C}$VHS structure but usually not with any relabeled $\mathbb{C}$VHS.

4.3. **G-$\mathbb{C}$VHS and moduli spaces.** Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ be a $\mathbf{G}$-$\mathbb{C}$VHS on X.

On the Betti side it canonically defines a point in $M(\Gamma, \mathbf{G})$: the isomorphism class of the flat G-bundle $\mathcal{P}$ defined in 4.1.1.

On the Dolbeault side : the adjoint bundle $\mathrm{Ad}\,\mathrm{P} = \mathrm{P} \times_{G,\mathrm{Ad}} \mathfrak{g} = f^*((L_{\mathbb{C}} \times_{Q,\mathrm{Ad}\circ\tau} \mathfrak{g})_{|D})$ identifies with the graded bundle $\mathrm{Gr}_F\,\mathrm{Ad}\,\mathcal{P}$ of the weight 0 μ-complex variation of Hodge structure $\mathrm{Ad}\,\mathcal{P} = \mathcal{E}_{\mathfrak{g}}$ associated to the representation $\mathbf{L}_{\mathbb{C}} \longrightarrow \mathbf{GL}(\mathfrak{g})$:

$$\mathrm{Ad}\,\mathrm{P} = \bigoplus_{p\in\mathbb{Z}} (\mathrm{Ad}\,\mathrm{P})^{p,-p} \quad .$$

Definition 4.3.1. *Define $\theta_f \in (\mathrm{Ad}\,\mathrm{P})^{-\mu,\mu} \otimes \Omega^1_X$ as the differential of f.*

Thus (P, θ) is a semi-stable G-Higgs-bundle and the $\mathbf{G}$-$\mathbb{C}$VHS $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}, \rho, f : \tilde{X} \longrightarrow D)$ canonically defines the point $[(\mathrm{P}, \theta)]$ (called a system of $\mathbf{G}$-Hodge bundles by Simpson) in $\mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})$.

Proposition 4.3.2. [25, cor 4.2] *Let $[\rho] \in \mathbf{M}(\Gamma, \mathbf{G})(\mathbb{C})$ with ρ reductive. Then $\phi_{\mathbf{G}}([\rho]) \in \mathbf{M}_{\mathrm{Dol}}(X, \mathbf{G})(\mathbb{C})$ is $\mathbb{C}^*$-fixed if and only if ρ is the monodromy of a $\mathbf{G}$-complex variation of Hodge structure $f : \tilde{X} \longrightarrow D$. Moreover $\phi_{\mathbf{G}}([\rho]) = [(\mathrm{P}, \theta)]$.*

5. Cohomology

5.1. **Cohomology of μ-$\mathbb{C}$VHS.** Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ be a $\mathbf{G}$-$\mathbb{C}$VHS. Let $\lambda : \mathbf{G} \longrightarrow \mathbf{GL}(E)$ be a complex representation of $\mathbf{G}$. The Higgs bundle $\phi_{\mathbf{GL}(E)}(\lambda \circ \rho)$ associated by Simpson's correspondence to the monodromy $\lambda \circ \rho : \Gamma \longrightarrow G$ of the weight 0 μ-$\mathbb{C}$VHS $\mathcal{E}_\lambda$ defined in 4.2.5 is $(\mathrm{Gr}_F\,\mathcal{E}_\lambda, \theta_\lambda)$, where $\theta_\lambda = \mathrm{ad}\,\lambda(\theta) \in \Omega^1_X \otimes \mathrm{End}(\mathrm{Gr}_F\,\mathcal{E}_\lambda)$ and $\mathrm{ad}\,\lambda : \mathfrak{g} \longrightarrow \mathrm{End}(E)$ is deduced from $\lambda : \mathbf{G} \longrightarrow \mathbf{GL}(E)$.

The Dolbeault complex $(\Omega^\bullet_X(\mathrm{Gr}_F\,\mathcal{E}_\lambda), \theta_\lambda)$ decomposes as a direct sum :

(5.1)
$$(\Omega^\bullet_X(\mathrm{Gr}_F\,\mathcal{E}_\lambda), \theta_\lambda) = \bigoplus_{p\in\mathbb{Z}} \left(\mathrm{Gr}^p_F\mathcal{E}_\lambda \xrightarrow{\theta_\lambda} \mathrm{Gr}^{p-\mu}_F\mathcal{E}_\lambda \otimes \Omega^1_X \xrightarrow{\theta_\lambda} \mathrm{Gr}^{p-2\mu}_F\mathcal{E}_\lambda \otimes \Omega^2_X \xrightarrow{\theta_\lambda} \cdots \right) \quad .$$

Thus the isomorphism from theorem 2.3.1

$$H^i(X, \mathcal{E}_\lambda) = \mathbb{H}^i(X, (\Omega^\bullet_X(\mathrm{Gr}_F\,\mathcal{E}_\lambda), \theta_\lambda)) \tag{5.2}$$

particularizes to

Proposition 5.1.1. *Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ be a $\mathbf{G}$-complex variation of Hodge structure and $\lambda : \mathbf{G} \longrightarrow \mathbf{GL}(E)$ be a representation of $\mathbf{G}$. Then*

$$H^i(X, \mathcal{E}_\lambda) = \bigoplus_{p\in\mathbb{Z}} \mathbb{H}^i(X, (\mathrm{Gr}^p_F\mathcal{E}_\lambda \xrightarrow{\theta_\lambda} \mathrm{Gr}^{p-\mu}_F\mathcal{E}_\lambda \otimes \Omega^1_X \xrightarrow{\theta_\lambda} \mathrm{Gr}^{p-2\mu}_F\mathcal{E}_\lambda \otimes \Omega^2_X \xrightarrow{\theta_\lambda} \cdots)) \quad . \tag{5.3}$$

Notice that equality (5.3) is nothing else than the classical result of Deligne [31] putting a complex Hodge structure of weight μi on the cohomology H^i of the weight 0 μ-$\mathbb{C}$VHS $\mathcal{E}_\lambda$ on X : the Hodge filtration $F^\bullet$ on $\mathcal{E}_\lambda$ defines a Hodge filtration $F^\bullet\Omega^\bullet_X(\mathcal{E}_\lambda)$ on the holomorphic De Rham complex $\Omega^\bullet_X(\mathcal{E}_\lambda)$ by

$$F^r\Omega^k_X(\mathcal{E}_\lambda) = \Omega^k_X \otimes F^r\mathcal{E}_\lambda \quad .$$

This induces a Hodge filtration on $H^i(X, \mathcal{E}_\lambda) = \mathbb{H}^i(\Omega_X^\bullet(\mathcal{E}_\lambda))$:

$$H^i(X, \mathcal{E}_\lambda) = \bigoplus_{p+q=i\mu} H^{p,q}(X, \mathcal{E}_\lambda) \quad . \tag{5.4}$$

The Hodge to De Rham spectral sequence degenerates at E_1 thus

$$H^{p,i\mu-p}(X, \mathcal{E}_\lambda) = \mathbb{H}^i(X, \mathrm{Gr}_F^p \Omega_X^\bullet(\mathcal{E}_\lambda)) \quad . \tag{5.5}$$

But this last complex $\mathrm{Gr}_F^p \Omega_X^\bullet(E_\lambda))$ is nothing else than

$$\left(\mathrm{Gr}_F^p \mathcal{E}_\lambda \xrightarrow{\theta_\lambda} \mathrm{Gr}_F^{p-\mu} \mathcal{E}_\lambda \otimes \Omega_X^1 \xrightarrow{\theta_\lambda} \mathrm{Gr}_F^{p-2\mu} \mathcal{E}_\lambda \otimes \Omega_X^2 \xrightarrow{\theta_\lambda} \cdots \right) \quad .$$

Notice moreover that Deligne's construction is fonctorial : if $\mathcal{E}_\lambda \xrightarrow{\psi} \mathcal{E}_\nu$ is a morphism of weight 0 μ-$\mathbb{C}$VHS on X then the induced morphism on cohomology :

$$H^\bullet(X, \mathcal{E}_\lambda) \xrightarrow{\psi} H^\bullet(X, \mathcal{E}_\nu)$$

is a morphism of Hodge structures.

5.2. **The case of the adjoint representation.** Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{L}_\mathbb{C}, u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ be an $\mathbf{L}_\mathbb{C}$-$\mathbb{C}$VHS. We fix a real morphism $\eta : \mathbf{L} \longrightarrow \mathbf{G}$ and we particularize the previous section to the case where $\lambda = \mathrm{Ad} \circ \eta : \mathbf{L} \longrightarrow \mathbf{GL}(\mathfrak{g})$. In this case the Lie bracket on $\mathbb{R}$-VHS $\mathcal{E}_\mathfrak{g} := \mathcal{E}_\lambda$ induced by the Lie bracket on $\mathfrak{g}$ enriches the Hodge theory of $H^\bullet(X, \mathcal{E}_\mathfrak{g})$.

5.2.1. *Polarization.* In general $\mathfrak{g}$ is not a simple $\mathbf{L}$-module thus the $\mathbb{R}$-VHS $\mathcal{E}_\mathfrak{g}$ admits many nonequivalent polarization. We choose the one compatible with the Lie bracket on $\mathcal{E}_\mathfrak{g}$. Let $B_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ be the Killing form on $\mathfrak{g}$. The symmetric bilinear form on $\mathfrak{g}$ defined by

$$\beta_0(X, Y) = -B_\mathfrak{g}(C_G C_L^{-1} \cdot X, Y)$$

is L-invariant and its associated Hermitian form $\beta_0(C_G \cdot X, \sigma_G(X))$ is positive definite on $\mathfrak{g} \otimes \mathbb{C}$. Thus one can define :

Definition 5.2.1. *We still denote by $\beta_0 : \mathcal{E}_\mathfrak{g} \otimes \mathcal{E}_\mathfrak{g} \longrightarrow \mathbb{R}$ the polarization on $\mathcal{E}_\mathfrak{g}$ defined by $\beta_0 : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$.*

This polarization satisfies the following compatibility with the Lie bracket on $\mathcal{E}_\mathfrak{g}$:

$$\beta_0(Y, [X, Z]) + \beta_0([C_L C_G X, Y], Z) = 0 \quad .$$

5.2.2. *Cohomology.*

Definition 5.2.2. *A graded μ-Hodge $\mathbb{R}$-Lie algebra is a $\mathbb{Z}$-graded real Lie algebra $\mathfrak{m}_\bullet$ such that :*

- *each $\mathfrak{m}_i$, $i \in \mathbb{Z}$, is a weight $\mu \cdot i$ polarized $\mathbb{R}$-HS.*

- *for any $i, j \in \mathbb{Z}$, the Lie bracket*

$$[\cdot,\cdot] : \mathfrak{m}_i \otimes \mathfrak{m}_j \longrightarrow \mathfrak{m}_{i+j}$$

is a morphism of polarized $\mathbb{R}$-HS.

One immediately obtains the :

Lemma 5.2.3. *The cohomology $H^\bullet(X, \mathcal{E}_\mathfrak{g})$ is a graded μ-Hodge $\mathbb{R}$-Lie algebra.*

Moreover the natural polarization on $H^\bullet(X, \mathcal{E}_\mathfrak{g})$ obtained from β_0 is naturally enriched. As the formation of the Hodge structure on the cohomology of an $\mathbb{R}$VHS is functorial and

$$\beta_0 : \mathcal{E}_\mathfrak{g} \otimes \mathcal{E}_\mathfrak{g} \longrightarrow \mathbb{R}(0) = \mathbb{R}$$

is a morphism of weight 0 $\mathbb{R}$VHS (where we consider $\mathbb{R}$ as the constant local system on X) we obtain the following canonical morphism of graded $\mathbb{R}$-Hodge structures :

$$\begin{tikzcd}
H^\bullet(X, \mathcal{E}_\mathfrak{g}) \otimes H^\bullet(X, \mathcal{E}_\mathfrak{g}) \arrow[r, "\sim"] \arrow[ddr, "\beta"'] & H^\bullet(X \times X, p_1^*(\mathcal{E}_\mathfrak{g}) \otimes p_2^*(\mathcal{E}_\mathfrak{g})) \arrow[d, "\Delta^*"] \\
& H^\bullet(X, \mathcal{E}_\mathfrak{g} \otimes \mathcal{E}_\mathfrak{g}) \arrow[d, "\beta_0"] \\
& H^\bullet(X, \mathbb{R})
\end{tikzcd} ,$$

where the horizontal isomorphism is Künneth's one.

The positivity properties of β_0 and its compatibility with the Lie bracket immediately implies the :

Lemma 5.2.4. *The morphism β satisfies the following two properties :*

1. $\forall x, y, z \in H^\bullet(\Gamma, \mathfrak{g})$, $\qquad \beta(x, [y, z]) + (-1)^{d(x)\cdot d(y)} \beta([C_G \cdot C_L^{-1} x, y], z) = 0$ *where $d(x)$ denotes the degree of x divided by μ.*
2. *for any $x \neq 0$ in $H^{p,q}(\Gamma, \mathfrak{g})$, $\beta(C_G \cdot x, \overline{x}) > 0$ in $H^{p+q,p+q}(X)$ where positivity in $H^{p+q,p+q}(X)$ is induced by weak positivity on forms.*

6. Eichler-Shimura for cohomological complex hyperbolic representations

6.1. **Locally homogeneous $\mathbb{C}$VHS and Eichler-Shimura type isomorphisms.** From now on we assume for simplicity that the group $\mathbf{L}$ is simple.

Definition 6.1.1. *Let $(i : \mathbf{L} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}, \rho : \Gamma \longrightarrow L, f : \tilde{X} \longrightarrow D)$ be a $\mathbf{G}$-$\mathbb{C}$VHS. It is called locally homogeneous if f is a biholomorphism.*

Equivalently : the group $\mathbf{L}$ is a real form of $\mathbf{G}$ of Hermitian type, $V = K$ is a maximal compact subgroup of L, the Hodge datum $u : \mathbf{U}(1) \longrightarrow \mathbf{L}^{\mathrm{ad}}$ is the canonical isomorphism of $U(1)$ with the connected center $Z(K^{\mathrm{ad}})$ of the maximal compact subgroup K^{ad} of L^{ad} such that for the (complexified) Cartan decomposition $\mathfrak{g} = \mathfrak{l}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$ the K-module $\mathfrak{p}_\mathbb{C}$

decomposes as $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $u(z)$ acting via multiplication by z on $\mathfrak{p}_+$ and by z^{-1} on $\mathfrak{p}_-$. The variety $X = \Gamma\backslash L/K$ is an Hermitian locally symmetric space.

We refer to [30] for a detailed study of locally homogeneous $\mathbb{C}$VHS. Notice that the index μ of definition 4.2.1 is nothing else in this case than the index $\mu(L)$ of [30, p.247], namely the degree of the covering map $Z(K) \longrightarrow Z(K^{\mathrm{ad}})$.

From the general point of view of non-Abelian Hodge theory the main interest of locally homogeneous $\mathbb{C}$VHS is a computational one : all the Higgs bundles appearing geometrically are of automorphic nature (i.e. given by a representation of K). As a result Dolbeault complexes split into direct sums of (shifted) sheaves. In particular the Hodge factors in the decomposition (5.3) greatly simplify.

Definition 6.1.2. *Let $K_{\mathbb{C}} \subset L_{\mathbb{C}}$ be the complexified group of K. We denote by*

$$F : K_{\mathbb{C}} - \mathrm{mod} \longrightarrow \mathrm{Bun}(X)$$

the functor from the category of finite dimensional $K_{\mathbb{C}}$-modules to the category of holomorphic vector bundles on X which associates to $(\pi, E_{\mathbb{C}}) \in K_{\mathbb{C}} - \mathrm{mod}$ the holomorphic vector bundle $F(\pi) := \Gamma\backslash(G \times_{Q,\pi\circ\tau} E_{\mathbb{C}})_{|D}$ (with the obvious definition of F on morphisms).

In particular the holomorphic tangent bundle TX identifies with the automorphic bundle $F(\mathfrak{p}_+)$, the bundle Ω^1_X with $F(\mathfrak{p}_-)$.

Let $\lambda : \mathbf{G} \longrightarrow \mathbf{GL}(E_{\mathbb{C}})$ be a representation. As $K_{\mathbb{C}} = V_{\mathbb{C}}$ is the centralizer in G of $u(S^1)$, every term $\mathrm{Gr}^p_F \mathcal{E}_\lambda \otimes \Omega^i_X$ in the decomposition (5.1) is an automorphic vector bundle and the differential θ_λ preserves the automorphic structure. For example the $\mathbf{G}$-Higgs field $\theta_\rho : TX = F(\mathfrak{p}_+) \longrightarrow \mathrm{Ad}\,\mathcal{P}_f = F(\mathfrak{g})$ of the Higgs bundle $\mathcal{E}_{\mathrm{Ad}} = \mathrm{Ad}\,\mathcal{P}_f$ is given by the obvious inclusion of $K_{\mathbb{C}}$-modules $\mathfrak{p}_+ \hookrightarrow \mathfrak{g}$. As $K_{\mathbb{C}}$ is reductive, the complex $\mathrm{Gr}^p_F(\Omega^\bullet_X(\mathcal{E}_\lambda))$ completely splits. As a corollary, one obtains Eichler-Shimura type isomorphisms :

Corollary 6.1.3. *Let $X = \Gamma\backslash L/K$ be a co-compact locally Hermitian symmetric space. Let $\lambda : \mathbf{G} \longrightarrow \mathbf{GL}(E_{\mathbb{C}})$ be a finite dimensional representation. Then for any positive integer i one has a canonical decomposition*

$$H^i(\Gamma, E_{\mathbb{C}}) = \bigoplus_{0 \leq j \leq i} H^j(X, F(\tau_j)) \ ,$$

where τ_j is a sub-$K_{\mathbb{C}}$-module of $E_{\mathbb{C}}$.

An explicit formula for the τ_j can be given in terms of highest weight, but we won't give it here.

6.2. **The complex hyperbolic case.**

6.2.1. *Notations.* Let $n > 1$ be a positive integer. Let V denote the $(n+1)$-dimensional $\mathbb{C}$-vector space endowed with the Hermitian form $h(\mathbf{z}, \mathbf{w}) = z_0\overline{w_0} + \cdots + z_{n-1}\overline{w_{n-1}} - z_n\overline{w_n}$. Let $\mathbf{L} = \mathbf{SU}(n,1) := \mathbf{SU}(V,h)$ and $\Gamma \overset{i}{\hookrightarrow} SU(n,1)$ be a co-compact torsion-free lattice. We denote by $\mathbf{U}(n)$ the maximal compact subgroup of L, $\mathbf{K}_\mathbb{C} \simeq \mathbf{GL}(n,\mathbb{C}) = \{\begin{pmatrix} X & 0 \\ 0 & \chi^{-1}(X) \end{pmatrix}, X \in \mathbf{GL}(n,\mathbb{C})\} \subset \mathbf{SL}(n+1,\mathbb{C}) = \mathbf{L}_\mathbb{C}$. Here $\chi : \mathbf{U}(n) \longrightarrow \mathbf{U}(1)$ denotes the determinant.

The canonical Hodge datum $u : \mathbf{U}(1) \longrightarrow \mathbf{SU}(n,1)^{\mathrm{ad}}$ is given by

$$u(z) = [\begin{pmatrix} z\mathrm{Id}_n & 0 \\ 0 & 1 \end{pmatrix}] \ ,$$

where $[A]$ denote the class in $\mathbf{U}(n,1)^{\mathrm{ad}}$ of a matrix $A \in \mathbf{U}(n,1)$. The index μ is $n+1$ and $\tilde{u} : \mathbf{U}(1) \longrightarrow \mathbf{SU}(n,1)$ is defined by

$$\tilde{u}(z) = \begin{pmatrix} z\mathrm{Id}_n & 0 \\ 0 & z^{-n} \end{pmatrix} \ .$$

6.2.2. *The result.*

Definition 6.2.1. *Let j be a positive integer. We say that a finite dimensional $\mathbf{SU}(V,h)$-module π is j-cohomological if for some co-compact lattice $\Gamma \subset SU(n,1)$ one has $H^j(\Gamma, \pi) \neq 0$. We denote by $\mathrm{Coh}^j_\mathbb{R}$ (respectively $\mathrm{Coh}^j_\mathbb{C}$) the set of isomorphism classes of* real *(respectively* complex*) irreducible j-cohomological representations of $\mathbf{SU}(n,1)$.*

As explained in the introduction by Raghunathan's theorem :

$$\mathrm{Coh}^1_\mathbb{R} = \{S^k V_\mathbb{R} \text{ for } k \geq 0\}, \qquad \mathrm{Coh}^1_\mathbb{C} = \{S^k V \text{for } k \in \mathbb{Z}\}$$

with the usual notation $S^kV = S^{-k}V^*$ for $k < 0$.

The main result of this section is a description of the complex Hodge structures on $H^1(\Gamma, S^kV)$ or $H^1(\Gamma, S^kV^*)$. A crucial point in the proof of theorem 1.3.3 is that they are extremely simple, with only one non-vanishing Hodge type (recall that the functor F was defined in section 6.1) :

Theorem 6.2.2. *Let $k > 0$ be a positive integer. Then*

$$H^1(\Gamma, S^kV) = H^1(\Gamma, S^kV)^{(-k, \mu+k)} = H^1(X, F(S^k\mathfrak{p}_+ \otimes \chi^{-k}))$$
$$H^1(\Gamma, S^kV^*) = H^1(\Gamma, S^kV^*)^{(\mu+k, -k)} = H^0(X, F(S^{k+1}\mathfrak{p}_+^* \otimes \chi^k)) \ .$$

6.3. **Decomposition of S^pV and S^pV^* as $U(n)$-modules.**

Lemma 6.3.1. *Let V be the standard real $\mathbf{SU}(n,1)$-module. Let $\chi : K = U(n) \longrightarrow S^1$ be the character* det*. Then for all integers $k \in \mathbb{Z}$ the following decomposition of complex K-modules holds :*

$$S^kV = (\bigoplus_{i=0}^{k} S^i\mathfrak{p}_+) \otimes \chi^{-k}, \qquad S^kV^* = (\bigoplus_{i=0}^{k} S^i\mathfrak{p}_+^*) \otimes \chi^k. \tag{6.1}$$

Proof. The decomposition of V as complex K-module is its decomposition in $K_{\mathbb{C}}$-modules. As the embedding $K_{\mathbb{C}} = GL(n, \mathbb{C}) \xrightarrow{i} L_{\mathbb{C}} = SL(n+1, \mathbb{C})$ is given by

$$i(X) = \begin{pmatrix} X & 0 \\ 0 & \chi^{-1}(X) \end{pmatrix} \quad ,$$

V is the direct sum of the standard $K_{\mathbb{C}}$-module Z of rank n and χ^{-1}. Now notice that $\mathfrak{p}_+$ is the tensor product $Z \otimes \chi$. We thus get

$$V = (\mathfrak{p}_+ \oplus 1) \otimes \chi^{-1} \quad .$$

The lemma follows. $\square$

6.3.1. *Hodge structure.* Let $\lambda : \mathbf{L}_{\mathbb{C}} \longrightarrow \mathbf{GL}(V)$. The weight 0 complex Hodge structure V has the following Hodge decomposition :

$$V = V^{-1} \oplus V^{+n} \quad \text{with} \quad V^{-1} = \mathfrak{p}_+ \cdot \chi^{-1} \;\text{ and }\; V^n = \chi^{-1} \quad .$$

On symmetric powers this naturally leads to :

$$\mathrm{Gr}_F^p S^k V = \begin{cases} S^j \mathfrak{p}_+ \otimes \chi^{-k} & \text{if } p = (k-j)n - j \text{ and } 0 \leq j \leq k \\ 0 & \text{otherwise} \end{cases} \quad . \tag{6.2}$$

On the dual V^* :

$$V = V^1 \oplus V^{-n} \quad \text{with} \quad V^1 = \mathfrak{p}_+^* \cdot \chi \;\text{ and }\; V^{-n} = \chi \quad .$$

This similarly yields :

$$\mathrm{Gr}_F^p S^k V^* = \begin{cases} S^j \mathfrak{p}_+^* \otimes \chi^{k} & \text{if } p = n(j-k) + j \text{ and } 0 \leq j \leq k \\ 0 & \text{otherwise} \end{cases} \quad . \tag{6.3}$$

6.4. **The Dolbeault complex.** As an intermediary step towards theorem 6.2.2, we first prove the

Proposition 6.4.1. *One has the following equalities :*

$$H^1(\Gamma, S^k V) = H^1(X, F(S^k \mathfrak{p}_+ \otimes \chi^{-k})) \oplus H^0(X, F(\frac{S^k \mathfrak{p}_+ \otimes \mathfrak{p}_+^*}{S^{k-1}\mathfrak{p}_+} \otimes \chi^{-k})), \tag{6.4}$$

and

$$H^1(\Gamma, S^k V^*) = H^1(X, F(\chi^k)) \oplus H^0(X, S^{k+1}\mathfrak{p}_+^* \otimes \chi^k) \quad . \tag{6.5}$$

Proof. We first deal with $S^k V$. We can apply equality (5.3) for $E_\lambda = S^k V$. As we compute hyper-cohomology in degree one, we can truncate our complexes in degree 2, thus obtaining :

$$H^1(\Gamma, S^k V) = \mathbb{H}^1(X, F(\chi^{-k} \otimes \mathcal{C}^k)) \quad , \quad \text{where } \mathcal{C}^k \text{ denotes the complex} \tag{6.6}$$

$$\left(\begin{tikzcd}[column sep=small]
1 \oplus \arrow[dr] & \mathfrak{p}_+ \oplus \arrow[dr] & \cdots \oplus \arrow[dr] & S^{k-2}\mathfrak{p}_+ \oplus \arrow[dr] & S^{k-1}\mathfrak{p}_+ \oplus \arrow[dr] & S^k\mathfrak{p}_+ \\
1 \otimes \mathfrak{p}_+^* \oplus \arrow[dr] & \mathfrak{p}_+ \otimes \mathfrak{p}_+^* \oplus \arrow[dr] & S^2\mathfrak{p}_+ \otimes \mathfrak{p}_+^* \oplus \arrow[dr] & \cdots \arrow[dr] & \oplus S^{k-1}\mathfrak{p}_+ \otimes \mathfrak{p}_+^* \oplus \arrow[dr] & S^k\mathfrak{p}_+ \otimes \mathfrak{p}_+^* \\
1 \otimes \Lambda^2\mathfrak{p}_+^* \oplus & \mathfrak{p}_+ \otimes \Lambda^2\mathfrak{p}_+^* \oplus & S^2\mathfrak{p}_+ \otimes \Lambda^2\mathfrak{p}_+^* \oplus & S^3\mathfrak{p}_+ \otimes \Lambda^2\mathfrak{p}_+^* \oplus & \cdots & \oplus S^k\mathfrak{p}_+ \otimes \Lambda^2\mathfrak{p}_+^*
\end{tikzcd} \right) .$$

In this complex the maps of $K_{\mathbb{C}}$-modules are the standard ones induced by $1 \longrightarrow \mathfrak{p}_+ \otimes \mathfrak{p}_+^*$ and contractions :

$$S^i\mathfrak{p}_+ \otimes \Lambda^k\mathfrak{p}_+^* \longrightarrow S^i\mathfrak{p}_+ \otimes (\mathfrak{p}_+ \otimes \mathfrak{p}_+^*) \otimes \Lambda^k\mathfrak{p}_+^* \longrightarrow S^{i+1}\mathfrak{p}_+ \otimes \Lambda^{k+1}\mathfrak{p}_+^* \ .$$

Notice that for all non-negative integers i the following short sequence of $K_{\mathbb{C}}$-modules is exact :

$$0 \longrightarrow S^i\mathfrak{p}_+ \longrightarrow S^{i+1}\mathfrak{p}_+ \otimes \mathfrak{p}_+^* \longrightarrow S^{i+2}\mathfrak{p}_+ \otimes \Lambda^2\mathfrak{p}_+^* \ .$$

As a result,

$$H^1(\Gamma, S^kV) = H^1(X, F(S^k\mathfrak{p}_+ \otimes \chi^{-k}) \oplus H^0(X, F(\frac{S^k\mathfrak{p}_+ \otimes \mathfrak{p}_+^*}{S^{k-1}\mathfrak{p}_+} \otimes \chi^{-k})).$$

Similarly applying equality (5.3) for $E_\lambda = S^kV^*$, we obtain :

(6.7) $\quad H^1(\Gamma, S^kV) = \mathbb{H}^1(X, F(\chi^k \otimes \mathcal{C}^k))$, where $\mathcal{C}^k$ denotes the complex

$$\left(\begin{array}{l}
1 \oplus \mathfrak{p}_+^* \oplus S^2\mathfrak{p}_+^* \oplus S^3\mathfrak{p}_+^* \oplus \cdots \oplus S^k\mathfrak{p}_+^* \\
1 \otimes \mathfrak{p}_+^* \oplus \mathfrak{p}_+^* \otimes \mathfrak{p}_+^* \oplus S^2\mathfrak{p}_+^* \otimes \mathfrak{p}_+^* \oplus \cdots \oplus S^{k-1}\mathfrak{p}_+^* \otimes \mathfrak{p}_+^* \oplus S^k\mathfrak{p}_+^* \otimes \mathfrak{p}_+^* \\
1 \otimes \Lambda^2\mathfrak{p}_+^* \oplus \mathfrak{p}_+^* \otimes \Lambda^2\mathfrak{p}_+^* \oplus \cdots \oplus S^{k-2}\mathfrak{p}_+^* \otimes \Lambda^2\mathfrak{p}_+^* \oplus S^{k-2}\mathfrak{p}_+^* \otimes \Lambda^2\mathfrak{p}_+^* \oplus S^k\mathfrak{p}_+^* \otimes \Lambda^2\mathfrak{p}_+^*
\end{array}\right) .$$

Once more the maps of $K_{\mathbb{C}}$-modules are the standard ones induced by $1 \longrightarrow \mathfrak{p}_+ \otimes \mathfrak{p}_+^*$ and contractions :

$$S^i\mathfrak{p}_+^* \otimes \Lambda^k\mathfrak{p}_+^* \longrightarrow S^i\mathfrak{p}_+^* \otimes (\mathfrak{p}_+ \otimes \mathfrak{p}_+^*) \otimes \Lambda^k\mathfrak{p}_+^* \longrightarrow S^{i-1}\mathfrak{p}_+^* \otimes \Lambda^{k+1}\mathfrak{p}_+^*$$

Once more for all non-negative integers i the following short sequence of $K_{\mathbb{C}}$-modules is exact :

$$0 \longrightarrow S^i\mathfrak{p}_+^* \longrightarrow S^{i-1}\mathfrak{p}_+^* \otimes \mathfrak{p}_+^* \longrightarrow S^{i-2}\mathfrak{p}_+^* \otimes \Lambda^2\mathfrak{p}_+^* \ .$$

As a result,

$$\begin{aligned} H^1(\Gamma, S^kV^*) &= H^1(X, F(\chi^k)) \oplus H^0(X, \ker(S^k\mathfrak{p}_+^* \otimes \mathfrak{p}_+^* \to S^{k-1}\mathfrak{p}_+^* \otimes \Lambda^2\mathfrak{p}_+^*) \otimes \chi^k) \\ &= H^1(X, F(\chi^k)) \oplus H^0(X, F(S^{k+1}\mathfrak{p}_+^* \otimes \chi^k)) \ . \end{aligned}$$

□

Remark 6.4.2. Notice that the complexes $\mathcal{C}^k$ appearing in (6.4) and (6.7) are nothing else than a truncated tautological Koszul complex. The only terms giving rise to cohomology are the ones on the boundary created by the truncation.

6.5. **Vanishing theorem : a Hodge type argument.** To conclude the proof of theorem 6.2.2 from proposition 6.4.1 we have to show the :

Proposition 6.5.1. $H^0(X, F(\frac{S^k\mathfrak{p}_+ \otimes \mathfrak{p}_+^*}{S^{k-1}\mathfrak{p}_+} \otimes \chi^{-k}))) = H^1(X, F(\chi^k)) = 0.$

Proof. This is provided by a Hodge type argument. Using the equation (5.5), we can compute the Hodge decomposition of the weight 1 complex Hodge structures $H^1(\Gamma, S^kV)$ and $H^1(\Gamma, S^kV^*)$:

$$(6.8)\qquad \begin{aligned} &H^1(\Gamma, S^kV) = H^1(\Gamma, S^kV)^{(-k,\mu+k)} \oplus H^1(\Gamma, S^kV)^{(-k+\mu,k)} \\ &\text{with} \begin{cases} H^1(\Gamma, S^kV)^{(-k,\mu+k)} = H^1(X, F(S^k\mathfrak{p}_+ \otimes \chi^{-k}) \\ H^1(\Gamma, S^kV)^{(-k+\mu,k)} = H^0(X, F(\frac{S^k\mathfrak{p}_+ \otimes \mathfrak{p}_+^*}{S^{k-1}\mathfrak{p}_+} \otimes \chi^{-k})) \end{cases} . \end{aligned}$$

Similarly :

$$(6.9)\qquad \begin{aligned} &H^1(\Gamma, S^kV^*) = H^1(\Gamma, S^kV)^{(k+\mu,-k)} \oplus H^1(\Gamma, S^kV)^{(-nk,\mu+nk)} \\ &\text{with} \begin{cases} H^1(\Gamma, S^kV^*)^{(k+\mu,-k)} = H^0(X, F(S^{k+1}\mathfrak{p}_+^* \otimes \chi^k)) \\ H^1(\Gamma, S^kV^*)^{(-nk,\mu+nk)} = H^1(X, F(\chi^k)) \end{cases} . \end{aligned}$$

As $V_\mathbb{R}$ is a *real* representation of $\mathbf{L}$, the bundle $\mathcal{E}_{S^kV_\mathbb{R}}$ is a $\mathbb{R}$VHS on X. As $S^kV_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = S^kV \oplus S^kV^*$, this Hodge structure is nothing else than

$$\mathcal{E}_{S^kV_\mathbb{R}} \otimes_\mathbb{R} \mathbb{C} = \mathcal{E}_{S^kV} \oplus \mathcal{E}_{S^kV^*} \quad .$$

In particular its cohomology

$$H^1(\Gamma, S^kV_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) = H^1(\Gamma, S^kV) \oplus H^1(\Gamma, S^kV^*)$$

has a weight μ *real* Hodge structure. The Hodge type symmetry $\overline{H^{p,q}} = H^{q,p}$ for *real* Hodge structures forces at once

$$H^1(\Gamma, S^kV)^{(-k+\mu,k)} = H^1(\Gamma, S^kV^*)^{(-nk,\mu+nk)} = 0$$

and the result.

□

Remark 6.5.2. I don't know of any direct proof of proposition 6.5.1.

7. All the deformations are trivial

Let X be a compact Kähler manifold and $\rho : \pi_1(X) \longrightarrow G$ a representation of the fundamental group $\pi_1(X)$. Then by definition $H^1(\pi_1(X), \mathrm{Ad}\,\rho) = H^1(X, \mathcal{E}_\mathfrak{g})$ and $H^2(\pi_1(X), \mathrm{Ad}\,\rho) \subset H^2(X, \mathcal{E}_\mathfrak{g})$. In the case where $\mathcal{E}_\mathfrak{g}$ is an $\mathbb{R}$VHS it is not known whether $H^2(\pi_1(X), \mathrm{Ad}\,\rho)$ is a Hodge substructure of $H^2(X, \mathcal{E}_\mathfrak{g})$. In the case we consider the manifold $X = \Gamma\backslash\mathbf{H}^n_\mathbb{C}$ is a $K(\Gamma, 1)$ and the cohomologies $H^\bullet(\pi_1(X), \mathrm{Ad}\,\rho)$ and $H^\bullet(X, \mathcal{E}_\mathfrak{g})$ coincide. We will freely use this identification.

7.1. **Hodge decomposition.** The sequence

$$H^1(\Gamma, \mathrm{Ad}\,\rho) \otimes H^1(\Gamma, \mathrm{Ad}\,\rho) \longrightarrow H^2(\Gamma, \mathrm{Ad}\,\rho \otimes \mathrm{Ad}\,\rho) \xrightarrow{[\cdot,\cdot]} H^2(\Gamma, \mathrm{Ad}\,\rho)$$

of weight 2μ real Hodge structures decomposes (after complexification) accordingly to Hodge types. For any $p \in \mathbb{Z}$ one obtains :

$$\begin{array}{c} \bigoplus_{\substack{a,b\in\mathbb{Z} \\ a+b=p}} H^1(\Gamma, \mathrm{Ad}\,\rho)^{(a,\mu-a)} \otimes H^1(\Gamma, \mathrm{Ad}\,\rho)^{(b,\mu-b)} \\ \downarrow \\ H^2(\Gamma, \mathrm{Ad}\,\rho \otimes \mathrm{Ad}\,\rho)^{(p,2\mu-p)} \\ \downarrow [\cdot,\cdot] \\ H^2(\Gamma, \mathrm{Ad}\,\rho)^{(p,2\mu-p)} \quad . \end{array} \tag{7.1}$$

Let us analyze the first term of the sequence (7.1). By Ragunathan's theorem 1.1.1 one has :

$$H^1(\Gamma, \mathrm{Ad}\,\rho) = \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} H^1(\Gamma, \mathfrak{g}_\pi) \quad .$$

For any $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$ the inclusion $\mathfrak{g}_\pi \hookrightarrow \mathfrak{g}$ is a morphism of real Hodge structures.

Definition 7.1.1. *Let $\pi = S^k V_{\mathbb{R}} \in \mathrm{Coh}^1_{\mathbb{R}}$. One defines $i_\pi = -k$, $W^{\mathbb{C}}_\pi = S^k \mathfrak{p}_+ \otimes \chi^{-k}$ and $W^{\mathbb{C}}_{\pi^*} = S^{k+1}\mathfrak{p}^*_+ \otimes \chi^k$.*

Definition 7.1.2. *For $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$ one defines $V^{\mathbb{C}}_\pi$ as the complex $W^{\mathbb{C}}_\pi$-isotypic component of the complex $\mathbf{U}(n)$-module $\mathfrak{g}_\pi \otimes \mathbb{C}$. Similarly one defines $V^{\mathbb{C}}_{\pi^*}$ as the complex $W^{\mathbb{C}}_{\pi^*}$-isotypic component of the complex $\mathbf{U}(n)$-module $\mathfrak{p}^*_+ \otimes_{\mathbb{C}} (\mathfrak{g}_\pi \otimes \mathbb{C})$. Thus $V_\pi \otimes \mathbb{C} = W^{\mathbb{C}}_\pi \oplus \overline{W^{\mathbb{C}}_\pi}$ as a $\mathbf{U}(n)$-module.*

By section 6 we know that as a real Hodge structure $H^1(\Gamma, \mathfrak{g}_\pi)$ has only two non-trivial Hodge types :

$$H^1(\Gamma, \mathfrak{g}_\pi) \otimes_{\mathbb{R}} \mathbb{C} = H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi} \oplus \overline{H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi}} \quad .$$

Explicitly :

$$H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi} = H^1(X, \mathcal{F}(V^{\mathbb{C}}_\pi)) \quad .$$

Thus

$$\begin{aligned} H^1(\Gamma, \mathrm{Ad}\,\rho) \otimes_{\mathbb{R}} \mathbb{C} &= \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} H^1(\Gamma, \mathfrak{g}_\pi) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} \left(H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi} \oplus \overline{H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi}}\right) \quad . \end{aligned} \tag{7.2}$$

In particular for any $a \in \mathbb{Z}$:

$$H^1(\Gamma, \mathrm{Ad}\,\rho)^{(a,\mu-a)} = \bigoplus_{\substack{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \\ i_\pi = a}} H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi} \oplus \bigoplus_{\substack{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \\ i_\pi = \mu - a}} \overline{H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi}} \quad . \tag{7.3}$$

Remark 7.1.3. Notice that the right hand side of equality (7.3) contains at most one non-zero term for $a \neq 0$: this follows immediately from the fact that $i_\pi \leq 0$ for $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$.

Let $z \in H^1(\Gamma, \mathrm{Ad}\,\rho)$. By equation (7.2) z uniquely decomposes as

$$z = \sum_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (z_\pi + \overline{z_\pi}) \ ,$$

with $z_\pi \in H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi}$. Thus :

$$[z,z] = \sum_{\substack{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\\ \pi' \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)}} [z_\pi, z_{\pi'}] \oplus 2 \sum_{\substack{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\\ \pi' \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)}} [z_\pi, \overline{z_{\pi'}}] \oplus \sum_{\substack{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\\ \pi' \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)}} \overline{[z_\pi, z_{\pi'}]} \ .$$

For $\pi \in \mathrm{Coh}^1_{\mathbb{R}}, \pi' \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$, notice that $[z_\pi, z_{\pi'}]$ is of type $(i_\pi + i_{\pi'}, 2\mu - (i_\pi + i_{\pi'}))$, $[z_\pi, \overline{z_{\pi'}}]$ is of type $(\mu + i_\pi - i_{\pi'}, \mu + i_{\pi'} - i_\pi)$ and $\overline{[z_\pi, z_{\pi'}]}$ is of type $(2\mu - (i_\pi + i_{\pi'}), i_\pi + i_{\pi'})$. As $i_\pi \leq 0$ for any $\pi \in \mathrm{Coh}^1_{\mathbb{R}}$ the (μ,μ)-component of $[z,z]$ is :

$$[z,z]^{(\mu,\mu)} = \sum_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} [z_\pi, \overline{z_\pi}] \ .$$

Thus we obtain :

Lemma 7.1.4. *If* $[z,z] = 0$ *in* $H^2(\Gamma, \mathrm{Ad}\,\rho)$ *then* $\sum_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} [z_\pi, \overline{z_\pi}] = 0$ *in* $H^2(\Gamma, \mathrm{Ad}\,\rho)^{(\mu,\mu)}$.

7.2. **Compact versus non-compact.** Fix $\mathbf{K_G}$ a maximal compact subgroup of $\mathbf{G}$ containing $\mathbf{U}(n)$, with Cartan involution C_G, and Cartan decomposition $\mathfrak{g} = \mathfrak{k}_{\mathbf{G}} \oplus \mathfrak{p}_{\mathbf{G}}$. As the Cartan decomposition of $\mathfrak{g}$ is $\mathbf{U}(n)$-stable each $\mathbf{U}(n)$-module V_π, $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$, decomposes into a "compact" and a "non-compact" part :

$$V_\pi = V_{\pi,c} \oplus V_{\pi,n} \ ,$$

where $V_{\pi,c} = V_\pi \cap \mathfrak{k}_{\mathbf{G}}$ and $V_{\pi,n} = V_\pi \cap \mathfrak{p}_{\mathbf{G}}$. We say that V_π is of compact type if $V_\pi = V_{\pi,c}$ and of non-compact type if $V_\pi = V_{\pi,n}$.

As a corollary the $\mathbf{SU}(n,1)$-module $\mathfrak{g}_\pi$, $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$, uniquely decomposes as $\mathfrak{g}_\pi = \mathfrak{g}_{\pi,c} \oplus \mathfrak{g}_{\pi,n}$, where $\mathfrak{g}_{\pi,c}$ (resp. $\mathfrak{g}_{\pi,n}$) is the $\mathbf{SU}(n,1)$-module generated by $V_{\pi,c}$ (resp. $V_{\pi,n}$). Once more the inclusions $\mathfrak{g}_{\pi,c} \hookrightarrow \mathfrak{g}$ and $\mathfrak{g}_{\pi,n} \hookrightarrow \mathfrak{g}$ are morphisms of real Hodge structures. Thus the decomposition 7.3 refines to :

$$\begin{aligned} H^1(\Gamma,\rho) = & \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (H^1(\Gamma, \mathfrak{g}_{\pi,c})^{i_\pi, \mu - i_\pi} \oplus \overline{H^1(\Gamma, \mathfrak{g}_{\pi,c})^{i_\pi, \mu - i_\pi}}) \\ & \oplus \\ & \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (H^1(\Gamma, \mathfrak{g}_{\pi,n})^{i_\pi, \mu - i_\pi} \oplus \overline{H^1(\Gamma, \mathfrak{g}_{\pi,n})^{i_\pi, \mu - i_\pi}}) \ . \end{aligned} \tag{7.4}$$

Explicitly : $H^1(\Gamma, \mathfrak{g}_{\pi,c})^{i_\pi, \mu - i_\pi} \simeq H^1(X, \mathcal{F}(V^{\mathbb{C}}_{\pi,c}))$ and $H^1(\Gamma, \mathfrak{g}_{\pi,n})^{i_\pi, \mu - i_\pi} \simeq H^1(X, \mathcal{F}(V^{\mathbb{C}}_{\pi,n}))$.

Each $z_\pi \in H^1(\Gamma, \mathfrak{g}_\pi)^{i_\pi, \mu - i_\pi}$ thus uniquely decomposes as

$$z_\pi = z_{\pi,c} + z_{\pi,n} \ ,$$

with $z_{\pi,c} \in H^1(\Gamma, \mathfrak{g}_{\pi,c})^{i_\pi, \mu - i_\pi}$ and $z_{\pi,n} \in H^1(\Gamma, \mathfrak{g}_{\pi,n})^{i_\pi, \mu - i_\pi}$.

Taking the C_G-fixed part of $[z,z]$, we deduce from lemma 7.1.4 :

Lemma 7.2.1. *If $[z,z]=0$ in $H^2(\Gamma, \mathrm{Ad}\,\rho)$ then*

$$\sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} ([z_{\pi,c}, \overline{z_{\pi,c}}] + [z_{\pi,n}, \overline{z_{\pi,n}}]) = 0 \in H^2(\Gamma, \mathrm{Ad}\,\rho)^{(\mu,\mu)} \quad . \tag{7.5}$$

7.3. **Proof of theorem 1.3.7.** Although theorem 1.3.7 is a corollary of theorem 1.3.3 we first give a proof of the easier theorem 1.3.7.

With the notations of the introduction one has for any $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$:

$$H^1(\Gamma, \mathfrak{g}_{\pi,c}) = \bigoplus_{\chi_0\in\Phi_{\pi,c}} \chi_0^{d_{\chi_0}} \otimes H^1(\Gamma,\pi) \quad \text{and} \quad H^1(\Gamma, \mathfrak{g}_{\pi,n}) = \bigoplus_{\chi_0\in\Phi_{\pi,n}} \chi_0^{d_{\chi_0}} \otimes H^1(\Gamma,\pi)$$

as a $\mathbf{T}_0(\rho)$-module. Let $z_{\pi,c} = \sum_{\chi_0\in\Phi_{\pi,c}} z_{\pi,c,\chi_0}$ and $z_{\pi,n} = \sum_{\chi_0\in\Phi_{\pi,n}} z_{\pi,n,\chi_0}$ be the corresponding decomposition. The restriction of equation 7.2.1 to the 0-eigenspace of $\mathfrak{t}_0(\rho)$ in $H^2(\Gamma, \mathrm{Ad}\,\rho)^{(\mu,\mu)}$ leads to :

Lemma 7.3.1. *If $[z,z]=0$ in $H^2(\Gamma, \mathrm{Ad}\,\rho)$ then*

$$\sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (\sum_{\chi_0\in\Phi_{\pi,c}} [z_{\pi,c,\chi_0}, \overline{z_{\pi,c,\chi_0}}] + \sum_{\chi_0\in\Phi_{\pi,n}} [z_{\pi,n,\chi_0}, \overline{z_{\pi,n,\chi_0}}]) = 0 \in H^2(\Gamma, \mathrm{Ad}\,\rho)^{(\mu,\mu)} \quad . \tag{7.6}$$

Let $H \in i\mathfrak{t}_0(\rho) \subset i\mathfrak{z}_0(\rho)$. As $\mathfrak{z}_0(\rho)$ is a trivial $\mathbf{SU}(n,1)$-module, the bundle $\mathcal{E}_{\mathfrak{t}_0(\rho)}$ is a trivial sub-$\mathbb{R}$VHS of $\mathcal{E}_{\mathfrak{g}}$. In particular any element $H \in i\mathfrak{t}_0(\rho)$ defines a flat section, still denoted H, in $H^0(X, \mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}) = H^0(\Gamma, \mathrm{Ad}\,\rho)_{\mathbb{C}}$. From the definition of the polarization morphism β defined in section 5.2 and from equation (7.3.1) one obtains :

(7.7)

$$\begin{aligned}
0 &= \beta(H, \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (\sum_{\chi_0\in\Phi_{\pi,c}} [z_{\pi,c,\chi_0}, \overline{z_{\pi,c,\chi_0}}] + \sum_{\chi_0\in\Phi_{\pi,n}} [z_{\pi,n,\chi_0}, \overline{z_{\pi,n,\chi_0}}]))\\
&= \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (\sum_{\chi_0\in\Phi_{\pi,c}} \beta([H, z_{\pi,c,\chi_0}], \overline{z_{\pi,c,\chi_0}}) + \sum_{\chi_0\in\Phi_{\pi,n}} \beta([H, z_{\pi,n,\chi_0}], \overline{z_{\pi,n,\chi_0}}))\\
&= \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}} (\sum_{\chi_0\in\Phi_{\pi,c}} \beta([H, z_{\pi,c,\chi_0}], \overline{z_{\pi,c,\chi_0}}) + \sum_{\chi_0\in\Phi_{\pi,n}} \beta([H, z_{\pi,n,\chi_0}], \overline{z_{\pi,n,\chi_0}}))\\
&= \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}} (\sum_{\chi_0\in\Phi_{\pi,c}} \chi_0(H)\beta(z_{\pi,c,\chi_0}, \overline{z_{\pi,c,\chi_0}}) + \sum_{\chi_0\in\Phi_{\pi,c}} \chi_0(H)\beta(z_{\pi,n,\chi_0}, \overline{z_{\pi,n,\chi_0}}))\\
&\sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}} (\sum_{\chi_0\in\Phi_{\pi,c}} \chi_0(H)\beta(C_G\cdot z_{\pi,c,\chi_0}, \overline{z_{\pi,c,\chi_0}}) - \sum_{\chi_0\in\Phi_{\pi,c}} \chi_0(H)\beta(C_G\cdot z_{\pi,n,\chi_0}, \overline{z_{\pi,n,\chi_0}})); \quad .
\end{aligned}$$

The second line follows from the compatibility of β with the Lie bracket and the fact that $C_G C_L^{-1}$ is trivial on H.

From now on we suppose the cone $\Lambda_0(\rho)$ is non-empty and pick $H \in \Lambda_0(\rho)$. Thus $\chi_0(H) > 0$ for $\chi_0 \in \Phi_{\pi,c}$, $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \setminus 1_{\mathbb{R}}$ and $\chi_0(H) < 0$ for $\chi_0 \in \Phi_{\pi,n}$, $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \setminus 1_{\mathbb{R}}$. We deduce from equation (7.7) and from the positivity properties of β that all the z_{π,c,χ_0}'s and z_{π,n,χ_0}'s vanish for $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho) \setminus 1_{\mathbb{R}}$. Thus $z = z_{1_{\mathbb{R}}} + \overline{z_{1_{\mathbb{R}}}}$ and any integrable deformation z occurs in $H^1(\Gamma, \mathfrak{z}_{\mathfrak{g}}(\mathfrak{su}(n,1)))$, which implies theorem 1.3.7.

7.4. **Proof of theorem 1.3.3.**

7.4.1. *Holomorphic interpretation.* Theorem 1.3.3 can be seen as a refinement of theorem 1.3.7. By equation (5.3) the sequence (7.1) for $p = \mu$ can be written holomorphically as :

$$\begin{array}{c} \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} \mathbb{H}^1(X, (\mathrm{Gr}^{i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})) \otimes \mathbb{H}^1(X, (\mathrm{Gr}^{\mu - i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})) \\ \downarrow \\ \bigoplus_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} \mathbb{H}^2(X, (\mathrm{Gr}^{i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho}) \overset{\mathbf{L}}{\otimes} (\mathrm{Gr}^{\mu - i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})) \\ \downarrow [\cdot,\cdot] \\ \mathbb{H}^2(X, (\mathrm{Gr}^{\mu} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g} \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})) \ . \end{array} \tag{7.8}$$

As we have seen in section 6, for $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$ the complex $(\mathrm{Gr}^{i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})$ admits as a direct factor the complex $(\mathcal{F}(V^{\mathbb{C}}_\pi) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots)$ which generates its 1-hyper-cohomology. Similarly the complex $(\mathrm{Gr}^{\mu - i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})$ admits as a direct factor the complex $(0 \longrightarrow \mathcal{F}((V^{\mathbb{C}}_{\pi^*})) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots)$ which generates its 1-hyper-cohomology. As a corollary in the sequence (7.8) the image of

$$\mathbb{H}^1(X, (\mathrm{Gr}^{i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})) \otimes \mathbb{H}^1(X, (\mathrm{Gr}^{\mu - i_\pi} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g}_\pi \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})$$

is contained in the 2-hyper-cohomology of the sub-complex

$$\mathcal{C}_\pi = (0 \longrightarrow \mathcal{F}([V^{\mathbb{C}}_\pi, (V^{\mathbb{C}}_{\pi^*})]) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots)$$

of $(\mathrm{Gr}^{\mu} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g} \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})$. Notice that all these complexes $\mathcal{C}_\pi$ are natural sub-complexes of the sub-quotient $(0 \longrightarrow \Omega^1_X \otimes \mathcal{F}(\mathfrak{l}(\rho)_{\mathbb{C}}) = \mathrm{Gr}^{\mu} \Omega^1_X(\mathcal{E}_{\mathfrak{g} \otimes \mathbb{C}}) \longrightarrow 0 \longrightarrow \cdots)$ of $(\mathrm{Gr}^{\mu} \Omega^\bullet_X(\mathcal{E}_{\mathfrak{g} \otimes \mathbb{C}}), \theta_{\mathrm{Ad}\,\rho})$. As a corollary lemma 7.2.1 implies the

Lemma 7.4.1. *If $[z, z] = 0$ in $H^2(\Gamma, \mathrm{Ad}\,\rho)$ then*

$$\sum_{\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} ([z_{\pi,c}, \overline{z_{\pi,c}}] + [z_{\pi,n}, \overline{z_{\pi,n}}]) = 0 \in H^1(X, \Omega^1_X \otimes \mathcal{F}(\mathfrak{l}(\rho)_{\mathbb{C}})) \ . \tag{7.9}$$

7.4.2. One proceeds similarly to section 7.3. With the notations of the introduction one has for any $\pi \in \mathrm{Coh}^1_{\mathbb{R}}(\rho)$:

$$H^1(\Gamma, \mathfrak{g}_{\pi,c})^{(i_\pi, \mu - i_\pi)} = \bigoplus_{\chi \in \Phi_{\pi,c}} \chi^{d_\chi} \otimes H^1(X, F(V^{\mathbb{C}}_{\pi,c}))$$

$$\overline{H^1(\Gamma, \mathfrak{g}_{\pi,c})^{(i_\pi, \mu - i_\pi)}} = \bigoplus_{\chi \in \Phi_{\pi,c}} \chi^{d_\chi} \otimes H^0(X, \mathcal{F}(V^{\mathbb{C}}_{\pi^*,c})) \subset \bigoplus_{\chi \in \Phi_{\pi,c}} \chi^{d_\chi} H^0(X, \Omega^1_X \otimes \mathcal{F}(V^{\mathbb{C}}_{\pi,c})^*)$$

as a $\mathbf{T}(\rho)$-module. Similarly for $H^1(\Gamma, \mathfrak{g}_{\pi,n})^{(i_\pi,\mu-i_\pi)}$ and $\overline{H^1(\Gamma, \mathfrak{g}_{\pi,c})^{(i_\pi,\mu-i_\pi)}}$. Let

$$z_{\pi,c} = \sum_{\chi\in\Phi_{\pi,c}} z_{\pi,c,\chi} \ , \qquad \overline{z_{\pi,c}} = \sum_{\chi\in\Phi_{\pi,c}} \overline{z_{\pi,c,\chi}} \ ,$$
$$z_{\pi,n} = \sum_{\chi\in\Phi_{\pi,n}} z_{\pi,n,\chi} \ , \qquad \overline{z_{\pi,n}} = \sum_{\chi\in\Phi_{\pi,n}} \overline{z_{\pi,c,\chi}}$$

be the corresponding decompositions.

The restriction of equation 7.4.1 to the 0-eigenspace of $\mathfrak{t}(\rho)$ in $H^0(X, \Omega^1_X \otimes \mathcal{F}(\mathfrak{l}(\rho)_\mathbb{C}))$ leads to :

Lemma 7.4.2. *If $[z,z]=0$ in $H^2(\Gamma, \mathrm{Ad}\,\rho)$ then*

$$\sum_{\pi\in\mathrm{Coh}^1_\mathbb{R}(\rho)} \Big(\sum_{\chi\in\Phi_{\pi,c}} [z_{\pi,c,\chi}, \overline{z_{\pi,c,\chi}}] + \sum_{\chi\in\Phi_{\pi,n}} [z_{\pi,n,\chi}, \overline{z_{\pi,n,\chi}}]\Big) = 0 \in H^0(X, \Omega^1_X \otimes \mathcal{F}(\mathfrak{l}(\rho)_\mathbb{C})) \ . \tag{7.10}$$

The Hermitian metric $\beta_0(C_G \cdot X, \overline{X})$ on $\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}$ induces the harmonic Hermitian metric still denoted β_0 on $\mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}$. The cohomology of the holomorphic Hermitian vector bundle $\mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}$ can thus be computed using $\Delta_{\overline{\partial}}$-harmonic forms :

$$H^{p,q}(X, \mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}) \simeq \mathcal{H}^{p,q}_{\overline{\partial}}(X, \mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}) \ .$$

From know on we will always use this identification. In particular $z_{\pi,c,\chi}$ is an element of $\mathcal{H}^{0,1}(X, \mathrm{Gr}^{i_\pi}\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}})$. Moreover the Hermitian metric $\beta_0(C_G \cdot X, \overline{X})$ on $\mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}$ extends to a natural L^2-metric $\tilde{\beta}(C_G.x, \overline{x})$ on $\mathcal{H}^{p,q}_{\overline{\partial}}(X, \mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}})$. One easily checks the following lemma analogous to lemma 5.2.4 :

Lemma 7.4.3. *The pairing $\tilde{\beta}$ satisfies the following two properties :*

1. $\forall x,y,z \in H^\bullet(X, \mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}})$, $\qquad \tilde{\beta}(x,[y,z]) + (-1)^{d(x)\cdot d(y)}\tilde{\beta}([C_G \cdot C_L^{-1}x, y], z) = 0$ *where $d(x)$ denotes the degree of x divided by μ.*
2. *for any $x \neq 0$ in $H^{p,q}(X, \mathrm{Gr}^\bullet\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}})$, $\tilde{\beta}(C_G \cdot x, \overline{x}) > 0$.*

Let $H \in i\mathfrak{t}(\rho) \subset i\mathfrak{z}(\rho)$. As $\mathfrak{z}(\rho)$ is a trivial $\mathbf{U}(n)$-module, the bundle $\mathcal{F}(\mathfrak{t}(\rho))$ is a trivial holomorphic sub-bundle of $\mathrm{Gr}^0\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}}$. In particular any element $H \in i\mathfrak{t}(\rho)$ defines a holomorphic section, still denoted H, in $H^0(X, \mathrm{Gr}^0\mathcal{E}_{\mathfrak{g}\otimes\mathbb{C}})$. From lemma 7.4.2 one obtains :

(7.11)

$$\begin{aligned}
0 &= \tilde{\beta}(H, \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (\sum_{\chi\in\Phi_{\pi,c}} [z_{\pi,c,\chi}, \overline{z_{\pi,c,\chi}}] + \sum_{\chi\in\Phi_{\pi,n}} [z_{\pi,n,\chi}, \overline{z_{\pi,n,\chi}}])) \\
&= \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)} (\sum_{\chi\in\Phi_{\pi,c}} \tilde{\beta}([H, z_{\pi,c,\chi}], \overline{z_{\pi,c,\chi}}) + \sum_{\chi\in\Phi_{\pi,n}} \tilde{\beta}([H, z_{\pi,n,\chi}], \overline{z_{\pi,n,\chi}})) \\
&= \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}} (\sum_{\chi\in\Phi_{\pi,c}} \tilde{\beta}([H, z_{\pi,c,\chi}], \overline{z_{\pi,c,\chi}}) + \sum_{\chi\in\Phi_{\pi,n}} \tilde{\beta}([H, z_{\pi,n,\chi}], \overline{z_{\pi,n,\chi}})) \\
&= \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}} (\sum_{\chi\in\Phi_{\pi,c}} \chi(H)\tilde{\beta}(z_{\pi,c,\chi}, \overline{z_{\pi,c,\chi}}) + \sum_{\chi\in\Phi_{\pi,c}} \chi_0(H)\tilde{\beta}(z_{\pi,n,\chi}, \overline{z_{\pi,n,\chi}})) \\
& \sum_{\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}} (\sum_{\chi\in\Phi_{\pi,c}} \chi(H)\tilde{\beta}(C_G\cdot z_{\pi,c,\chi}, \overline{z_{\pi,c,\chi}}) - \sum_{\chi\in\Phi_{\pi,c}} \chi(H)\tilde{\beta}(C_G\cdot z_{\pi,n,\chi}, \overline{z_{\pi,n,\chi}}));\ .
\end{aligned}$$

If we assume that the cone $\Lambda(\rho)$ is non-empty and pick H in $\Lambda(\rho)$ then as before the previous equation forces :

$$\forall\, \pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}, \quad z_{\pi,c} = z_{\pi,n} = 0\ .$$

Thus $z = z_{1_{\mathbb{R}}} + \overline{z_{1_{\mathbb{R}}}}$ and any integrable deformation z occurs in $H^1(\Gamma, \mathfrak{z}_{\mathfrak{g}}(\mathfrak{su}(n,1)))$, which implies theorem 1.3.3.

7.5. **Proof of theorem 1.3.4.** If all the V_π, $\pi\in \mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}}$, are of compact type, then by definition the generator $-E_\rho$ of $i\rho_*(\mathfrak{u}(1))$ belongs to $\Lambda(\rho)$ which is thus non-empty. Thus Theorem 1.3.4 follows immediately from theorem 1.3.3.

7.6. **Proof of theorem 1.3.8.** For the standard embedding ρ of $\mathbf{SU}(n,1)$ in $\mathbf{SU}(n+p, 1+q)$ the centralizer $\mathbf{Z}_p(\rho)$ of $\mathbf{SU}(n,1)$ in $\mathbf{SU}(n+p, 1+q)$ is $\mathbf{U}(p,q)$ whose maximal compact subgroup is $\mathbf{U}(p)\times\mathbf{U}(q)$. In this case :

- $\mathrm{Coh}^1_{\mathbb{R}}(\rho)\setminus 1_{\mathbb{R}} = \{V_{\mathbb{R}}\}$,
- $\mathfrak{g}_{V_{\mathbb{R}},c} \simeq (\mathbb{C}^p)_{\mathbb{R}}\otimes_{\mathbb{R}} V_{\mathbb{R}}$ as a $\mathbf{U}(p)\times\mathbf{U}(q)\times\mathbf{SU}(n,1)$-module (with the standard action of $\mathbf{U}(p)$ on $\mathbb{C}^p$),
- $\mathfrak{g}_{V_{\mathbb{R}},n} \simeq (\mathbb{C}^q)_{\mathbb{R}}\otimes_{\mathbb{R}} V_{\mathbb{R}}$ as a $\mathbf{U}(p)\times\mathbf{U}(q)\times\mathbf{SU}(n,1)$-module (with the standard action of $\mathbf{U}(q)$ on $\mathbb{C}^q$).

It follows trivially that the cone $\Lambda_0(\rho)\subset\mathbb{R}^p\times\mathbb{R}^q$ is non-empty.

Appendix A. Some Hodge theory

A.1. **Real Hodge structure.**

Definition A.1.1. *A real Hodge structure on a real (finite-dimensional) vector space $E_{\mathbb{R}}$ is a bi-graduation of $E_{\mathbb{C}} := E_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$*

$$E_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} E^{p,q}$$

such that $\overline{E^{p,q}} = E^{q,p}$ *(where* $\overline{x}$ *denote the complex-conjugate of* x*). One says that the real structure is of pure weight* $n \in \mathbb{Z}$ *if* $E^{p,q} = 0$ *for* $p + q \neq n$*.*

Of course any real Hodge structure on $E_{\mathbb{R}}$ decomposes uniquely as a direct sum of real Hodges structures of pure weight $E_{\mathbb{R}} = \bigoplus_{n \in \mathbb{Z}} E_{\mathbb{R},n}$.

A first variant of this definition is :

Definition A.1.2. *A weight* n *real Hodge structure on* $E_{\mathbb{R}}$ *is a finite decreasing filtration* $F^\bullet$ *on* $E_{\mathbb{C}}$ *(the Hodge filtration) such that for any* $p \in \mathbb{Z}$

$$F^p E_{\mathbb{C}} \oplus \overline{F}^{n+1-p} E_{\mathbb{C}} = E_{\mathbb{C}} \quad .$$

The equivalence of definitions A.1.1 and A.1.2 is given by $F^p E_{\mathbb{C}} = \bigoplus_{p' \geq p} E^{p', n-p'}$ and $E^{p,n-p} = F^p E_{\mathbb{C}} \cap \overline{F}^{n-p} E_{\mathbb{C}}$.

A second variant if given by :

Definition A.1.3. *A real Hodge structure on* $E_{\mathbb{R}}$ *is a structure of* $\mathbb{S}$*-module* $h : \mathbb{S} \longrightarrow \mathbf{GL}(E_{\mathbb{R}})$ *on* $E_{\mathbb{R}}$*, where* $\mathbb{S}$ *denotes the real algebraic torus* $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$*.*

Thus $\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ with complex conjugation $\overline{(z,w)} = (\overline{w}, \overline{z})$ and the embedding $\mathbb{S}(\mathbb{R}) = \mathbb{C}^* \longrightarrow \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ is given by $z \mapsto (z, \overline{z})$. The equivalence between definitions A.1.1 and A.1.3 is given by defining the action of $(z,w) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ on $E^{p,q}$ by multiplication by $z^{-p} w^{-q}$.

The real torus $\mathbb{S}$ is the non-trivial extension

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{w} \mathbb{S} \longrightarrow \mathbf{U}(1) \longrightarrow 1 \quad ,$$

where the weight homomorphism on real points $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^* \xrightarrow{w} \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ is given by $r \mapsto r^{-1}$. The pure weight n Hodge substructure $E_{\mathbb{R},n}$ is the eigenspace of $h \circ w : \mathbb{G}_m \longrightarrow GL(E_{\mathbb{R}})$ for the character $z \longrightarrow z^n$.

The real torus $\mathbb{S}$ can also be seen as the non-trivial extension

$$1 \longrightarrow \mathbf{U}(1) \longrightarrow \mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_m \longrightarrow 1 \quad ,$$

where $\mathrm{Nm}(z,w) = zw$.

Definition A.1.4. *A polarization of the real Hodge structure* $E_{\mathbb{R}}$ *is a* $\ker \mathrm{Nm}$*-invariant bilinear form* β *on* $E_{\mathbb{R}}$ *such that the Hermitian form* $\beta(C \cdot v, \overline{v})$ *is positive definite on* $E_{\mathbb{C}}$ *and the* $E^{p,q}$ *are orthogonal for it (where* $C = h(i) \in GL(E_{\mathbb{R}})$ *is the Weil operator of the Hodge structure, acting by* i^{q-p} *on* $E^{p,q}$*).*

A.2. **Complex Hodge structure.** This notion is essentially trivial but of interest when considered in variations.

Definition A.2.1. *A weight n complex Hodge structure on a complex (finite-dimensional) vector space $E_{\mathbb{C}}$ is a decomposition $E_{\mathbb{C}} = \bigoplus_{p+q=n} E^{p,q}$. Alternatively, this is a decreasing filtration $F^\bullet$ on $E_{\mathbb{C}}$ or a complex representation $h : \mathbb{S}_{\mathbb{C}} \longrightarrow \mathbf{GL}(E_{\mathbb{C}})$.*

Definition A.2.2. *A polarization of a complex Hodge structure $E_{\mathbb{C}}$ is a* $\ker \mathrm{Nm}(\mathbb{R})$*-invariant sesquilinear pairing $\psi : E_{\mathbb{C}} \times E_{\mathbb{C}} \longrightarrow \mathbb{C}$ such that $\psi(C \cdot v, v) > 0$ for $v \neq 0$.*

Example A.2.3. If $E_{\mathbb{R}}$ is a weight n real Hodge structure with polarization β then $E_{\mathbb{C}}$ is a weight n complex Hodge structure with polarization $\psi(v,w) = \beta_{\mathbb{C}}(v, \overline{w})$ (where $\beta_{\mathbb{C}}$ denotes the complexified bilinear form extension of β).

A.2.1. Let $\mathbb{R}(i)$ be the 1-dimensional structure of weight $2i$ defined by the action $h(z,w) = z^{-i}w^{-i}$ on $\mathbb{C}$. Any real Hodge structure of pure weight is isomorphic (modulo tensorisation by some $\mathbb{R}(i)$) to a real Hodge structure of weight 0 or 1. On the other hand any complex Hodge structure of pure weight is isomorphic (modulo tensorisation by some $\mathbb{C}(i)$) to a complex Hodge structure of weight 0. This more generally indicates that the notion of weight is much less useful for complex Hodge structures than for real Hodge structures.

A.3. **Weight 0 Hodge structures.** Let $E_{\mathbb{C}}$ be a weight 0 complex Hodge structure. For simplicity we denote by E^i the weight space $E^{i,-i}$.

Let $\mu : \mathbf{G}_m(\mathbb{C}) \longrightarrow \mathbb{S}(\mathbb{C})$ be the co-character $z \mapsto (z,1)$. The weight 0 complex Hodge structure $h : \mathbb{S}_{\mathbb{C}} \longrightarrow \mathbf{GL}(E_{\mathbb{C}})$ is entirely determined by the algebraic morphism $u = h \circ \mu : \mathbf{G}_m(\mathbb{C}) \longrightarrow \mathbf{GL}(E_{\mathbb{C}})$ via the formula :

$$\forall z, w \in \mathbb{C}^*, \quad h(z,w) = u(z) \cdot u(w)^{-1} \quad .$$

The u-action on E^i is via multiplication by z^{-i}. Notice that the Weil operator C is nothing else than $u(-1)$.

If $\psi : E_{\mathbb{C}} \times E_{\mathbb{C}} \longrightarrow \mathbb{C}$ is a polarization for h and $\mathbf{U}(E_{\mathbb{C}})$ denotes the real form of $\mathbf{GL}(E_{\mathbb{C}})$ defined by the positive definite Hermitian form $\psi(Cz,w)$ then the morphism u is defined over $\mathbb{R}$: $u : \mathbf{U}(1) \longrightarrow \mathbf{U}(E_{\mathbb{C}})$ as $u(\alpha) = h(\alpha^{1/2})$ for $\alpha \in \mathbf{U}(1)$.

If moreover $E_{\mathbb{C}} = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $u : \mathbf{U}(1) \longrightarrow \mathbf{GL}(E_{\mathbb{R}})$ then $E_{\mathbb{R}}$ is a weight 0 real Hodge structure.

This is the point of view emphasized in the definition of a Hodge datum.

A.4. **Variation of Hodge structures.**

Definition A.4.1. *Let X be a complex analytic space. A polarizable real variation of Hodge structures of weight $n \in \mathbb{Z}$ on X is a flat real vector bundle $(\mathcal{E}, D)$ on X, a $\mathcal{C}^\infty$ decomposition $\mathcal{E}_{\mathbb{C}} = \bigoplus_{p+q=n} \mathcal{E}^{p,q}$ of $\mathcal{C}^\infty$-vector bundles and a flat bilinear form $\beta : \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}$ such that :*

1. *For any $x \in X$ the induced decomposition on the fiber at x $(\mathcal{E}_x)_{\mathbb{C}} = \bigoplus_{p+q=n} \mathcal{E}_x^{p,q}$ is a weight n real Hodge structure on $\mathcal{E}_x$ polarized by β_x.*
2. *The fiber bundle $F^p\mathcal{E}_{\mathbb{C}} = \bigoplus_{r \geq p} \mathcal{E}^{r,n-r}$ is a holomorphic sub-bundle of $\mathcal{E}_{\mathbb{C}}$.*
3. *If D is the flat connection on $\mathcal{E}$ then $D(F^p\mathcal{E}_{\mathbb{C}}) \subset F^{p-1}\mathcal{E}_{\mathbb{C}} \otimes \Omega^1_X$.*

Definition A.4.2. *Let X be a complex analytic space. A polarizable complex variation of Hodge structures of weight $n \in \mathbb{Z}$ on X is a flat complex vector bundle $(\mathcal{E}, D)$ on X, a $\mathcal{C}^\infty$ decomposition $\mathcal{E} = \bigoplus_{p+q=n} \mathcal{E}^{p,q}$ of $\mathcal{C}^\infty$-vector bundles and a flat sesquilinear form $\psi : \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{C}$ such that :*

(1) *For any $x \in X$ the decomposition $\mathcal{E}_x = \bigoplus_{p+q=n} E_x^{p,q}$ is a weight n complex Hodge structure on $\mathcal{E}_x$ polarized by ψ_x.*
(2) *For any $p \in \mathbb{Z}$ the fiber bundle $F^p\mathcal{E} = \bigoplus_{r \geq p} \mathcal{E}^{r,n-r}$ is a holomorphic sub-bundle of $\mathcal{E}$ and the fiber bundle $\overline{F}^p\mathcal{E} := \bigoplus_{r \leq n-p} \mathcal{E}^{r,n-r}$ is an anti-holomorphic sub-bundle of $\mathcal{E}$.*
(3) *If D is the flat connection on $\mathcal{E}$ then $D(F^p\mathcal{E}) \subset F^{p-1}\mathcal{E} \otimes \Omega^1_X$ and $D(\overline{F}^p\mathcal{E}) \subset \overline{F}^{p-1}\mathcal{E} \otimes \overline{\Omega^1_X}$.*

Example A.4.3. If $(\mathcal{E}, D, \beta)$ is a weight n polarizable real variation of Hodge structure on X then $(\mathcal{E}_\mathbb{C}, D, \psi)$ is a weight n polarizable complex variation of Hodge structure on X, with $\psi(x, y) = \beta_\mathbb{C}(x, \overline{y})$.

Appendix B. Labeling

The difference between a μ-$\mathbb{C}$VHS and a usual $\mathbb{C}$VHS is essentially a difference in the labeling of the Hodge types.

First notice that the labeling of the Hodge types in a $\mathbb{C}$HS is ambiguous : if $E_\mathbb{C} = \bigoplus_{p \in \mathbb{Z}} E < p >$ is a weight 0 complex Hodge structure on $E_\mathbb{C}$ and $\psi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is an affine function of the form $\psi(p) = a + bp$ then $E_\mathbb{C} = \bigoplus_{p \in \mathbb{Z}} E < \psi(p) >$ is also one, isomorphic to the previous one up to the relabeling ψ of the Hodge types. In general there is no canonical way to choose a labeling. A strong restriction appears when we want to glue together different complex Hodge structures $E_\mathbb{C}$ into a complex variation of Hodge structure $(\mathcal{E}_\mathbb{C}, D)$ on a variety X, as the Griffiths's transversality condition $D(F^p\mathcal{E}_\mathbb{C}) \subset F^{p-1}\mathcal{E}_\mathbb{C} \otimes \Omega^1_X$ uniquely determines the dilation factor b of ψ. Thus when considering an irreducible weight 0 complex variation of Hodge structures the Hodge-types are uniquely fixed up to a translation $a \in b\mathbb{Z}$.

This leads to the :

Definition B.0.4. (a) *Let $\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E_\mathbb{C})$ be an irreducible complex representation. Let $\Lambda = \alpha + \mu(u)\mathbb{Z} \subset \mathbb{Z}$ where α denotes any weight of $\tilde{u}$ in $E_\mathbb{C}$. An element $\beta \in \Lambda$ is called an admissible labeling for λ. For any such labeling $\beta \in \Lambda$ one labels the complex Hodge structure on $E_\mathbb{C}$ by $E_\mathbb{C} = \bigoplus_{p \in \mathbb{Z}} E^p_{(\beta)}$, with*

$$E^p_{(\beta)} = E < \beta + p\mu > \quad .$$

(b) *Let $\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E_\mathbb{C})$ be a complex representation. Let $\lambda = \oplus_i \lambda_i$ be its decomposition into isotypical components. An admissible labeling for λ is the datum of an admissible labeling for every isotypical component λ_i. The corresponding complex*

Hodge structure is the direct sum of the corresponding complex Hodge structure on each isotypical component.

(c) *A labeled representation* $(\lambda : \mathbf{L} \longrightarrow \mathbf{GL}(E_{\mathbb{C}}), \beta)$ *is a complex representation* λ *of* $\mathbf{L}$ *with an admissible labeling* β*.*

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Bruno Klingler

email: klingler@math.jussieu.fr

Institut de Mathématiques de Jussieu, Paris, France and Institute for Advanced Study, Princeton NJ, USA