THE AX–SCHANUEL CONJECTURE FOR VARIATIONS OF MIXED HODGE STRUCTURES

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Abstract. We prove in this paper the Ax–Schanuel conjecture for all admissible variations of mixed Hodge structures.

Contents

1. Introduction

In this paper we prove the Ax-Schanuel conjecture for all admissible, graded polarized, integral variation of mixed Hodge structures over a smooth complex quasi-projective variety S.

Let $(\mathbb{V}_{\mathbb{Z}}, W_{\bullet}, \mathcal{F}^{\bullet}) \to S^{\mathrm{an}}$ be an admissible, graded-polarized, integral variation of mixed Hodge structures on the complex manifold S^{an} associated to S. Let $[\Phi]\colon S^{\mathrm{an}} \to \Gamma\backslash\mathcal{M}$ be the associated complex analytic period map, where $\mathcal{M}$ denotes the period domain classifying graded-polarized mixed Hodge structures of the relevant type and Γ is an arithmetic subgroup in the group of automorphisms of $\mathcal{M}$. The classifying space $\mathcal{M}$ admits a natural realization as a real semi-algebraic subset, open in the usual topology, of a complex algebraic variety $\mathcal{M}^{\vee}$. The Ax–Schanuel conjecture is a functional transcendence statement comparing the algebraic structure on $\mathcal{M}^{\vee}$ and the algebraic structure on S, via $[\Phi]$ and $u\colon \mathcal{M} \to \Gamma\backslash\mathcal{M}$. Consider the commutative diagram in the category of complex analytic spaces

$$\begin{tikzcd} S^{\mathrm{an}} \times \mathcal{M}^\vee & S^{\mathrm{an}} \times \mathcal{M} \arrow[l, hook'] & \Delta := S^{\mathrm{an}} \times_{\Gamma\backslash\mathcal{M}} \mathcal{M} \arrow[l, hook'] \arrow[r, "p_{\mathcal{M}}"] \arrow[d, "p_S"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[d, "u"] \\ & & S^{\mathrm{an}} \arrow[r, "{[\Phi]}"'] & \Gamma\backslash\mathcal{M} \end{tikzcd} .$$

We prove the following result, conjectured in [Kli17, Conj. 7.5] (we refer to Definition 2.5 for the definition of weak Mumford–Tate subdomains of $\mathcal{M}$):

Theorem 1.1. *Let $\mathcal{Z}$ be a complex analytic irreducible subset of Δ. Then*

$$\dim \mathcal{Z}^{\mathrm{Zar}} - \dim \mathcal{Z} \geq \dim p_{\mathcal{M}}(\mathcal{Z})^{\mathrm{ws}}, \tag{1.1}$$

where $\mathcal{Z}^{\mathrm{Zar}}$ denotes the Zariski closure of $\mathcal{Z}$ in $S \times \mathcal{M}^\vee$, and $p_{\mathcal{M}}(\mathcal{Z})^{\mathrm{ws}}$ is the smallest weak Mumford–Tate subdomain of $\mathcal{M}$ containing $p_{\mathcal{M}}(\mathcal{Z})$.

In the course of the proof, we also explain how to construct $p_{\mathcal{M}}(\mathcal{Z})^{\mathrm{ws}}$. Let S' be the Zariski closure of $p_S(\mathcal{Z})$. Let N be the connected algebraic monodromy group of $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)|_{S'} \to S'^{\mathrm{an}}$. Then $p_{\mathcal{M}}(\mathcal{Z})^{\mathrm{ws}}$ is the $N(\mathbb{R})^+\mathcal{R}_u(N)(\mathbb{C})$-orbit of any point $\tilde{z} \in p_{\mathcal{M}}(\mathcal{Z})$, where $\mathcal{R}_u(N)$ is the unipotent radical of N; see Remark 6.3.

Theorem 1.1 closes a long series of works. The idea of functional transcendence statements related to Hodge theory first appeared in the context of Shimura varieties, where $[\Phi]$ is the identity. Motivated by Pila's pionner work [Pil11] on the André–Oort conjecture for copies of moduli curves, the Ax–Lindemann conjecture (a special case of the Ax–Schanuel conjecture) was proved for various cases in [PT13, UY14, PT14] and ultimately for all pure Shimura varieties in [KUY16]; this was extended to mixed Shimura varieties in [Gao17]. After the proof of the André–Oort conjecture [Tsi18] (see [Gao16] for mixed Shimura varieties), and in order to attack the more general Zilber–Pink conjecture, Theorem 1.1 was proved for copies of moduli curves in [PT16] and for any pure Shimura variety in [MPT19]; this was extended to mixed Shimura varieties of Kuga type in [Gao20b]. In [Kli17, Conj. 7.5] the second author suggested that these functional transcendence statements should hold much more generally for all admissible, graded polarizable, integral variation of mixed Hodge structures over a smooth complex quasi-projective variety S and formulated Theorem 1.1; this was proved in [BT19] if the variation of Hodge structures in question is pure.

All these works have been important ingredients in the proofs of various diophantine results: the André–Oort conjecture for mixed Shimura varieties, results in the direction of the more general Zilber–Pink conjecture [DR18], use of [MPT19] to prove the submersivity of the Betti map in [ACZ20], use of [BT19] for Shafarevich type results in [LV20, LS20], use of [Gao20b] to fully study the Betti rank in [Gao20a] which eventually was applied to prove a rather uniform bound on the number of rational points on curves [DGH20]. Hast [Has21] recently proved a transcendence property of the unipotent Albanese map assuming Theorem 1.1. We expect Theorem 1.1 to have more applications in diophantine geometry, for instance in direction of the general Hodge-theoretical atypical intersection conjecture [Kli17, Conj. 1.9] and its special case [Kli17, Conj. 5.2].

The strategy for proving Theorem 1.1 is similar in spirit to previous works, in particular [BT19], [MPT19] and [Gao20b]. However its implementation in the mixed non-Shimura case contains serious new difficulties.

For readers' convenience, we start the paper by recalling basic knowledge on variations of mixed Hodge structures and mixed Mumford–Tate domains in Section 2, Section 3, Section 4 and Section 5. Unlike for the pure or the Shimura case, references to some of the results recalled hereby are not easy to find. We also give proofs in these sections and Appendix A to some results which are surely known to experts but whose proofs we cannot find in existing references. For example, mixed Mumford–Tate domains are complex spaces and are stable under intersection; as an upshot, the classifying space $\mathcal{M}$ in Theorem 1.1 can be replaced by a suitable mixed Mumford–Tate domain $\mathcal{D}$. We also use mixed Hodge data developed in [Kli17] to prove that we are able to take quotients by normal groups in the category of mixed Mumford–Tate domains, and each such quotient is a holomorphic map. All these results are fundamental to the proof of Theorem 1.1. In fact, with these preparation, we can prove a particular case of Theorem 1.1, called the *logarithmic Ax theorem*, in Section 6.

Another formalism we do for our strategy is the fibered structure of mixed Mumford–Tate domains. We also need to discuss the real points of mixed Mumford–Tate domains; they correspond to mixed Hodge structures split over $\mathbb{R}$. This is done in Section 7.

Then we move on to prove Theorem 1.1. We start by some dévissages in Section 8, and reduce to the case where the projection of $\mathcal{Z}$ in S is Zariski-dense in S and that $\mathcal{Z}$ is an irreducible component of the intersection of its Zariski-closure with Δ: see Lemma 8.1. In order to obtain a better group theoretical control of $\mathcal{Z}$, we also replace the classifying space $\mathcal{M}$ by its refinement $\mathcal{D}$, the mixed Mumford–Tate domain associated to the generic Mumford–Tate group P of the variation $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$.

The first step in the proof of Theorem 1.1 consists of proving that the inequality (1.1) holds true if the $\mathbb{Q}$-stabilizer $\mathcal{Z}^{\mathrm{Zar}}$ (for the action of P on the second factor of $S^{\mathrm{an}} \times \mathcal{D}$), denoted by $H_{\mathcal{Z}^{\mathrm{Zar}}}$, is zero dimensional; see Proposition 9.1. To do so we use o-minimal geometry (more precisely the result of [BBKT20] generalizing [BKT20] saying that mixed period maps are definable in some o-minimal structure, and the celebrated Pila-Wilkie theorem [Pil11, 3.6]) to prove a counting result Theorem 9.2.

More precisely, take a suitable semi-algebraic fundamental set $\mathfrak{F}$ for $\mathcal{D} \to \Gamma\backslash\mathcal{D}$. As in all proofs of Ax–Schanuel type transcendence results via o-minimality, we start by constructing a definable subset Θ of $P(\mathbb{R})$ which contains all integer elements $\gamma \in \Gamma$ such that $\gamma(S \times \mathfrak{F}) \cap \mathcal{Z} \neq \emptyset$. We wish to prove that Θ contains semi-algebraic curves with arbitrarily many integer elements; this will yield the non-triviality of $H_{\mathcal{Z}^{\mathrm{Zar}}}$ unless (1.1) already holds true by induction. The Pila-Wilkie theorem then reduces the question to showing that the number of elements in $\Gamma \cap \Theta$ of height at most T grows at least polynomially in T. The latter is precisely Theorem 9.2.

The first main new difficulty lies in the proof of this counting result. It occupies the full section Section 9 and is quite technical. While in the pure case it follows from an explicit description of the semi-algebraic fundamental set $\mathfrak{F}$ for Γ in terms of Siegel sets furnished by reduction theory and from the non-positive curvature in the horizontal direction for pure Mumford–Tate domains (see [BT19]), in the mixed case we have only an implicit knowledge of $\mathfrak{F}$: its construction in [BBKT20] relies fundamentally on the rather mysterious retraction of $\mathcal{D}$ on its subvariety $\mathcal{D}_{\mathbb{R}}$ of real split mixed Hodge structures furnished by the $\mathfrak{sl}_2$-splitting of mixed Hodge structures. Instead, we use the natural fibered structure

$$\mathcal{D} = \mathcal{D}_m \to \mathcal{D}_{m-1} \to \cdots \to \mathcal{D}_0$$

of mixed Mumford–Tate domains associated to the weight filtration of the variation of Hodge structures. Each step is a vector bundle. Considering the successive projections $\mathcal{Z}_k$ of $\mathcal{Z}$ to the storeys $S \times \mathcal{D}_k$, we proceed as follows:

- assuming that the required estimate holds for $\mathcal{Z}_k$ we prove that we can "lift" this estimate to $\mathcal{Z}_{k+1}$: see Proposition 9.7 and Section 9.6. As in [Gao20b], there are two cases to consider for this lifting process, namely the "horizontal" case treated by Lemma 9.10 and the "vertical" case treated by Lemma 9.11.
- we initiate the process at the smallest integer k_0 such that the projection of $\mathcal{Z}$ to $\mathcal{D}_{k_0}$ is not a point. If $k_0 = 0$ the required estimate follows from [BT19] as $\mathcal{D}_0$ is a pure Mumford–Tate domain; for technical reasons some suitable arrangement is needed; see case 2 of Section 9.7. On the other hand there is some non-trivial work to be done if $k_0 > 0$ (the unipotent case, or equivalently when the maximal pure quotient of the variation is constant): see Section 9.5, more precisely Proposition 9.4.

The second step in the proof of Theorem 1.1 consists of dealing with the case where the group $H_{\mathcal{Z}^{\mathrm{Zar}}}$ is positive dimensional. In that case one wants to reduce to the first step by working in the quotient Mumford–Tate domain $\mathcal{D}/H_{\mathcal{Z}^{\mathrm{Zar}}}$. Such a quotient exists as a Mumford–Tate domain only if the group $H_{\mathcal{Z}^{\mathrm{Zar}}}$ is normal in the generic Mumford–Tate group P. Following the guideline of [MPT19], we prove in Section 10 that $H_{\mathcal{Z}^{\mathrm{Zar}}}$ is normal in the algebraic monodromy group of this variation of mixed Hodge structures. While this immediately implies that $H_{\mathcal{Z}^{\mathrm{Zar}}}$ is normal in P in the pure case, it turns out to be more subtle in the mixed case. We solve this problem in Section 11 and the argument is ultimately Hodge-theoretic.

Right before this paper is publicized, we received a preprint [Chi21] from Chiu independently proving the same result. The approach and idea are similar to ours, but some techniques are different.

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2. Mixed Hodge structures, classifying space, and Mumford–Tate domains

2.1. Mixed Hodge structure.

In this subsection we recall some definitions and properties of $\mathbb{Q}$-mixed Hodge structures.

Definition 2.1. Let V be a finite dimensional $\mathbb{Q}$-vector space and $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$ its complexification.

(i) A $\mathbb{Q}$-pure Hodge structure on V of weight n is a decreasing filtration $F^\bullet$ (the *Hodge filtration*) on $V_{\mathbb{C}}$ such that $V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{n+1-q} V_C}$ for all $p \in \mathbb{Z}$.

(ii) A $\mathbb{Q}$-mixed Hodge structure on V consists of two filtrations, an increasing filtration $W_\bullet$ on V (the *weight filtration*) and a decreasing filtration $F^\bullet$ on $V_{\mathbb{C}}$ (the *Hodge filtration*) such that for each $k \in \mathbb{Z}$ the $\mathbb{Q}$-vector space $\mathrm{Gr}_k^W V = W_k/W_{k-1}$ is a pure Hodge structure of weight k for the filtration on $\mathrm{Gr}_k^W V \otimes_{\mathbb{Q}} \mathbb{C}$ deduced from $F^\bullet$.

The numbers $h^{p,q}(V) = \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W (V_{\mathbb{C}})$ are called the *Hodge numbers* of $(V, W_\bullet, F^\bullet)$.

$\mathbb{Q}$-mixed Hodge structures, defined in terms of two filtrations, can be equivalently described in terms of *bigradings*. This is classical in the pure case, where a weight n $\mathbb{Q}$-pure Hodge structure on V is equivalently given by a direct sum decomposition $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$ (the *Hodge decomposition*) into $\mathbb{C}$-vector spaces, such that the complex conjugate $\overline{V^{q,p}}$ coincides with $V^{p,q}$ for all $p, q \in \mathbb{Z}$ with $p+q=n$. The relation between the Hodge filtration and the Hodge decomposition is given by $F^p V_{\mathbb{C}} = \oplus_{p' \geq p} V^{p', n-p'}$. In the general mixed case Deligne [Del71, 1.2.8] proved the following:

Proposition 2.2. *A $\mathbb{Q}$-mixed Hodge structure on V is the datum of a bigrading*

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} I^{p,q} \tag{2.1}$$

satisfying that each complex vector subspace $W_k V_{\mathbb{C}} = \bigoplus_{p+q\leq k} I^{p,q}$ of $V_{\mathbb{C}}$ is defined over $\mathbb{Q}$ and

$$I^{p,q} \equiv \overline{I^{q,p}} \bmod \bigoplus_{r<p,s<q} I^{r,s}. \tag{2.2}$$

The Hodge filtration is then defined by $F^p V_{\mathbb{C}} = \bigoplus_{r\geq p} I^{r,q}$.

We will use a third, more group-theoretic, point of view on $\mathbb{Q}$-mixed Hodge structures. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$ be the Deligne torus, this is the real algebraic group such that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$, with the action of the complex conjugation twisted by the automorphism that interchanges the two factors. The character group of $\mathbb{S}$, denoted by $X_*(\mathbb{S})$, identifies with $\mathbb{Z}\oplus\mathbb{Z}$ under

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\sim} & X_*(\mathbb{S}) \\ (p,q) & \mapsto & \big(z\in\mathbb{S}(\mathbb{R}) = \mathbb{C}^* \mapsto z^{-p}\overline{z}^{-q} \in \mathbb{C}^*\big). \end{array}$$

Given a $\mathbb{Q}$-vector space V a bigrading $V_{\mathbb{C}} = \oplus_{p,q\in\mathbb{Z}} I^{p,q}$ is thus equivalent to a homomorphism $h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$. In particular we deduce from the paragraph above that any mixed Hodge structure on V defines a homomorphism $h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$. In [Pin89] Pink identified the conditions such a homomorphism has to satisfy to define a mixed Hodge structure on V:

Proposition 2.3. *[Pin89, 1.4 and 1.5] Let V be a finite dimensional $\mathbb{Q}$-vector space. A morphism $h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ defines a MHS on V if and only if there exists a connected $\mathbb{Q}$-algebraic subgroup $P \subset \mathrm{GL}(V)$ such that h factors through $P_{\mathbb{C}}$ and which satisfies the following conditions:*

- (i) *The composite $\mathbb{S}_{\mathbb{C}} \xrightarrow{h} P_{\mathbb{C}} \to (P/W_{-1})_{\mathbb{C}}$ is defined over $\mathbb{R}$, where W_{-1} denotes the unipotent radical of P.*
- (ii) *The composite $\mathbb{G}_{m,\mathbb{R}} \xrightarrow{w} \mathbb{S} \xrightarrow{h} P \to (P/W_{-1})_{\mathbb{R}}$ is a cocharacter of the center of $(P/W_{-1})_{\mathbb{R}}$ defined over $\mathbb{Q}$.*
- (iii) *The weight filtration on $\mathrm{Lie}\,P$ defined by $\mathrm{Ad}_P \circ h$ satisfies $W_0 \,\mathrm{Lie}\,P = \mathrm{Lie}\,P$ and $W_{-1}(\mathrm{Lie}\,P) = \mathrm{Lie}\,W_{-1}$.*

If $h \in \mathcal{M}$ let us define the *Mumford–Tate group* $\mathrm{MT}(h)$ of the $\mathbb{Q}$-mixed Hodge structure (M,h) as the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ whose complexification contains $h(\mathbb{S}_{\mathbb{C}})$. One easily checks that the groups P satisfying the conditions of Proposition 2.3 are precisely the ones containing $\mathrm{MT}(h)$. Condition (iii) implies in particular that the unipotent radical $\mathcal{R}_u(P)$ of any such P coincides with $\mathcal{R}_u(P^{\mathcal{M}}) \cap P$.

We finish this subsection by recalling the definition of polarizations.

Definition 2.4. Let $(V, W_\bullet, F^\bullet)$ be a $\mathbb{Q}$-mixed Hodge structure. A *(graded) polarization* is a collection of non-degenerate $(-1)^k$-symmetric bilinear forms

$$Q_k\colon \mathrm{Gr}_k^W(V) \otimes \mathrm{Gr}_k^W(V) \to \mathbb{Q}$$

such that

- (i) $Q_k(F^p\mathrm{Gr}_k^W V_{\mathbb{C}}, F^{k-p+1}\mathrm{Gr}_k^W V_{\mathbb{C}}) = 0$ for each k (first Riemann bilinear relation);
- (ii) the Hermitian form on $\mathrm{Gr}_k^W(V)_{\mathbb{C}}$ given by $Q_k(Cu, \overline{v})$ is positive-definite, where C is the Weil operator ($C|_{I^{p,q}} = i^{p-q}$ for all p,q).

One easily checks that the Mumford-Tate group of a polarizable pure $\mathbb{Q}$-Hodge structure is reductive.

2.2. **Classifying space.** In this subsection, we discuss the classifying space of all $\mathbb{Q}$-mixed Hodge structures with given weight filtration, graded polarization and Hodge numbers.

Let V be a finite dimensional $\mathbb{Q}$-vector space, endowed with the following additional data:

(i) a finite increasing filtration $W_\bullet$ of V;
(ii) a collection of non-degenerate $(-1)^k$-symmetric bilinear forms
$$Q_k\colon \mathrm{Gr}_k^W(V) \otimes \mathrm{Gr}_k^W(V) \to \mathbb{Q} \ ;$$
(iii) a partition $\{h^{p,q}\}_{p,q\in\mathbb{Z}}$ of $\dim V_\mathbb{C}$ into non-negative integers.

Given these data, one forms the classifying space $\mathcal{M}$ parametrizing $\mathbb{Q}$-mixed Hodge structures $(V, W_\bullet, F^\bullet)$ with the following properties:

(1) the (p,q)-constituent $V^{p,q} := \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W V_\mathbb{C}$ has complex dimension $h^{p,q}$;
(2) $Q_k(F^p\mathrm{Gr}_k^W V_\mathbb{C}, F^{k-p+1}\mathrm{Gr}_k^W V_\mathbb{C}) = 0$ for each k (first Riemann bilinear relation);
(3) $(V, W_\bullet, F^\bullet)$ is graded-polarized by Q_k.

Let us summarize the construction and basic properties of $\mathcal{M}$; see [Kap95], [Pea00, below (3.7) to Lemma 3.9] for more details. First one defines the complex algebraic variety $\mathcal{M}^\vee$ parametrizing mixed Hodge structures satisfying only the conditions (1) and (2) above (see [Pea00, Lem. 3.8]). This is a homogeneous space under $P^\mathcal{M}(\mathbb{C})$, where $P^\mathcal{M}$ is the $\mathbb{Q}$-algebraic group defined as follows: for any $\mathbb{Q}$-algebra R,

$$P^\mathcal{M}(R) := \{g \in \mathrm{GL}(V_R) : g(W_k) \subseteq W_k \text{ and } \mathrm{Gr}_k^W(g) \in \mathrm{Aut}_R(Q_k) \text{ for all } k \in \mathbb{Z}\}. \tag{2.3}$$

The classifying space $\mathcal{M}$ is defined as the real semi-algebraic open subset of $\mathcal{M}^\vee$ consisting of mixed Hodge structures which satisfy moreover condition (3) above (see [Pea00, Lem. 3.9 and above]). The fact that $\mathcal{M}$ is open in $\mathcal{M}^\vee$ endows $\mathcal{M}$ with a natural complex analytic structure. The real semi-algebraic group

$$\{g \in P^\mathcal{M}(\mathbb{C}) : \mathrm{Gr}_k^W(g) \in \mathrm{Aut}_\mathbb{R}(Q_k) \text{ for all } k \in \mathbb{Z}\} \tag{2.4}$$

identifies with $P^\mathcal{M}(\mathbb{R})^+W_{-1}^\mathcal{M}(\mathbb{C})$, where $W_{-1}^\mathcal{M}$ is the unipotent radical of $P^\mathcal{M}$, see [Pea00, Remark below Lem. 3.9]. It acts transitively on $\mathcal{M}$.

2.3. **Adjoint Hodge structure.** For each $h \in \mathcal{M}$ Proposition 2.3 defines a natural $\mathbb{Q}$-mixed Hodge structure on $\mathrm{Lie}\, P^\mathcal{M}$ via $\mathrm{Ad}^\mathcal{M} \circ h\colon \mathbb{S}_\mathbb{C} \to P_\mathbb{C}^\mathcal{M} \to \mathrm{GL}(\mathrm{Lie}\, P^\mathcal{M})_\mathbb{C}$: the *adjoint Hodge structure* associated with h. One easily checks that the corresponding weight filtration and graded polarization are independent of h. Indeed the weight filtration $W_\bullet$ on $\mathrm{Lie}\, P^\mathcal{M} \subseteq \mathrm{End}(V) = V \otimes V^\vee$ is the one deduced from the weight filtration $W_\bullet$ on V. Similarly for the graded-polarization.

2.4. **(Weak) Mumford–Tate domains.** Proposition 2.3 suggests to attack the problem of classifying mixed Hodge structures by rather considering mixed Hodge structures with prescribed Mumford-Tate group. This leads abstractly to the notion of mixed Hodge data, see Section 4.1; and geometrically to the notion of (weak) Mumford-Tate domain refining the classifying space $\mathcal{M}$.

Definition 2.5. (i) A subset $\mathcal{D}$ of the classifying space $\mathcal{M}$ is called a Mumford–Tate domain if there exists an element $h \in \mathcal{D}$ such that $\mathcal{D} = P(\mathbb{R})^+W_{-1}(\mathbb{C})h$, where $P = \mathrm{MT}(h)$ and $W_{-1} = \mathcal{R}_u(P)$ is the unipotent radical of P.

(ii) A subset $\mathcal{D}$ of the classifying space $\mathcal{M}$ is called a weak Mumford–Tate domain if there exist an element $h \in \mathcal{D}$ and a normal subgroup N of $P = \mathrm{MT}(h)$ such that $\mathcal{D} = N(\mathbb{R})^+\mathcal{R}_u(N)(\mathbb{C})h$, where $\mathcal{R}_u(N)$ is the unipotent radical of N.

In the definition, as $N \lhd P$, we have $\mathcal{R}_u(N) = W_{-1} \cap N$. One easily checks that $\mathcal{M}$ is a Mumford-Tate domain in itself, for $P = P^{\mathcal{M}}$. A closer look at the geometry of general Mumford–Tate domains is given in Appendix A.1. In particular we will prove the following results (well-known in the pure case):

Proposition 2.6. *Every weak Mumford–Tate domain in $\mathcal{M}$ is a complex analytic subspace of $\mathcal{M}$.*

Lemma 2.7. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be Mumford–Tate domains in $\mathcal{M}$. Then every irreducible component of $\mathcal{D}_1 \cap \mathcal{D}_2$ is again a Mumford–Tate domain in $\mathcal{M}$.*

This lemma has the following immediate corollary.

Corollary 2.8. *Let $\mathcal{Z}$ be a complex analytic irreducible subset of $\mathcal{M}$. Then there exists a smallest Mumford–Tate domain, denoted by $\mathcal{Z}^{\mathrm{sp}}$ and called the* special closure *of $\mathcal{Z}$, which contains $\mathcal{Z}$.*

We close this subsection with some discussion on the *generic Mumford–Tate group* of a complex analytic irreducible subvariety of $\mathcal{M}$. In particular the discussion applies to weak Mumford–Tate domains. The trivial local system $\mathbb{V} = \mathcal{M} \times V$ underly a natural family of mixed Hodge structures: for each $h \in \mathcal{M}$ the triple $(V, (W_\bullet)_h, (\mathcal{F}^\bullet)_h)$ is a mixed $\mathbb{Q}$-Hodge structure. For any complex analytic irreducible subset $\mathcal{Z}$ of $\mathcal{M}$, the first part of the proof of [And92, §4, Lemma 4] applies: for a very general element $h \in \mathcal{Z}$, the Mumford–Tate group $P(h)$ does not depend on h. Such an h is said to be *Hodge–generic* in $\mathcal{Z}$ and its Mumford-Tate group is called the *generic Mumford–Tate group* of $\mathcal{Z}$. We write $\mathrm{MT}(\mathcal{Z})$ to denote the generic Mumford–Tate group of $\mathcal{Z}$. It satisfies the following property: $\mathrm{MT}(h') < \mathrm{MT}(\mathcal{Z})$ for any $h' \in \mathcal{Z}$.

Lemma 2.9. *Let $\mathcal{D} = P(\mathbb{R})^+W_{-1}(\mathbb{C})h$ be a Mumford-Tate domain in $\mathcal{M}$ (thus $h \in \mathcal{D}$, $P = \mathrm{MT}(h)$ and W_{-1} is the unipotent radical of P). Then $P = \mathrm{MT}(\mathcal{D})$.*

Proof. By definition of $\mathrm{MT}(\mathcal{D})$ the group P is a subgroup of $\mathrm{MT}(\mathcal{D})$. Thus we are reduced to proving the converse inclusion.

Each $h' \in \mathcal{D}$ is of the form ghg^{-1} for some $g \in P(\mathbb{R})^+W_{-1}(\mathbb{C})$, and hence the homomorphism $h' =: \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $gP_{\mathbb{C}}g^{-1} = P_{\mathbb{C}}$. This implies that $\mathrm{MT}(h') < P$ for all $h' \in \mathcal{D}$. Looking at a Hodge generic point h' we are done. ☐

The following lemma, whose proof is given Appendix A.1, is useful to determine when an orbit is a Mumford–Tate domain.

Lemma 2.10. *Let P be a $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ with $W_{-1} = \mathcal{R}_u(P)$ and let $\mathcal{D}$ be a $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-orbit in $\mathcal{M}$. If some $h \in \mathcal{D}$ satisfies that $h\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $P_{\mathbb{C}}$ then $\mathcal{D}$ is a Mumford–Tate domain and $\mathrm{MT}(\mathcal{D}) \lhd P$.*

3. Variation of mixed Hodge structures

Let $f\colon X \to S$ be a morphism of algebraic varieties. If f satisfies a sharp notion of topological local constancy (suffice it to say here it is automatically satisfied if f is proper smooth, and is true over a Zariski-open subset of S for any morphism of varieties), then f gives rise to a family of mixed Hodge structures (pure when f is proper smooth) on $H^n(X_s, \mathbb{Q})$, as s varies over S^{an}, subject to certain rules. This leads to the notion of a (graded-polarizable) variation of mixed Hodge structures, which we now recall:

Definition 3.1. Let S be a connected complex manifold. A *variation of mixed Hodge structures* (abbreviated VMHS) on S is a triple $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ consisting of:

(i) a local system $\mathbb{V}_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules of finite rank on S;
(ii) a finite increasing filtration $W_\bullet$ of the local system $\mathbb{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}_S} \mathbb{Q}_S$ by local subsystems (weight filtration);
(iii) a finite decreasing filtration $\mathcal{F}^\bullet$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles (Hodge filtration),

satisfying the following conditions:

(1) for each $s \in S$, the triple $(\mathbb{V}_s, W_\bullet(s), \mathcal{F}^\bullet(s))$ is a mixed Hodge structure;
(2) the connection $\nabla \colon \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}} := \mathbb{V} \otimes_{\mathbb{Q}} \mathbb{C}$ satisfies the Griffiths' transversality condition

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S.$$

Definition 3.2. A VMHS $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ on S is called *graded-polarizable* if the induced variations of pure $\mathbb{Q}$-Hodge structures (VHS) $\mathrm{Gr}^W_k \mathbb{V}$, $k \in \mathbb{Z}$, are all polarizable, *i.e.* for each $k \in \mathbb{Z}$ there exists a morphism of local systems

$$\mathcal{Q}_k \colon \mathrm{Gr}^W_k \mathbb{V} \otimes \mathrm{Gr}^W_k \mathbb{V} \to \mathbb{Q}_S$$

inducing on each fiber a polarization of the corresponding $\mathbb{Q}$-Hodge structure of weight k.

From now on all VMHS are assumed to be graded-polarizable.

3.1. **Mumford-Tate group and monodromy group.** Let S be a connected complex manifold and $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ a VMHS on S. The pull-back $\pi^*\mathbb{V}_{\mathbb{Z}}$ of $\mathbb{V}_{\mathbb{Z}}$ along the universal covering map $\pi \colon \widetilde{S} \to S$ is canonically trivialized: $\pi^*\mathbb{V}_{\mathbb{Z}} \simeq \widetilde{S} \times V_{\mathbb{Z}}$, with $V_{\mathbb{Z}} = H^0(\widetilde{S}, \pi^*\mathbb{V}_{\mathbb{Z}})$.

For $s \in S$, we denote by $\mathrm{MT}_s \subseteq \mathrm{GL}(\mathbb{V}_s)$ the Mumford–Tate group of the Hodge structure $\mathbb{V}_s$ and by $H^{\mathrm{mon}}_s \subseteq \mathrm{GL}(\mathbb{V}_s)$ the *connected algebraic monodromy group* at s, that is the connected component of identity of the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(\mathbb{V}_s)$ containing the image under monodromy of $\pi_1(S, s)$.

By definition the algebraic monodromy group H^{mon}_s is locally constant on S. By [And92, §4, Lemma 4], following [Del87, § 7.5] in the pure case, the Mumford-Tate group $\mathrm{MT}_s \subset \mathrm{GL}(\mathbb{V}_s)$ is locally constant on $S^\circ = S \setminus \Sigma$ where Σ denotes a meager subset of S; and H^{mon}_s is a subgroup of P_s for all $s \in S^\circ$ as $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ is graded-polarizable. We call S° the *Hodge-generic locus.* For $s \in S^\circ$ the group MT_{s_0} is called the *generic Mumford–Tate group* $\mathrm{MT}(S)$ of $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$.

3.2. **Admissible VMHS.** *Admissible* VMHSs are the ones with good asymptotic properties. The concept was introduced by Steenbrick–Zucker [SZ85, Properties 3.13] on a curve and Kashiwara [Kas86, 1.8 and 1.9] in general. All VMHSs which arise from geometry are admissible [EZ86] and all VHSs are automatically admissible. We recall briefly the definition.

Definition 3.3 (admissible VMHS). A VMHS $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ over the punctured unit disc Δ^* is called admissible if

(i) it is graded-polarizable;
(ii) the monodromy T around zero is quasi-unipotent and the logarithm N of the unipotent part of T admits a weight filtration $M(N, W_\bullet)$ relative to $W_\bullet$ (see [Kas86, §3.1]);

(iii) Let $\overline{\mathcal{V}}$, resp. $W_k\overline{\mathcal{V}}$, be Deligne's canonical extension of $\mathcal{V}$, resp. of $\mathcal{O}_{\Delta^*} \otimes_{\mathbb{Q}} W_k\mathbb{V}$, to Δ. The Hodge filtration $\mathcal{F}^\bullet$ extends to a locally free filtration $\overline{\mathcal{F}}^\bullet$ of $\overline{\mathcal{V}}$ such that $\mathrm{Gr}^p_{\overline{\mathcal{F}}}\mathrm{Gr}^W_k\overline{\mathcal{V}}$ is locally free.

Let S be a connected complex manifold compactifiable by a compact complex analytic space $\overline{S}$. A graded-polarizable variation of mixed Hodge structure $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ on S is said admissible with respect to $\overline{S}$ if for every holomorphic map $i\colon \Delta \to \overline{S}$ which maps Δ^* to S, the variation $i^*(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ is admissible.

Let S be a smooth complex quasi-projective variety. The property for a VHMS on S^{an} to be admissible with respect to a smooth projective compactification $\overline{S}^{\mathrm{an}}$ is easily seen to be independent of the choice of $\overline{S}$. Hence we can and will talk of admissible VMHSs on S^{an}. *From now on, and in order to simplify notations, we will not distinguish between S and S^{an}, the meaning being clear from the context.*

Admissible VMHSs have the following advantage (see André [And92, §5, Theorem 1], following [Del87, §7.5] in the pure case):

Theorem 3.4. *(Deligne, André) Let $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ be an admissible VMHS over a smooth connected complex quasi-projective variety S. Then for any Hodge-generic point $s \in S^\circ$, the connected algebraic monodromy group H^{mon}_s is a normal subgroup of the derived group $\mathrm{MT}(S)^{\mathrm{der}}$ of the generic Mumford-Tate group of S.*

4. Mixed Hodge data

Classifying mixed Hodge structures with prescribed Mumford-Tate group leads to the formalism of mixed Hodge data introduced in [Kli17], following [Pin89] in the Shimura case. This group theoretical formalism is useful to relate VMHS and Mumford-Tate domains.

4.1. Mixed Hodge data.

Definition 4.1. A connected mixed Hodge datum is a pair $(P, \mathcal{X})$, where P is a connected linear algebraic group over $\mathbb{Q}$ whose unipotent radical we denote by W_{-1}, and $\mathcal{X} \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ is a $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-conjugacy class such that one (and then any) $h \in \mathcal{X}$ satisfies property (i), (ii) and (iii) of Proposition 2.3. A morphism $(P, \mathcal{X}) \to (P', \mathcal{X}')$ of mixed Hodge data is a morphism $P \to P'$ of $\mathbb{Q}$-algebraic groups inducing an equivariant map $\mathcal{X} \to \mathcal{X}'$.

Let $(P, \mathcal{X})$ be a mixed Hodge datum. As a homogeneous space under $P(\mathbb{R})^+W_{-1}(\mathbb{C})$, the set $\mathcal{X}$ is naturally endowed with a structure of real semi-algebraic variety. In general however it does not carry any complex structure. To relate $\mathcal{X}$ to complex geometry, let us fix $\rho\colon P \to \mathrm{GL}(V)$ a $\mathbb{Q}$-representation. By Proposition 2.3, for each $h \in \mathcal{X}$ the map $\rho \circ h$ endows V with a rational mixed Hodge structure, whose weight filtration and Hodge numbers are easily seen to be independent of $h \in \mathcal{X}$. We thus obtain a $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-equivariant map

$$\varphi_\rho\colon \mathcal{X} \to \mathcal{M},$$

for $\mathcal{M}$ a classifying space as in Section 2.2. By [Pin89, 1.7], φ_ρ factors through a complex manifold $\mathcal{D}$ which is independent of ρ [1]. From now on we will just write

$$\varphi\colon \mathcal{X} \to \mathcal{D} \tag{4.1}$$

[1]Take ρ to be a faithful representation of P, then we can take $\mathcal{D} = \varphi_\rho(\mathcal{X})$.

and call this map *the classifying map of the Hodge datum* $(P, \mathcal{X})$. The group $P(\mathbb{R})^+W_{-1}(\mathbb{C})$ acts on $\mathcal{D}$ preserving its complex structure, and the action of $W_{-1}(\mathbb{C})$ on $\mathcal{D}$ is holomorphic.

Lemma 4.2. *([Pin89, 1.8(b)]) For each $x \in \mathcal{D}$, the fiber $\varphi^{-1}(x)$ is a principal homogeneous space under $\exp(F_x^0(\mathrm{Lie}\, W_{-1})_{\mathbb{C}})$.*

In particular φ is an isomorphism in the pure case.

4.2. **Mixed Hodge data and Mumford-Tate domains.** We now relate mixed Hodge data and Mumford-Tate domains by showing that the complex space $\mathcal{D}$ in (4.1) is a Mumford-Tate domain, and that conversely any Mumford-Tate domain appears as a target in (4.1) for some connected mixed Hodge datum. We start with the case where $\mathcal{D} = \mathcal{M}$ is a classifying space.

Lemma 4.3. *Let $\mathcal{M}$ be a classifying space of mixed Hodge structure as in Section 2.2, $P^{\mathcal{M}}$ the corresponding group, and $W_{-1}^{\mathcal{M}}$ its unipotent radical.*

There exists a mixed Hodge datum $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$ such that the classifying map (4.1) for $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$ reads $\varphi^{\mathcal{M}} \colon \mathcal{X}^{\mathcal{M}} \to \mathcal{M}$. For any $h \in \mathcal{X}^{\mathcal{M}}$, the mixed Hodge structures on $\mathrm{Lie}\, P^{\mathcal{M}}$ induced by h and by $\varphi^{\mathcal{M}}(h)$ coincide.

Proof. Take $h \in \mathcal{M}$. Thus $h \in \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}}^{\mathcal{M}})$ satisfies conditions (i), (ii) and (iii) of Proposition 2.3. In particular $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$ is a mixed Hodge datum, where $\mathcal{X}^{\mathcal{M}} := P^{\mathcal{M}}(\mathbb{R})^+W_{-1}^{\mathcal{M}}(\mathbb{C})h \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}}^{\mathcal{M}})$. The existence of $\varphi^{\mathcal{M}}$ follows from [Pin89, 1.7]; it is precisely the φ from (4.1) for $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$. $\square$

Proposition 4.4. *Let $\mathcal{M}$ be a classifying space of mixed Hodge structure as in Section 2.2, with associated connected mixed Hodge datum $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$ and classifying map $\varphi^{\mathcal{M}} \colon \mathcal{X}^{\mathcal{M}} \to \mathcal{M}$ as in Lemma 4.3.*

- (i) *For each Mumford–Tate domain $\mathcal{D}$ in $\mathcal{M}$, there exists a sub-mixed Hodge datum $(\mathrm{MT}(\mathcal{D}), \mathcal{X})$ of $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$ such that $\varphi^{\mathcal{M}}(\mathcal{X}) = \mathcal{D}$. Moreover $\varphi := \varphi^{\mathcal{M}}|_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{D}$ is precisely the classifying map (4.1) for $(\mathrm{MT}(\mathcal{D}), \mathcal{X})$.*
- (ii) *Conversely for any sub-mixed Hodge datum $(P, \mathcal{X})$ of $(P^{\mathcal{M}}, \mathcal{X}^{\mathcal{M}})$, the image $\varphi^{\mathcal{M}}(\mathcal{X})$ is a Mumford–Tate domain in $\mathcal{M}$ (whose generic Mumford–Tate group is a normal subgroup of P).*

Proof. For (i): for simplicity we write P for $\mathrm{MT}(\mathcal{D})$ and W_{-1} for $\mathcal{R}_u(P)$. Take a point $x \in \mathcal{D}$; it gives rise to a homomorphism $h_x \colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$. View $h_x \in \mathcal{X}^{\mathcal{M}}$, then $\varphi^{\mathcal{M}}(h_x) \in \mathcal{D}$ by definition of $\varphi^{\mathcal{M}}$. Let $\mathcal{X} = P(\mathbb{R})^+W_{-1}(\mathbb{C})h_x \subset \mathcal{M}$. As $\varphi^{\mathcal{M}}$ is $P^{\mathcal{M}}(\mathbb{R})^+W_{-1}^{\mathcal{M}}(\mathbb{C})$-equivariant, we have $\varphi^{\mathcal{M}}(\mathcal{X}) = P(\mathbb{R})^+W_{-1}(\mathbb{C})\varphi^{\mathcal{M}}(h_x) = P(\mathbb{R})^+W_{-1}(\mathbb{C})x = \mathcal{D}$. By Proposition 2.3 the pair $(P, \mathcal{X})$ is a mixed Hodge datum and by construction $\varphi = \varphi^{\mathcal{M}}|_{\mathcal{X}}$ is precisely the map in (4.1).

For (ii): Denote by $\mathcal{D} = \varphi^{\mathcal{M}}(\mathcal{X})$. Then $\mathcal{D}$ is a $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-orbit because the map $\varphi^{\mathcal{M}}$ is $P^{\mathcal{M}}(\mathbb{R})^+W_{-1}^{\mathcal{M}}(\mathbb{C})$-equivariant. Moreover for any $x \in \mathcal{D}$, the corresponding homomorphism $h_x \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $P_{\mathbb{C}}$ by definition of mixed Hodge data. Thus $\mathcal{D}$ is a Mumford–Tate domain and $\mathrm{MT}(\mathcal{D}) \lhd P$ by Lemma 2.10. $\square$

5. Quotients

5.1. **Quotient of mixed Hodge datum.** Given a connected mixed Hodge datum $(P, \mathcal{X})$ and a normal subgroup $N \lhd P$, the *quotient mixed Hodge datum*

$$q_N \colon (P, \mathcal{X}) \to (P, \mathcal{X})/N \tag{5.1}$$

is defined as follows. Given $h \in \mathcal{X} \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ we denote by $\overline{h} \in \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, (P/N)_{\mathbb{C}})$ the homomorphism $\mathbb{S}_{\mathbb{C}} \xrightarrow{h} P_{\mathbb{C}} \to (P/N)_{\mathbb{C}}$. Note that $\mathcal{R}_u(P/N) = W_{-1}/(W_{-1} \cap N)$. Denote by $\mathcal{X}/N = (P/N)(\mathbb{R})^+(W_{-1}/W_{-1} \cap N)(\mathbb{C})\overline{h} \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, (P/N)_{\mathbb{C}})$. One easily checks that $(P, \mathcal{X})/N \colon = (P/N, \mathcal{X}/N)$ is a connected mixed Hodge datum, independent of the choice of $h \in \mathcal{X}$. The morphism $q_N \colon (P, \mathcal{X}) \to (P/N, \mathcal{X}/N)$ is what we desire. Moreover $q_N \colon \mathcal{X} \to \mathcal{X}/N$ is clearly real algebraic.

5.2. **Quotient of Mumford–Tate domains.** Next we prove that Mumford–Tate domains are stable under taking quotients. This operation is important to understand the structure of Mumford–Tate domains.

Let $V_{\mathbb{Z}}$ be a free finite rank $\mathbb{Z}$-module and $V := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the associated $\mathbb{Q}$-vector space. Let $\mathcal{M}$ be the classifying space of certain polarized mixed Hodge structures and let $P^{\mathcal{M}}$ be the $\mathbb{Q}$-group, both from Section 2.2.

Proposition 5.1. *Let $\mathcal{D}$ be a Mumford–Tate domain in $\mathcal{M}$ with $P = \mathrm{MT}(\mathcal{D})$, and let $(P, \mathcal{X})$ and $\varphi \colon \mathcal{X} \to \mathcal{D}$ be as in (4.4)(i). Let N be a normal subgroup of P. Then there exists a quotient $p_N \colon \mathcal{D} \to \mathcal{D}/N$, in the category of complex varieties, such that*

- (i) *$\mathcal{D}/N$ is a Mumford–Tate domain in some classifying space of mixed Hodge structures, and $\mathrm{MT}(\mathcal{D}/N) = P/N$.*
- (ii) *Each fiber of p_N is an $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})$-orbit, where $W_{-1} = \mathcal{R}_u(P)$.*
- (iii) *For the quotient mixed Hodge datum $q_N \colon (P, \mathcal{X}) \to (P/N, \mathcal{X}/N)$ defined in (5.1), the classifying map (4.1) for $(P/N, \mathcal{X}/N)$ has image $\mathcal{D}/N$, thus defining $\varphi_{/N} \colon \mathcal{X}/N \to \mathcal{D}/N$.*
- (iv) *The following commutative diagram commutes*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "q_N"] \arrow[d, "\varphi"'] & \mathcal{X}/N \arrow[d, "\varphi_{/N}"] \\ \mathcal{D} \arrow[r, "p_N"] & \mathcal{D}/N. \end{tikzcd} \tag{5.2}$$

Proof. Consider the quotient mixed Hodge datum $q_N \colon (P, \mathcal{X}) \to (P/N, \mathcal{X}/N)$ defined in (5.1). Any $\overline{h} \in \mathcal{X}/N \subseteq \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, (P/N)_{\mathbb{C}})$ induces a $\mathbb{Q}$-mixed Hodge structure on $\mathrm{Lie}(P/N)$, via $\mathrm{Ad}_{P/N} \circ \overline{h} \colon \mathbb{S}_{\mathbb{C}} \to (P/N)_{\mathbb{C}} \to \mathrm{GL}(\mathrm{Lie}(P/N))_{\mathbb{C}}$, which satisfies the three properties listed in Definition 4.1 with P replaced by P/N and h replaced by $\overline{h}$.

Fix a faithful representation $\overline{\rho} \colon P/N \to \mathrm{GL}(V')$ defined over $\mathbb{Q}$. Then the morphism $\overline{\rho} \circ \overline{h}$ induces a $\mathbb{Q}$-mixed Hodge structure on V' by Proposition 2.3 for each $\overline{h} \in \mathcal{X}/N$, and the weight filtration and the Hodge numbers does not depend on the choice of $\overline{h} \in \mathcal{X}/N$. Thus we obtain a map

$$\varphi_{/N} \colon \mathcal{X}/N \to \{\text{mixed Hodge structures on } V'\}.$$

Set $\mathcal{D}/N = \varphi_{/N}(\mathcal{X}/N)$. Then we get $\varphi_{/N} \colon \mathcal{X}/N \to \mathcal{D}/N$, which by [Pin89, 1.7] is $(P/N)(\mathbb{R})^+(W_{-1}/(W_{-1} \cap N))(\mathbb{C})$-equivariant (here $W_{-1} = \mathcal{R}_u(P)$ and hence $\mathcal{R}_u(P/N) = W_{-1}/(W_{-1} \cap N)$). This establishes (iii) for the space $\mathcal{D}/N$.

By [Pin89, 1.12] the $\mathbb{Q}$-mixed Hodge structures on V' thus obtained are graded-polarized by the some collection of non-degenerate bilinear forms (same for all $\overline{h}$). So $\mathcal{D}/N$ is a contained in some classifying space $\mathcal{M}'$. This establishes (i).

Now let us construct the map $p_N \colon \mathcal{D} \to \mathcal{D}/N$ and prove properties (ii) and (iv). Take $x \in \mathcal{D}$, and take any $h_x \in \varphi^{-1}(x)$. Then $\varphi^{-1}(x) = \exp(F_x^0(\mathrm{Lie}\, W_{-1})_{\mathbb{C}})h_x$ by Lemma 4.2. Denote for simplicity by $F_x^0 = \exp(F_x^0(\mathrm{Lie}\, W_{-1})_{\mathbb{C}})$; it is a subgroup of $P_{\mathbb{C}}$. Then $q_N(\varphi^{-1}(x)) = q_N(F_x^0 h_x) = (F_x^0/(N(\mathbb{C}) \cap F_x^0))q_N(h_x)$.

On the other hand define $\overline{x} := \varphi_{/N}(q_N(h_x))$. Then $\varphi_{/N}^{-1}(\overline{x}) = \exp(F_{\overline{x}}^0(\mathrm{Lie}\, W_{-1}/(W_{-1}\cap N))_{\mathbb{C}})q_N(h_x)$ again by Lemma 4.2.

We claim that $F_x^0/(N(\mathbb{C})\cap F_x^0) = \exp(F_{\overline{x}}^0(\mathrm{Lie}\, W_{-1}/(W_{-1}\cap N))_{\mathbb{C}})$. Indeed it suffices to check for Lie algebras, *i.e.* it suffices to prove $F_x^0(\mathrm{Lie}\, W_{-1})_{\mathbb{C}}/(\mathrm{Lie}\, N_{\mathbb{C}}\cap F_x^0(\mathrm{Lie}\, W_{-1})_{\mathbb{C}}) \simeq F_{\overline{x}}^0(\mathrm{Lie}\, W_{-1}/(W_{-1}\cap N))_{\mathbb{C}}$ canonically. As $N \lhd P$, we have $\mathrm{Ad}_P(\mathrm{Lie}\, N) \subseteq \mathrm{Lie}\, N$. So $\mathrm{Lie}\, N$ is a sub-mixed Hodge structure of the adjoint Hodge structure on $\mathrm{Lie}\, P$. Thus $\mathrm{Lie}\, N_{\mathbb{C}} \cap F_x^0(\mathrm{Lie}\, W_{-1})_{\mathbb{C}} = F_x^0(\mathrm{Lie}\, W_{-1}\cap N)_{\mathbb{C}}$. Thus we proved the desired claim.

By the last three paragraphs, we have $q_N(\varphi^{-1}(x)) = \varphi_{/N}^{-1}(\overline{x})$. So the map $\mathcal{D}\to\mathcal{D}/N$, $x\mapsto \overline{x} := \varphi_{/N}(q_N(h_x))$ is well-defined. Call this map p_N. Then property (iv) holds true by construction of p_N. Property (ii) then is not hard to check.

Now the map is complex analytic by property (ii). $\square$

6. Period Map and Logarithmic Ax

6.1. **Period map.** Let S be an irreducible algebraic variety defined over $\mathbb{C}$. Assume that S carries a graded-polarized VMHS $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet) \to S$. Then it induces a period map $[\Phi]\colon S\to \Gamma\backslash\mathcal{M}$ where $\mathcal{M}$ is the classifying space and Γ is an arithmetic subgroup of $P^{\mathcal{M}}(\mathbb{Q})$. It is known that $[\Phi]$ satisfies the Griffiths tranversality.

The period map $[\Phi]$ factors through a quotient space in the following way. Take a complex analytic irreducible component $\widetilde{S}$ of $u^{-1}([\Phi](S))$, where $u\colon \mathcal{M}\to\Gamma\backslash\mathcal{M}$. Let $\mathcal{D} = \widetilde{S}^{\mathrm{sp}}$, the smallest Mumford–Tate domain containing $\widetilde{S}$; see Corollary 2.8. Let $P = \mathrm{MT}(\widetilde{S})$ and $W_{-1} = \mathcal{R}_u(P)$, then $\mathcal{D}$ is a $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-orbit. Now we have $[\Phi](S)\subseteq u(\mathcal{D})$.

Let $\Gamma_P = \Gamma\cap P(\mathbb{Q})$, then $[\Phi]$ factors through $S\to\Gamma_P\backslash\mathcal{D}$.

Let $\Delta = S\times_{\Gamma\backslash\mathcal{M}}\mathcal{M}$. We claim that $\Delta = S\times_{\Gamma_P\backslash\mathcal{D}}\mathcal{D}$. Indeed $\supseteq$ is clear, and $\subseteq$ follows from $[\Phi](S)\subseteq\Gamma_P\backslash\mathcal{D}$ and the definition $\Delta = S\times_{\Gamma\backslash\mathcal{M}}\mathcal{D}$.

So to prove Theorem 1.1, it suffices to work in the following diagram

$$
S\times\mathcal{D}\supseteq \quad
\begin{tikzcd}
\Delta \arrow[r] \arrow[d, "u_S"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D} \arrow[d, "u"] \\
S \arrow[r, "{[\Phi]}"] & \Gamma_P\backslash\mathcal{D}
\end{tikzcd}
. \tag{6.1}
$$

This is our setup for the rest of the paper.

6.2. **Quotient for the period map.** Assume $N\lhd P$. We have constructed the quotient Mumford–Tate domain $p_N\colon\mathcal{D}\to\mathcal{D}/N$ in Proposition 5.1. For the arithmetic group $\Gamma_{P/N} := \Gamma_P/(\Gamma_P\cap N(\mathbb{Q}))$, we then have a map $[p_N]\colon\Gamma_P\backslash\mathcal{D}\to\Gamma_{P/N}\backslash(\mathcal{D}/N)$. Composing with $[\Phi]\colon S\to\Gamma_P\backslash\mathcal{D}$, we obtain

$$
[\Phi_{/N}]\colon S\to\Gamma_{P/N}\backslash(\mathcal{D}/N). \tag{6.2}
$$

Proposition 5.1 says that $\mathcal{D}/N$ is a Mumford–Tate domain in the classifying space of some mixed Hodge structures. Thus $[\Phi_{/N}]$ is again a period map.

Let us summarize the notations involving this operation of taking quotient in the following diagram:

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[r, "p_N"] \arrow[d, "u"'] & \mathcal{D}/N \arrow[d, "u_{/N}"] \\
S \arrow[r, "{[\Phi]}"] \arrow[rr, bend right, "{[\Phi_{/N}]}"'] & \Gamma_P\backslash\mathcal{D} \arrow[r, "{[p_N]}"] & \Gamma_{P/N}\backslash(\mathcal{D}/N)
\end{tikzcd}
\tag{6.3}
$$

6.3. **Bi-algebraic system.** Recall that $\mathcal{M}$ is a semi-algebraic open subset in some algebraic variety $\mathcal{M}^\vee$ over $\mathbb{C}$. So $\mathcal{D}$ is a semi-algebraic open subset in some algebraic variety $\mathcal{D}^\vee$ over $\mathbb{C}$.

Definition 6.1. (i) A subset of $\mathcal{D}$ is said to be *irreducible algebraic* if it is a complex analytic irreducible component of $U \cap \mathcal{D}$, with U an algebraic subvariety of $\mathcal{D}^\vee$.
(ii) An irreducible algebraic subset W of $\mathcal{D}$ is said to be *bi-algebraic* if $[\Phi]^{-1}(u(W))$ is algebraic.

By [BBKT20, Cor. 6.7], every weak Mumford–Tate domain is bi-algebraic.

6.4. **Logarithmic Ax.** In this subsection we prove a particular case of Theorem 1.1.

Theorem 6.2. *Let $\widetilde{S}$ be as above There is a smallest weak Mumford–Tate domain in $\mathcal{D}$, denoted by $\widetilde{S}^{\mathrm{ws}}$, which contains $\widetilde{S}$. Moreover, let $\mathcal{Z} \subseteq \Delta$ be a complex analytic irreducible set. Then*
(i) $\mathcal{Z}^{\mathrm{Zar}} \subseteq S \times \widetilde{S}^{\mathrm{ws}}$.
(ii) *Theorem 1.1 holds if $u_S(\mathcal{Z}) = S$.*

In the proof, we will see that $\widetilde{S}^{\mathrm{ws}}$ is an $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})$-orbit, where N is the connected algebraic monodromy group of $(\mathbb{V}, W_\bullet, \mathcal{F}^\bullet) \to S$.

Proof. Let N be the connected algebraic monodromy group of $(\mathbb{V}, W_\bullet, \mathcal{F}^\bullet) \to S$. Then $N \lhd P$ by Theorem 3.4. Thus $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ is a weak Mumford–Tate domain, for any $\widetilde{s} \in \widetilde{S}$.

As $N \lhd P$, we have the quotient period map $[\Phi_{/N}]\colon S \to \Gamma_{S/N}\backslash(\mathcal{D}/N)$. constructed in (6.2). Note that $[\Phi_{/N}]$ gives rise to a new VMHS over S, whose the connected algebraic monodromy group is trivial. So $[\Phi_{/N}](S)$ is a point by [BZ98, Thm. 7.12]. Thus using the notations in (6.3), we have that $p_N(\widetilde{S})$ is a point. So $\widetilde{S} \subseteq N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ for any $\widetilde{s} \in \widetilde{S}$.

In particular $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ is independent of the choice of $\widetilde{s} \in \widetilde{S}$.

On the other hand, the group $P(\mathbb{R})^+W_{-1}(\mathbb{C})$ acts on $S \times \mathcal{D}$ via its action on the second factor. Let $\rho\colon \pi_1(S, s) \to \mathrm{GL}(V)$ be the monodromy representation. Then $\mathrm{Im}(\rho)$ is a subgroup of Γ. By construction of $\widetilde{S}$, we have $\mathrm{Im}(\rho)(s, \widetilde{s}) \subseteq \mathcal{Z}$ for any $(s, \widetilde{s}) \in \mathcal{Z}$. Taking Zariski closures of both sides and recalling that $N = (\mathrm{Im}(\rho)^{\mathrm{Zar}})^\circ$, we have $\{s\} \times N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s} \subseteq \mathcal{Z}^{\mathrm{Zar}}$.

Let us start by proving part (ii). In the course of this proof, we will also show the existence of $\widetilde{S}^{\mathrm{ws}}$.

Assume $u_S(\mathcal{Z}) = S$. Then for each $s \in S$, there exists $\widetilde{s} \in \widetilde{S}$ such that $(s, \widetilde{s}) \in \mathcal{Z}$. Thus by the discussion above, we have $\{s\} \times N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s} \subseteq \mathcal{Z}^{\mathrm{Zar}}$. As this holds true for each $s \in S$, we then have $S \times N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s} \subseteq \mathcal{Z}^{\mathrm{Zar}}$.

To sum it up, we have $\mathcal{Z} \subseteq S \times \widetilde{S} \subseteq S \times N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s} \subseteq \mathcal{Z}^{\mathrm{Zar}}$. By taking Zariski closures, we have $\mathcal{Z}^{\mathrm{Zar}} = S \times N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ and $\widetilde{S}^{\mathrm{Zar}} = N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$.

By definition, $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ is a weak Mumford–Tate domain. Moreover if $\mathcal{W}$ is a weak Mumford–Tate domain which contains $\widetilde{S}$, then $\mathcal{W}$ contains $\widetilde{S}^{\mathrm{Zar}} = N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ because $\mathcal{W}$ is algebraic. So $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$ is the smallest weak Mumford–Tate domain which contains $\widetilde{S}$. Thus $\widetilde{S}^{\mathrm{ws}}$ exists and is precisely $N(\mathbb{R})^+(W_{-1} \cap N)(\mathbb{C})\widetilde{s}$. Now part (ii) is established.

Let us prove part (i) now. Apply part (ii) to $\mathcal{Z} = \Delta$. Then we have $\Delta^{\mathrm{Zar}} = S \times \widetilde{S}^{\mathrm{ws}}$. Thus (i) holds for an arbitrary $\mathcal{Z} \subseteq \Delta$. □

Remark 6.3. If we assume $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$, then $\widetilde{S}^{\mathrm{ws}}$ is the smallest weak Mumford–Tate domain which contains $p_{\mathcal{D}}(\mathcal{Z})$. Indeed, we have $p_{\mathcal{D}}(\mathcal{Z}) \subseteq \widetilde{S}^{\mathrm{ws}}$ by Theorem 6.2.(i). So it suffices to prove the following statement: for any W a weak Mumford–Tate domain in $\mathcal{D}$ which contains $p_{\mathcal{D}}(\mathcal{Z})$, we have $\widetilde{S}^{\mathrm{ws}} \subseteq W$. This is true: $u(W) \supseteq u(p_{\mathcal{D}}(\mathcal{Z})) = [\Phi](u_S(\mathcal{Z}))$, so $[\Phi]^{-1}(u(W)) \supseteq u_S(\mathcal{Z})$, so $[\Phi]^{-1}(u(W)) \supseteq S$ because $[\Phi]^{-1}(u(W))$ is algebraic (by [BBKT20, Cor. 6.7]) and $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$. Therefore $\widetilde{S}^{\mathrm{ws}} \subseteq W$ and hence we are done.

7. Fibered structure and real points

Let $\mathcal{D}$ be a Mumford–Tate domain in some classifying space $\mathcal{M}$ with $P = \mathrm{MT}(\mathcal{D})$. Let the connected mixed Hodge datum $(P, \mathcal{X})$ and the $P(\mathbb{R})^+W_{-1}(\mathbb{C})^+$-equivariant map $\varphi\colon \mathcal{X} \to \mathcal{D}$ be as in Proposition 4.4.(i). In particular by Lemma 4.2, the fiber $\varphi^{-1}(x)$ is a principal homogeneous space under $\exp(F^0_x(\mathrm{Lie}\, W_{-1})_{\mathbb{C}})$ for each $x \in \mathcal{D}$.

7.1. **Fibered structure of Mumford–Tate domains.** Let $0 = W_{-(m+1)} \subseteq W_{-m} \subseteq \cdots \subseteq W_{-1}$ be the sequence of unipotent normal subgroups of P defined in (B.1).

First for each $k \in \{0, \ldots, m\}$, let $\mathcal{X}_k = \mathcal{X}/W_{-(k+1)}$ and let

$$p_k\colon \mathcal{D} \to \mathcal{D}/W_{-(k+1)} =: \mathcal{D}_k \tag{7.1}$$

be the quotient constructed in Proposition 5.1. Notice that $\mathcal{X}_m = \mathcal{X}$ and p_m is the identity on $\mathcal{D}$.

Observe that we have $(P/W_{-k}, \mathcal{X}_k) = (P/W_{-(k+1)}, \mathcal{X}_{k+1})/(W_{-(k+1)}/W_{-(k+2)})$ and $\mathcal{D}_k = \mathcal{D}_{k+1}/(W_{-(k+1)}/W_{-(k+2)})$. Denote by $q_{k+1,k}\colon (P/W_{-(k+1)}, \mathcal{X}_{k+1}) \to (P/W_{-k}, \mathcal{X}_k)$ and $p_{k+1,k}\colon \mathcal{D}_{k+1} \to \mathcal{D}_k$ the quotients. Then by Proposition 5.1 we have the following commutative diagram

$$\begin{tikzcd} \mathcal{X} = \mathcal{X}_m \arrow[r, "q_{m,m-1}"] \arrow[d, "\varphi_m := \varphi"'] & \mathcal{X}_{m-1} \arrow[r, "q_{m-1,m-2}"] \arrow[d, "\varphi_{m-1}"'] & \mathcal{X}_{m-2} \arrow[r, "q_{m-2,m-3}"] \arrow[d, "\varphi_{m-2}"'] & \cdots \arrow[r, "q_{2,1}"] & \mathcal{X}_1 \arrow[r, "q_{1,0}"] \arrow[d, "\varphi_1"] & \mathcal{X}_0 \arrow[d, "\varphi_0"] \\ \mathcal{D} = \mathcal{D}_m \arrow[r, "p_{m,m-1}"] & \mathcal{D}_{m-1} \arrow[r, "p_{m-1,m-2}"] & \mathcal{D}_{m-2} \arrow[r, "p_{m-2,m-3}"] & \cdots \arrow[r, "p_{2,1}"] & \mathcal{D}_1 \arrow[r, "p_{1,0}"] & \mathcal{D}_0 \end{tikzcd} . \tag{7.2}$$

By Lemma 4.2, φ_0 is bijective. But the other φ_i's are not injective in general.

Let $k \in \{0, \ldots, m-1\}$. Recall that $W_{-(k+1)}/W_{-(k+2)} = \mathrm{Lie}\, W_{-(k+1)}/W_{-(k+2)}$ is a vector group. Thus for any $x_k \in \mathcal{D}_k$, the notation $F^0_{x_k}(W_{-(k+1)}/W_{-(k+2)})_{\mathbb{C}}$ makes sense.

Lemma 7.1. *For each $k \in \{0, \ldots, m\}$ and any point $x_k \in \mathcal{D}_k$, we have that*

(i) *the fiber $\varphi_k^{-1}(x_k)$ is a principal homogeneous space under $F^0_{x_k}(W_{-(k+1)}/W_{-(k+2)})_{\mathbb{C}}$.*

(ii) *(for $k \leq m-1$) the fiber $p^{-1}_{k+1,k}(x_k)$ is a principal homogeneous space under*

$$(W_{-(k+1)}/W_{-(k+2)})(\mathbb{C})/F^0_{x_k}(W_{-(k+1)}/W_{-(k+2)})_{\mathbb{C}}.$$

Proof. Part (i) follows directly from Lemma 4.2.

For (ii): By [Pin89, 1.8(a)], each fiber of $q_{k+1,k}$ is a principal homogeneous space under $(W_{-(k+1)}/W_{-(k+2)})(\mathbb{C})$. Combined with part (i) we can conclude. □

7.2. **Real points.** Define $\mathcal{D}_{\mathbb{R}}$ to be the set of $x \in \mathcal{D}$ such that the mixed Hodge structure parametrized by x is split over $\mathbb{R}$. Namely, $\mathcal{D}_{\mathbb{R}} = \varphi(\mathcal{X}_{\mathbb{R}})$ with $\mathcal{X}_{\mathbb{R}} = \{h\colon \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}} : h \text{ is defined over } \mathbb{R}\} \subseteq \mathcal{X}$.

It is known that $\mathcal{D}_{\mathbb{R}} = P(\mathbb{R})^+x$ for some $x \in \mathcal{D}$; see [Pea00, last Remark of §3].

Moreover for any $x \in \mathcal{D}_{\mathbb{R}}$, it is not hard to check that $F^0_x(\mathrm{Lie}\, W_{-1})_{\mathbb{C}} \cap \mathrm{Lie}\, P_{\mathbb{R}} = \{0\}$. So by Lemma 4.2, $p_0\colon P \to G = P/W_{-1}$ induces

$$\mathrm{Stab}_{P(\mathbb{R})^+}(x) \simeq \mathrm{Stab}_{G(\mathbb{R})^+}(\pi(x)). \tag{7.3}$$

Consider the real semi-algebraic $P(\mathbb{R})^+$-equivariant retraction induced by the $\mathfrak{sl}_2$-splitting [BP13, Thm. 2.18] (see also [BBKT20, Cor. 3.12])

$$r\colon \mathcal{D} \to \mathcal{D}_{\mathbb{R}}. \tag{7.4}$$

For each $k \in \{0, \dots, m-1\}$, $\mathcal{D}_k$ is a Mumford–Tate domain and hence we can define $\mathcal{D}_{k,\mathbb{R}}$ as above. Then $\mathcal{D}_{k,\mathbb{R}}$ is a $(P/W_{-(k+1)})(\mathbb{R})^+$-orbit, and there is a real semi-algebraic $(P/W_{-(k+1)})(\mathbb{R})^+$-equivariant retraction $r_k\colon \mathcal{D}_k \to \mathcal{D}_{k,\mathbb{R}}$ induced by the $\mathfrak{sl}_2$-splitting.

Let $p_k\colon \mathcal{D} \to \mathcal{D}_k$ be from (7.1). The following diagram is commutative by [BBKT20, Lem. 6.6]:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "p_k"] \arrow[d, "r"'] & \mathcal{D}_k \arrow[d, "r_k"] \\ \mathcal{D}_{\mathbb{R}} \arrow[r, "p_k|_{\mathcal{D}_{\mathbb{R}}}"] & \mathcal{D}_{k,\mathbb{R}}. \end{tikzcd} \tag{7.5}$$

We close this subsection with the following proposition, which states that $\mathcal{D}_{\mathbb{R}}$ can be split (non-canonically) into the product of a Mumford–Tate domain for pure Hodge structures and some vector spaces.

Proposition 7.2. *There exists a real algebraic isomorphism*

$$\mathcal{D}_{\mathbb{R}} \simeq \mathcal{D}_0 \times (W_{-1}/W_{-2})(\mathbb{R}) \times \cdots \times (W_{-(m-1)}/W_{-m})(\mathbb{R}) \times W_{-m}(\mathbb{R}) \tag{7.6}$$

with the following properties.

(i) *For any $g = (g_0, w_1, \dots, w_m) \in P(\mathbb{R})^+$ under the identification (B.6) and any $x = (x_0, x_1, \dots, x_m) \in \mathcal{D}_{\mathbb{R}}$ under (7.6), the action of $P(\mathbb{R})^+$ on $\mathcal{D}_{\mathbb{R}}$ is given by the formula*

$$gx = (g_0x_0, w_1+g_0x_1, w_2+g_0x_2+\mathrm{calb}_2(w_1, g_0x_1), \dots, w_m+g_0x_m+\mathrm{calb}_m(\mathbf{w}_{m-1}, g_0\mathbf{x}_{m-1})) \tag{7.7}$$

where $\mathbf{w}_k = (w_1, \dots, w_k)$ and $\mathbf{x}_k = (x_1, \dots, x_k)$ for all $k \geq 1$, and $\mathrm{calb}_2, \dots, \mathrm{calb}_m$ are the $\mathbb{Q}$-polynomials of degree at most $k-1$ given by Lemma B.3.

(ii) *The decomposition (7.6) is compatible with taking quotients of $W_{-(k+1)}$ on both sides for each $k \in \{0, \dots, m-1\}$, i.e., the following diagram commutes*

$$\begin{tikzcd} \mathcal{D}_{\mathbb{R}} \arrow[r, "\sim"] \arrow[d, "p_k|_{\mathcal{D}_{\mathbb{R}}}"'] & \mathcal{D}_0 \times (W_{-1}/W_{-2})(\mathbb{R}) \times \cdots \times (W_{-(m-1)}/W_{-m})(\mathbb{R}) \times W_{-m}(\mathbb{R}) \arrow[d] \\ \mathcal{D}_{k,\mathbb{R}} \arrow[r, "\sim"] & \mathcal{D}_0 \times (W_{-1}/W_{-2})(\mathbb{R}) \times \cdots \times (W_{-k}/W_{-(k+1)})(\mathbb{R}) \end{tikzcd}$$

where the top arrow is (7.6), the bottom arrow is (7.6) applied to $\mathcal{D}_{k,\mathbb{R}}$, and the right arrow is omitting the last $m-k$ factors.

Proof. First note that $\mathcal{D}_{0,\mathbb{R}} = \mathcal{D}_0$ because every pure Hodge structure is split over $\mathbb{R}$. Now (B.6) and (7.3) together induce a real algebraic isomorphism as in (7.6). Part (ii) is clear. Part (i) follows from the group law given by (B.7). ☐

7.3. **Fundamental set.**

Theorem 7.3. *Let $r\colon \mathcal{D} \to \mathcal{D}_{\mathbb{R}}$ be the retraction defined in (7.4).*

There exists an $\mathbb{R}_{\mathrm{alg}}$-definable subset $\mathfrak{F}_{\mathbb{R}}$ of $\mathcal{D}_{\mathbb{R}} \simeq \mathcal{D}_0 \times \prod_{1 \leq k \leq m} (W_{-k}/W_{-k-1})(\mathbb{R})$ of the following form

$$\mathfrak{F}_{\mathbb{R}} = \mathfrak{F}_0 \times \prod_{1 \leq k \leq m} (-M, M)^{\dim (W_{-k}/W_{-(k+1)})(\mathbb{R})}, \tag{7.8}$$

for some real number $M > 0$, such that the followings hold. We have

(i) $u|_{r^{-1}(\mathfrak{F}_{\mathbb{R}})}$ *is surjective;*
(ii) $[\Phi]$ *is* $\mathbb{R}_{\mathrm{an,exp}}$*-definable for the* $\mathbb{R}_{\mathrm{alg}}$*-structure on* $\Gamma_P\backslash\mathcal{D}$ *defined by* $r^{-1}(\mathfrak{F}_{\mathbb{R}})$*.*

Proof. [BBKT20, Prop. 3.13 and Thm. 4.4]. In the pure case this is the main result of [BKT20]. ☐

8. Dévissage and Preparation

In this section, we do some preparations. Recall the setup (6.1)

$$S \times \mathcal{D} \supseteq \quad \begin{tikzcd} \Delta \arrow[r, "p_{\mathcal{D}}|_{\Delta}"] \arrow[d, "u_S"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D} \arrow[d, "u"] \\ S \arrow[r, "{[\Phi]}"] & \Gamma_P\backslash\mathcal{D} \end{tikzcd} .$$

Lemma 8.1. *If Theorem 1.1 holds true under the following two additional assumptions:*
(i) $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$*.*
(ii) $\mathcal{Z}$ *is a complex analytic irreducible component of* $\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$*.*

then it holds true in full generality.

Proof. Let $\mathcal{Z}$ be as in Theorem 1.1. Notice that $\mathcal{Z}^{\mathrm{Zar}} \subseteq u_S(\mathcal{Z})^{\mathrm{Zar}} \times \mathcal{D}$. The assumptions and the conclusion of Theorem 1.1 do not change if we replace S by $u_S(\mathcal{Z})^{\mathrm{Zar}}$. So we may assume $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$.

Let $\mathcal{Z}'$ be a complex analytic irreducible component of $\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$ which contains $\mathcal{Z}$. Note that $\mathcal{Z} \subseteq \mathcal{Z}' \subseteq \mathcal{Z}^{\mathrm{Zar}}$. Thus by taking the Zariski closures, we obtain $\mathcal{Z}'^{\mathrm{Zar}} = \mathcal{Z}^{\mathrm{Zar}}$.

Thus $p_{\mathcal{D}}(\mathcal{Z}'^{\mathrm{Zar}}) = p_{\mathcal{D}}(\mathcal{Z}^{\mathrm{Zar}})$, for the projection $p_{\mathcal{D}} \colon S \times \mathcal{D} \to \mathcal{D}$. So for the algebraic structure on $\mathcal{D}$ defined by Definition 6.1, we have $p_{\mathcal{D}}(\mathcal{Z}')^{\mathrm{Zar}} = p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{Zar}}$ because the projection $p_{\mathcal{D}}$ is algebraic. But each weak Mumford–Tate domain is algebraic. So

$$p_{\mathcal{D}}(\mathcal{Z}') \subseteq p_{\mathcal{D}}(\mathcal{Z}')^{\mathrm{Zar}} = p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{Zar}} \subseteq p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{ws}} = \widetilde{S}^{\mathrm{ws}},$$

where the last equality follows from Remark 6.3. But $p_{\mathcal{D}}(\mathcal{Z}) \subseteq p_{\mathcal{D}}(\mathcal{Z}')$ because $\mathcal{Z} \subseteq \mathcal{Z}'$. So every weak Mumford–Tate domain containing $p_{\mathcal{D}}(\mathcal{Z}')$ must also contain $p_{\mathcal{D}}(\mathcal{Z})$, and thus contains $\widetilde{S}^{\mathrm{ws}}$ by Remark 6.3. Combined with the inclusion above, we get that $\widetilde{S}^{\mathrm{ws}}$ is also the smallest weak Mumford–Tate domain which contains $p_{\mathcal{D}}(\mathcal{Z}')$. So

$$\dim \mathcal{Z}'^{\mathrm{Zar}} - \dim \mathcal{Z}' \geq \dim p_{\mathcal{D}}(\mathcal{Z}')^{\mathrm{ws}} \Longrightarrow \dim \mathcal{Z}^{\mathrm{Zar}} - \dim \mathcal{Z} \geq \dim p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{ws}}$$

as $\dim \mathcal{Z} \leq \dim \mathcal{Z}'$ and $p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{ws}} = p_{\mathcal{D}}(\mathcal{Z}')^{\mathrm{ws}} = \widetilde{S}^{\mathrm{ws}}$. Replacing $\mathcal{Z}$ by $\mathcal{Z}'$, it is thus enough to prove Theorem 1.1 assuming furthermore (ii). ☐

Thus our main theorem is reduced to the following theorem, which we will prove in the rest of the paper.

Theorem 8.2. *Theorem 1.1 holds true under the additionnal assumption that* $\mathcal{Z}$ *is a complex analytic irreducible component of* $\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$ *and* $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$*.*

The rest of the paper is devoted to prove Theorem 8.2.

9. Bigness of the $\mathbb{Q}$-stabilizer

Recall our setup

$$S \times \mathcal{D} \supseteq \quad \begin{tikzcd} \Delta \arrow[r, "p_{\mathcal{D}}|_{\Delta}"] \arrow[d, "u_S"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D} \arrow[d, "u"] \\ S \arrow[r, "{[\Phi]}"] & \Gamma_P\backslash\mathcal{D} \end{tikzcd} . \tag{9.1}$$

We consider a subset $\mathcal{Z}$ of Δ satisfying the following properties: (i) $\mathcal{Z}$ is a complex analytic irreducible component of $\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$; (ii) $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$.

Let $H_{\mathcal{Z}^{\mathrm{Zar}}}$ be the $\mathbb{Q}$-stabilizer of $\mathcal{Z}^{\mathrm{Zar}}$, namely

$$H_{\mathcal{Z}^{\mathrm{Zar}}} = \left(\mathrm{Stab}_{P(\mathbb{R})}(\mathcal{Z}^{\mathrm{Zar}}) \cap \Gamma_P\right)^{\mathrm{Zar},\circ} = \left(\{\gamma \in \Gamma_P : \gamma \mathcal{Z}^{\mathrm{Zar}} = \mathcal{Z}^{\mathrm{Zar}}\}^{\mathrm{Zar}}\right)^{\circ}. \tag{9.2}$$

In this section we prove the following case of Theorem 8.2:

Proposition 9.1. *Theorem 8.2 holds true under the additional assumption $H_{\mathcal{Z}^{\mathrm{Zar}}}$ is the trivial group.*

9.1. **Auxiliary set.** The following set is important for the proof of Ax-Schanuel.

$$\Theta = \{g \in P(\mathbb{R}) : \dim(g^{-1}\mathcal{Z}^{\mathrm{Zar}} \cap (S \times \mathfrak{F})) = \dim \mathcal{Z}\}, \tag{9.3}$$

with $\mathfrak{F} = r^{-1}(\mathfrak{F}_{\mathbb{R}})$, where $\mathfrak{F}_{\mathbb{R}}$ is given by Theorem 7.3, or more precisely by (7.8).

It is clear that Θ is definable in $\mathbb{R}_{\mathrm{an,exp}}$, and

$$\{\gamma \in \Gamma_P : \gamma(S \times \mathfrak{F}) \cap \mathcal{Z} \neq \emptyset\} \subseteq \Theta.$$

Denote for simplicity by $\widetilde{Z} = p_{\mathcal{D}}(\mathcal{Z})$, then

$$p_{\mathcal{D}}\left(\gamma(S \times \mathfrak{F}) \cap \mathcal{Z}\right) = p_{\mathcal{D}}(p_{\mathcal{D}}^{-1}(\gamma\mathfrak{F}) \cap \mathcal{Z}) = \gamma\mathfrak{F} \cap \widetilde{Z}.$$

Thus for any $\gamma \in \Gamma_P$, we have

$$\gamma(S \times \mathfrak{F}) \cap \mathcal{Z} \neq \emptyset \Leftrightarrow \gamma\mathfrak{F} \cap \widetilde{Z} \neq \emptyset.$$

Therefore

$$\{\gamma \in \Gamma_P : \gamma\mathfrak{F} \cap \widetilde{Z} \neq \emptyset\} \subseteq \Theta. \tag{9.4}$$

Theorem 9.2. *Assume $\dim \widetilde{Z} > 0$. Then there exist constants $\epsilon > 0$, $c_\epsilon > 0$ and a sequence of real numbers $\{T_i\}_{i\in\mathbb{N}}$ with $T_i \to \infty$ such that*

$$\#\{\gamma \in \Theta \cap \Gamma_P : H(\gamma) \leq T_i\} \geq c_\epsilon T_i^\epsilon. \tag{9.5}$$

9.2. **Proof of Proposition 9.1 assuming Theorem 9.2.** If $\dim \widetilde{Z} = 0$, then $\dim \widetilde{Z}^{\mathrm{ws}} = 0$ and hence Theorem 8.2 clearly holds true. So we assume $\dim \widetilde{Z} > 0$.

We prove Proposition 9.1 by (downward) induction on $\dim \mathcal{Z}^{\mathrm{Zar}}$. The starting point for this induction is when $\mathcal{Z}^{\mathrm{Zar}} = S \times \widetilde{S}^{\mathrm{ws}}$ (see Theorem 6.2). In this case $\mathcal{Z} = S \times_{\Gamma_P \backslash \mathcal{D}} \widetilde{S}^{\mathrm{ws}}$, and so $\dim \mathcal{Z} = \dim S$. Thus Theorem 8.2 holds true in this case.

Let $c_\epsilon > 0$, $\epsilon > 0$ and $\{T_i\}$ be as in Theorem 9.2. Then by the Pila–Wilkie counting theorem [Pil11, 3.6], for each T_i there exists a connected semi-algebraic curve $C_i \subseteq \Theta$ which contains $\geq c_\epsilon T_i^\epsilon$ points in Γ_P of height at most T_i. For $T_i \gg 0$ we have $c_\epsilon T_i^\epsilon \geq 2$. For each $c \in C_i \cap \Gamma_P$, we have $c^{-1}(\mathcal{Z}^{\mathrm{Zar}} \cap \Delta) = c^{-1}\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$ since $\Gamma_P \Delta = \Delta$. So $c^{-1}\mathcal{Z}$ is a complex analytic irreducible component of $c^{-1}\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$.

We have the following alternative:

(i) $c^{-1}\mathcal{Z}^{\mathrm{Zar}}$ is independent of $c \in C_i \cap \Gamma$;
(ii) $c^{-1}\mathcal{Z}^{\mathrm{Zar}}$ is not independent of $c \in C_i \cap \Gamma$.

Assume we are in case (ii). Consider $\mathcal{Z}'$ the irreducible component of $(C_i^{-1}\mathcal{Z}^{\mathrm{Zar}})^{\mathrm{Zar}} \cap \Delta$ which contains $c\mathcal{Z}$. We then have $\dim \mathcal{Z}'^{\mathrm{Zar}} = \dim \mathcal{Z}^{\mathrm{Zar}} + 1$ by the assumption (ii). Take $c, c' \in C_i \cap \Gamma_P$ such that $c^{-1}\mathcal{Z}^{\mathrm{Zar}} \neq c'^{-1}\mathcal{Z}^{\mathrm{Zar}}$. Thus $\dim \mathcal{Z}' = \dim \mathcal{Z} + 1$. Moreover $\dim p_{\mathcal{D}}(\mathcal{Z}') = \dim p_{\mathcal{D}}(\mathcal{Z}) + 1$ as $P(\mathbb{R})$ acts on $S \times \mathcal{D}$ on the second factor.

Applying the induction hypothesis, we then have

$$\dim \mathcal{Z}'^{\mathrm{Zar}} - \dim \mathcal{Z}' \geq \dim p_{\mathcal{D}}(\mathcal{Z}')^{\mathrm{ws}}.$$

But the left hand side equals $\dim \mathcal{Z}^{\mathrm{Zar}} - \dim \mathcal{Z}$ and the right hand side is at least $\dim p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{ws}}$. Hence Theorem 8.2 holds true for $\mathcal{Z}$.

Assume we are in case (i). Fix $c_0 \in C_i \cap \Gamma_P$. Then $c^{-1}\mathcal{Z}^{\mathrm{Zar}} = c_0^{-1}\mathcal{Z}^{\mathrm{Zar}}$ for all $c \in C_i \cap \Gamma_P$. Hence $cc_0^{-1} \in \mathrm{Stab}_{P(\mathbb{R})}(\mathcal{Z}^{\mathrm{Zar}})$. This shows $\#(\mathrm{Stab}_{P(\mathbb{R})}(\mathcal{Z}^{\mathrm{Zar}}) \cap \Gamma_P) \geq C_i \cap \Gamma_P \geq c_\epsilon T_i^\epsilon$ for each i. Letting $T_i \to \infty$, we get $\#(\mathrm{Stab}_{P(\mathbb{R})}(\mathcal{Z}^{\mathrm{Zar}}) \cap \Gamma) = \infty$. Hence $\dim H_{\mathcal{Z}^{\mathrm{Zar}}} > 0$. This contradicts the triviality of $H_{\mathcal{Z}^{\mathrm{Zar}}}$.

9.3. **Preparation of the proof of Theorem 9.2.** We will prove Theorem 9.2, or more precisely (9.5), in the rest of this section. The proof is long. It will be divided in several steps for readers' convenience. In this subsection, we fix some notations and sketch the outline of the proof.

The proof of (9.5) uses the fibered structure of $\mathcal{D}$ and the discussion on its real points, both explained in Section 7. We start by recollecting basic knowledge on both aspects.

Recall the sequence of normal subgroups $0 = W_{-(m+1)} \subseteq W_{-m} \subseteq \cdots W_{-1} = \mathcal{R}_u(P)$ of P from (B.1), and the quotient Mumford–Tate domains $p_k \colon \mathcal{D} \to \mathcal{D}_k := \mathcal{D}/W_{-k-1}$, for each $k \in \{0, \ldots, m\}$, from (7.1). Notice that p_m is the identity map on $\mathcal{D}$.

Let $r \colon \mathcal{D} \to \mathcal{D}_{\mathbb{R}}$ be the $P(\mathbb{R})^+$-equivariant retraction of the inclusion $\mathcal{D}_{\mathbb{R}} \subseteq \mathcal{D}$ from (7.4). Applying (7.5) successively to $p_{k+1,k} \colon \mathcal{D}_{k+1} \to \mathcal{D}_k$ (defined in the diagram (7.2)), we obtain the following commutative diagram

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "p_{m,m-1}"] \arrow[d, "r"'] & \mathcal{D}_{m-1} \arrow[r, "p_{m-1,m-2}"] \arrow[d, "r_{m-1}"'] & \mathcal{D}_{m-2} \arrow[r, "p_{m-2,m-3}"] \arrow[d, "r_{m-2}"'] & \cdots \arrow[r, "p_{2,1}"] & \mathcal{D}_1 \arrow[r, "p_{1,0}"] \arrow[d, "r_1"'] & \mathcal{D}_0 \arrow[d, "r_0"'] \\
\mathcal{D}_{\mathbb{R}} \arrow[r] & \mathcal{D}_{m-1,\mathbb{R}} \arrow[r] & \mathcal{D}_{m-2,\mathbb{R}} \arrow[r] & \cdots \arrow[r] & \mathcal{D}_{1,\mathbb{R}} \arrow[r] & \mathcal{D}_{0,\mathbb{R}}
\end{tikzcd}
\tag{9.6}
$$

with each r_k a $(P/W_{-k-1})(\mathbb{R})^+$-equivariant retraction of $\mathcal{D}_{k,\mathbb{R}} \subseteq \mathcal{D}_k$. Recall that $\mathcal{D}_0$ is a Mumford–Tate domain in a classifying space of pure Hodge structures, and r_0 is the identity map. There is a metric on $\mathcal{D}_0$; see [BT19, §2].

In the proof, we often need to project subsets of $\mathcal{D}$ to different levels and consider the real points. So it is convenient to fix the following notations.

Notation 9.3. *For each $k \in \{0, 1, \cdots, m\}$,*

- *For any subset $A \subseteq \mathcal{D}$, denote by $A_k := p_k(A) \subseteq \mathcal{D}_k$. As convention $A_m = A$.*
- *For any subset $A \subseteq \mathcal{D}$, denote by $A_{\mathbb{R}} := r(A) \subseteq \mathcal{D}_{\mathbb{R}}$, and $A_{k,\mathbb{R}} = r_k(A_k) \subseteq \mathcal{D}_{k,\mathbb{R}}$.*

Let $\mathfrak{F} = r^{-1}(\mathfrak{F}_{\mathbb{R}})$ where $\mathfrak{F}_{\mathbb{R}} \subseteq \mathcal{D}_{\mathbb{R}}$ is given by Theorem 7.3, or more precisely by (7.8).

Before moving on, let us sketch how (9.5) is proved when $m = 0$, namely when $\mathcal{D} = \mathcal{D}_0$ and $P = P/W_{-1}$ is a reductive group. In this case, $\widetilde{Z} = \widetilde{Z}_0$, which has positive dimension by assumption. For each real number $T > 0$, take $\mathbf{B}_0(T) \subseteq \mathcal{D}_0$ to be the ball centered at a fixed point of radius $\log T$ in $\mathcal{D}_0$. Let $\widetilde{Z}_0(T)$ be a complex analytic irreducible component of $\widetilde{Z} \cap \mathbf{B}_0(T)$. Bakker and Tsimerman [BT19] proved that there exist constants $c_0, \epsilon_0 > 0$, independent of T, such that

$$\#\{\gamma \in \Gamma_P : \gamma\mathfrak{F} \cap \widetilde{Z}_0(T) \neq \emptyset,\ H(\gamma) \leq T\} \geq c_0 T^{\epsilon_0}.$$

By (9.4), the set on the left hand side is a subset of $\#\{\gamma \in \Theta \cap \Gamma_P : H(\gamma) \leq T\}$. This yields (9.5).

For a general m, we need to generalize this idea. A first thing to do is to find an appropriate generalization of $\mathbf{B}_0(T)$ for $\mathcal{D}$. To achieve this, we make use of the retractions r_k's (with $r_m = r$) and the following product structure on $\mathcal{D}_{\mathbb{R}}$ (7.6) (and the truncated version given by Proposition 7.2.(ii) for each $k \in \{0, 1, \cdots, m\}$)

$$\mathcal{D}_{k,\mathbb{R}} \simeq \mathcal{D}_{0,\mathbb{R}} \times (W_{-1}/W_{-2})(\mathbb{R}) \times (W_{-2}/W_{-3})(\mathbb{R}) \times \cdots \times (W_{-k}/W_{-k-1})(\mathbb{R}). \tag{9.7}$$

Now we are ready to give the generalization of the $\mathbf{B}_0(T)$ above. For each $k \in \{0, 1, \cdots, m\}$ and each real number $T > 0$, define the following subset $\mathbf{B}_k(T) \subseteq \mathcal{D}_k$ as follows.

- Let $\mathbf{B}_0(T) = B_0(T) \subseteq \mathcal{D}_0$ be the ball centered at a fixed point of radius $\log T$ in $\mathcal{D}_0$.
- For each $k \geq 1$, let $B_k(T)$ the $|\cdot|$-ball centered at 0 of radius T in $(W_{-k}/W_{-k-1})(\mathbb{R})$. Define $\mathbf{B}_k(T) = r_k^{-1}(\prod_{i=0}^k B_i(T))$. In particular, $p_{k+1,k}(\mathbf{B}_{k+1}(T)) = \mathbf{B}_k(T)$.

Next, we generalize the set $\widetilde{Z}_0(T)$ as follows. For each $k \in \{0, 1, \cdots, m\}$ and each real number $T > 0$:

- Let $\widetilde{Z}_k(T)$ be a complex analytic irreducible component of $\widetilde{Z}_k \cap \mathbf{B}_k(T) \subseteq \mathcal{D}$. Notice that $\widetilde{Z}_k(T) \to \widetilde{Z}_k \cap \mathbf{B}_k(T)$ when $T \to \infty$. Denote also by $\widetilde{Z}(T) = \widetilde{Z}_m(T)$.
- We may choose such $\widetilde{Z}_k(T)$'s that $p_{k+1,k}(\widetilde{Z}_{k+1}(T)) \subseteq \widetilde{Z}_k(T)$ for all k.[2]

9.4. **Sketch of the strategy of the proof of Theorem 9.2.** For simplicity, we use the same notation p_k to denote the projection $P \to P/W_{-k-1}$ and the projection $\mathcal{D} \to \mathcal{D}_k$.

Suppose $\dim \widetilde{Z}_0 = \dim p_0(\widetilde{Z}) > 0$. Then by applying the results of Bakker and Tsimerman as explained above, we find $\#\{\gamma_0 \in p_0(\Gamma_P) : \gamma_0\mathfrak{F}_0 \cap \widetilde{Z}_0(T) \neq \emptyset,\ H(\gamma_0) \leq T\} \geq c_0 T^{\epsilon_0}$. We wish to lift at least polynomially many such γ_0's to elements in $p_1(\Gamma_P)$ of height at most T with the following property: each such lift $\gamma_1 \in p_1(\Gamma_P)$ satisfies $\gamma_1\mathfrak{F}_1 \cap \widetilde{Z}_1(T) \neq \emptyset$, or equivalently $\gamma_1 r_1(\mathfrak{F}_1) \cap r_1(\widetilde{Z}_1(T)) \neq \emptyset$ (since $\mathfrak{F}_1 = r_1^{-1}(\mathfrak{F}_{1,\mathbb{R}})$ by definition of $\mathfrak{F}$). This last condition, expressed with Notation 9.3, becomes $\gamma_1\mathfrak{F}_{1,\mathbb{R}} \cap \widetilde{Z}_1(T)_{\mathbb{R}} \neq \emptyset$. The intersection is taken in $\mathcal{D}_{1,\mathbb{R}}$, which is isomorphic to $\mathcal{D}_0 \times (W_{-1}/W_{-2})(\mathbb{R})$ by (9.7). If the desired lifting can be realized, then we do similar liftings to $p_2(\Gamma_P)$, *etc.*, under we obtain at least polynomially many elements γ in $p_m(\Gamma_P) = \Gamma_P$ of height at most T such that $\gamma\mathfrak{F}_{\mathbb{R}} \cap \widetilde{Z}(T)_{\mathbb{R}} \neq \emptyset$.

At this stage, we can explain why the second bullet point in the constructions of the $\widetilde{Z}_k(T)$'s is needed: in the lifting process, we need that $\widetilde{Z}_{k+1}(T)_{\mathbb{R}}$ is mapped to $\widetilde{Z}_k(T)_{\mathbb{R}}$ under $p_{k+1,k}$.

There is a problem in the procedure described above, namely it is possible that $\widetilde{Z}_0$ is a point. In this case, we need to work with the smallest k_0 such that $\dim \widetilde{Z}_{k_0} > 0$, which serves as the base step of the lifting process. We need to find at least polynomially many elements $\gamma_{k_0} \in p_{k_0}(\Gamma_P)$ of height at most T such that $\gamma_{k_0}\mathfrak{F}_{k_0,\mathbb{R}} \cap \widetilde{Z}_{k_0}(T)_{\mathbb{R}} \neq \emptyset$. Whereas this is guaranteed by the result of Bakker and Tsimerman when $k_0 = 0$, it is not known when $k_0 \geq 1$. We will prove this result in Section 9.5, or more precisely Proposition 9.4.(ii).

Once we have established the base step, we need to realize the lifting. By (9.7), we have $\mathcal{D}_{k+1} \simeq \mathcal{D}_k \times (W_{-k-1}/W_{-k-2})(\mathbb{R})$. To realize the lifting process, we need to compare the growth of $\widetilde{Z}_{k+1}(T)$ in the vertical direction $(W_{-k-1}/W_{-k-2})(\mathbb{R})$ with its growth in the horizontal direction $\mathcal{D}_k$. This lifting process is done in Section 9.6, more precisely Lemma 9.10 (if $\widetilde{Z}_{k+1}(T)$ grows "faster" in the horizontal direction $\mathcal{D}_k$) and Lemma 9.11 (if $\widetilde{Z}_{k+1}(T)$ grows "faster" in the vertical direction $(W_{-k-1}/W_{-k-2})(\mathbb{R})$).

9.5. **Proof of Theorem 9.2: the base step and the statement for the lifting process.** The main goal of this subsection is to prove the base step for the lifting process, namely Proposition 9.4. At the end of this subsection we also state the result for the

[2]Notice that $\widetilde{Z}_k \cap \mathbf{B}_k(T) = p_k(\widetilde{Z}) \cap \mathbf{B}_k(T) = p_k(\widetilde{Z} \cap p_k^{-1}(\mathbf{B}_k(T)))$. Thus $\widetilde{Z}_k(T)$ equals $p_k(\widetilde{Z}(k,T))$ for some complex analytic irreducible component $\widetilde{Z}(k,T)$ of $\widetilde{Z} \cap p_k^{-1}(\mathbf{B}_k(T))$. By definition of $\mathbf{B}_k(T)$, we have $p_{k+1}^{-1}(\mathbf{B}_{k+1}(T)) \subseteq p_k^{-1}(\mathbf{B}_k(T))$ for each k. Thus the $\widetilde{Z}(k,T)$'s can be chosen such that $\widetilde{Z}(k+1,T) \subseteq \widetilde{Z}(k,T)$ for each k. For these choices, we then have $p_{k+1,k}(\widetilde{Z}_{k+1}(T)) \subseteq \widetilde{Z}_k(T)$.

lifting process (Proposition 9.7) and explain how it implies Theorem 9.2. The proof of the lifting process will be executed in the next subsection.

Let $k_0 \in \{0, \cdots, m\}$ be such that $\dim \widetilde{Z}_{k_0} > 0$, smallest for this property.

For simplicity, we introduce the following notation. For each real number $T \geq 0$, let

$$\Xi_{k_0}(T) = \{g \in (W_{-k_0}/W_{-k_0-1})(\mathbb{R}) : g\mathfrak{F}_{k_0} \cap \widetilde{Z}_{k_0}(T) \neq \emptyset\}. \tag{9.8}$$

We also denote by $\Gamma_{-k_0/-k_0-1} = (\Gamma_P \cap W_{-k_0}(\mathbb{Q}))/(\Gamma_P \cap W_{-k_0-1}(\mathbb{Q}))$; it acts on $\mathcal{D}_{k_0} = \mathcal{D}/W_{-k_0-1}$.

Proposition 9.4. (i) *There exist constants $\alpha_{k_0} > 0$ and $\alpha'_{k_0} > 0$ satisfying the following property. Any $\gamma_{-k_0/-k_0-1} \in \Xi_{k_0}(T) \cap \Gamma_{-k_0/-k_0-1}$ satisfies $H(\gamma_{-k_0/-k_0-1}) \leq \alpha_{k_0} T^{\alpha'_{k_0}}$ for all $T \gg 1$.*

(ii) *If $k_0 \geq 1$, then there exist constants $c_{k_0}, \epsilon_{k_0} > 0$ such that*

$$\#\{\gamma_{-k_0/-k_0-1} \in \Xi_{k_0}(T) \cap \Gamma_{-k_0/-k_0-1} : H(\gamma_{-k_0/-k_0-1}) \leq T\} \geq c_{k_0} T^{\epsilon_{k_0}}$$

for all $T \gg 1$.

Proof of Proposition 9.4. As $\widetilde{Z} \subseteq u^{-1}([i](S))$, we have that $\widetilde{Z}$ is Griffiths transverse. Hence $\widetilde{Z}_0 = p_0(\widetilde{Z})$ is Griffiths transverse global analytic.

If $k_0 = 0$, namely $\dim \widetilde{Z}_0 > 0$, then part (i) follows directly from [BT19, Thm. 4.2], which claims: if $\gamma_{0/-1} \in \Gamma_{0/-1}$ satisfies $\gamma_{0/-1}\mathfrak{F}_0 \cap \mathbf{B}_0(T) \neq \emptyset$, then $H(\gamma_{0/-1}) \leq \alpha_0 T^{\alpha'_0}$ for some $\alpha_0 > 0$ and $\alpha'_0 > 0$ when $T \gg 1$.

From now on, assume $k_0 \geq 1$. So $\widetilde{Z}_{k_0-1} = \overline{h}$ is a point in $\mathcal{D}_{k_0-1}$. Thus $\widetilde{Z}_{k_0} \subseteq p^{-1}_{k_0,k_0-1}(\overline{h})$. For $r_{k_0} \colon \mathcal{D}_{k_0} \to \mathcal{D}_{k_0-1}$, notice that $r_{k_0}(p^{-1}_{k_0,k_0-1}(\overline{h}))$ can be identified with $(W_{-k_0}/W_{-k_0-1})(\mathbb{R})$.

Lemma 9.5. *Recall $M > 0$ the real number in the definition of $\mathfrak{F}_{\mathbb{R}}$ from Theorem 7.3. Denote for simplicity $\mathfrak{F}'_{k_0} = (-M, M)^{\dim(W_{-k_0}/W_{-k_0-1})(\mathbb{R})} \subseteq (W_{-k_0}/W_{-k_0-1})(\mathbb{R})$. Then*

$$\{\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1} : \left(\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}\right) \cap \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T) \neq \emptyset\} = \Xi_{k_0}(T) \cap \Gamma_{-k_0/-k_0-1} \tag{9.9}$$

for $T \gg 1$.

Proof of Lemma 9.5. We have $\widetilde{Z}_{k_0,\mathbb{R}} = r_{k_0}(\widetilde{Z}_{k_0}) \subseteq r_{k_0}(p^{-1}_{k_0,k_0-1}(\overline{h})) = (W_{-k_0}/W_{-k_0-1})(\mathbb{R})$. Since $\mathbf{B}_{k_0}(T) = r^{-1}_{k_0}(\prod_{i=0}^{k_0} B_i(T))$, we then have

$$r_{k_0}(\widetilde{Z}_{k_0} \cap \mathbf{B}_{k_0}(T)) = r_{k_0}(\widetilde{Z}_{k_0}) \cap B_{k_0}(T) = \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T). \tag{9.10}$$

For any $\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1}$, we have, by the definition of $\mathfrak{F}_{\mathbb{R}}$ (7.8),

$$\gamma_{-k_0/-k_0-1}\mathfrak{F}_{k_0,\mathbb{R}} \cap (W_{-k_0}/W_{-k_0-1})(\mathbb{R}) = \gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}. \tag{9.11}$$

($\supseteq$) Let $\gamma_{-k_0/-k_0-1} \in \Xi_{k_0}(T) \cap \Gamma_{-k_0/-k_0-1}$, namely $\gamma_{-k_0/-k_0-1}\mathfrak{F}_{k_0} \cap \widetilde{Z}_{k_0}(T) \neq \emptyset$. As $\widetilde{Z}_{k_0}(T) \subseteq \widetilde{Z}_{k_0} \cap \mathbf{B}_{k_0}(T)$, we have $\gamma_{-k_0/-k_0-1}\mathfrak{F}_{k_0} \cap \widetilde{Z}_{k_0} \cap \mathbf{B}_{k_0}(T) \neq \emptyset$.

Apply r_{k_0} to both sides. We then get, by (9.10), $\gamma_{-k_0/-k_0-1}\mathfrak{F}_{k_0,\mathbb{R}} \cap \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T) \neq \emptyset$ because $\mathfrak{F}_{k_0} = r^{-1}_{k_0}(\mathfrak{F}_{k_0,\mathbb{R}})$. Thus $(\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}) \cap \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T) \neq \emptyset$ by (9.11). This proves $\supseteq$.

($\subseteq$) On the other hand note that $\widetilde{Z}_{k_0}(T) \to \widetilde{Z}_{k_0} \cap \mathbf{B}_{k_0}(T)$ when $T \to \infty$. So by (9.10) we have $\widetilde{Z}_{k_0}(T)_{\mathbb{R}} \to \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T)$ when $T \to \infty$.

Let $\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1}$ be such that $(\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}) \cap \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T) \neq \emptyset$. Then by the last paragraph we have $(\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}) \cap \widetilde{Z}_{k_0}(T)_{\mathbb{R}} \neq \emptyset$ when $T \to \infty$. By (9.11) and $\mathfrak{F}_{k_0} = r^{-1}_{k_0}(\mathfrak{F}_{k_0,\mathbb{R}})$, we then have $\gamma_{-k_0/-k_0-1}\mathfrak{F}_{k_0} \cap \widetilde{Z}_{k_0}(T) \neq \emptyset$. Thus $\gamma_{-k_0/-k_0-1} \in \Xi_{k_0}(T) \cap \Gamma_{-k_0/-k_0-1}$. This proves $\subseteq$ for $T \gg 1$. $\square$

Now we are ready to finish the proof of Proposition 9.4.

For (i): Recall the definition $\mathfrak{F}'_{k_0} = (-M, M)^{\dim(W_{-k_0}/W_{-k_0-1})(\mathbb{R})} \subseteq (W_{-k_0}/W_{-k_0-1})(\mathbb{R})$. It is clear that if $\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1}$ lies in the set on the left hand of (9.9), then we have $H(\gamma_{-k_0/-k_0-1}) \leq \frac{1}{M}T$. So by Lemma 9.5, part (i) holds true for $\alpha_{k_0} = \alpha'_{k_0} = 1$.

For (ii): By Lemma 9.5, it suffices to prove: There exist constants $c_{k_0} > 0$ and $\epsilon_{k_0} > 0$ such that

$$\#\{\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1} : (\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}) \cap \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T) \neq \emptyset,\ H(\gamma_{-k_0/-k_0-1}) \leq T\} \geq c_{k_0} T^{\epsilon_{k_0}}. \tag{9.12}$$

Consider $\{\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1} : (\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}) \cap \widetilde{Z}_{k_0,\mathbb{R}} \neq \emptyset\}$. We claim that it is infinite. Indeed, assume otherwise, then $\widetilde{Z}_{k_0,\mathbb{R}}$ is contained in a bounded subset of $(W_{-k_0}/W_{-k_0-1})(\mathbb{R})$. But $p^{-1}_{k_0,k_0-1}(\overline{h}) \simeq (W_{-k_0}/W_{-k_0-1})(\mathbb{C})/F^0_{\overline{h}}(W_{-k_0}/W_{-k_0-1})_{\mathbb{C}}$ by part (ii) of Lemma 7.1, and the composite (φ_{k_0} is the natural projection)

$$\begin{aligned}(W_{-k_0}/W_{-k_0-1})(\mathbb{C}) &\xrightarrow{\varphi_{k_0}} (W_{-k_0}/W_{-k_0-1})(\mathbb{C})/F^0_{\overline{h}}(W_{-k_0}/W_{-k_0-1})_{\mathbb{C}} = p^{-1}_{k_0,k_0-1}(\overline{h}) \\ &\xrightarrow{r_{k_0}} (W_{-k_0}/W_{-k_0-1})(\mathbb{R})\end{aligned}$$

is, up to an automorphism of $(W_{-k_0}/W_{-k_0-1})(\mathbb{R})$ sending bounded sets to bounded sets, the projection to the real part.[3] So $\varphi^{-1}_{k_0}(\widetilde{Z}_{k_0}) \subseteq \varphi^{-1}_{k_0}(r^{-1}_{k_0}(\widetilde{Z}_{k_0,\mathbb{R}}))$ is contained in a set whose real part is bounded. But $\varphi^{-1}_{k_0}(\widetilde{Z}_{k_0})$ is complex analytic, so $\varphi^{-1}_{k_0}(\widetilde{Z}_{k_0})$ is a point, and so is $\widetilde{Z}_{k_0}$. This contradicts $\dim \widetilde{Z}_{k_0} > 0$.

Now that $\mathfrak{F}'_{k_0}$ is a fundamental set for the action of $\Gamma_{-k_0/-k_0-1}$ on the Euclidean space $(W_{-k_0}/W_{-k_0-1})(\mathbb{R})$. So $\#\{\gamma_{-k_0/-k_0-1} \in \Gamma_{-k_0/-k_0-1} : (\gamma_{-k_0/-k_0-1} + \mathfrak{F}'_{k_0}) \cap \widetilde{Z}_{k_0,\mathbb{R}} \cap B_{k_0}(T) \neq \emptyset\} \geq T$ for all $T \gg 0$. This yields (9.12) by part (i). Hence we are done for part (ii). $\square$

We will furthermore fix the following notations, which generalize the notations above. For each $k \in \{k_0, \ldots, m\}$ and each real number $T \geq 0$:

- Let $\Gamma_{-k_0/-k-1} = (\Gamma_P \cap W_{-k_0}(\mathbb{Q}))/(\Gamma_P \cap W_{-k-1}(\mathbb{Q}))$. Then $\Gamma_{-k_0/-k-1}$ acts on $\mathcal{D}_k = \mathcal{D}/W_{-k-1}$.
- Let $\Xi_k(T) = \{g \in (W_{-k_0}/W_{-k-1})(\mathbb{R}) : g\mathfrak{F}_k \cap \widetilde{Z}_k(T) \neq \emptyset\}$.

Before going on, we prove the following lemma on the naive lifting of elements from $\Xi_k(T)$ to $\Xi_{k+1}(T)$. Here we call the lifting *naive* because an element of small height needs not be lifted to an element of small height.

Lemma 9.6. *For $T \gg 1$, the following holds true. For each $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$, there exists $\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T) \cap \Gamma_{-k_0/-k-2}$ such that $p_{k+1,k}(\gamma_{-k_0/-k-2}) = \gamma_{-k_0/-k-1}$ for the natural projection $p_{k+1,k} \colon P/W_{-k-2} \to P/W_{-k-1}$.*

Proof of Lemma 9.6. Let $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$. The definition of $\Xi_k(T)$ yields a point $\widetilde{z}'_k \in \gamma_{-k_0/-k-1}\mathfrak{F}_k \cap \widetilde{Z}_k(T)$. Take $\widetilde{z}'_{k+1} \in \widetilde{Z}_{k+1}(T)$ which maps to $\widetilde{z}'_k$ under the projection $p_{k+1,k} \colon \mathcal{D}_{k+1} \to \mathcal{D}_k$. Such an $\widetilde{z}'_{k+1}$ exists when $T \gg 1$.[4]

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[3]Recall that r_{k_0} is the retraction given by the $\mathfrak{sl}_2$-splitting. If r_{k_0} is replaced by the retraction induced by the Deligne δ-splitting, then this composite is precisely the projection to the real part. But the $\mathfrak{sl}_2$-splitting is defined by universal Lie polynomials in the Hodge components of the Deligne δ-splitting, so this claim holds true.

[4]We have that $\widetilde{Z}_k(T) = p_k(\widetilde{Z}(k,T))$ for some complex analytic irreducible component $\widetilde{Z}(k,T)$ of $\widetilde{Z} \cap p_k^{-1}(\mathbf{B}_k(T))$. In particular, $\widetilde{Z}(k+1,T) \subseteq \widetilde{Z}(k,T)$. Notice that $\widetilde{Z}(k,T) \to \widetilde{Z}$ when $T \to \infty$. So $\widetilde{Z}(k+1,T) \to \widetilde{Z}(k,T)$ when $T \to \infty$. So the desired $\widetilde{z}'_{k+1}$ exists when $T \gg 1$.

Let $\widetilde{z}_k = r_k(\widetilde{z}'_k) \in \gamma_{-k_0/-k-1}\mathfrak{F}_{k,\mathbb{R}} \cap \widetilde{Z}_k(T)_{\mathbb{R}}$. Similarly let $\widetilde{z}_{k+1} = r_{k+1}(\widetilde{z}'_{k+1})$. Then $\widetilde{z}_{k+1} \mapsto \widetilde{z}_k$ under the map $p_{k+1,k}|_{\mathcal{D}_{k+1,\mathbb{R}}} : \mathcal{D}_{k+1,\mathbb{R}} \to \mathcal{D}_{k,\mathbb{R}}$.

Take $\gamma_{-k_0/-k-2} \in \Gamma_{-k_0/-k-2}$ to be such that $\gamma_{-k_0/-k-2} \mapsto \gamma_{-k_0/-k-1}$ under the natural projection $P/W_{-k-2} \to P/W_{-k-1}$. Then $p_{k+1,k}(\gamma^{-1}_{-k_0/-k-2}\widetilde{z}_{k+1}) = \gamma^{-1}_{-k_0/-k-1}\widetilde{z}_k \in \mathfrak{F}_{k,\mathbb{R}}$. Notice that (9.7) induces a real-algebraic isomorphism $\mathcal{D}_{k+1,\mathbb{R}} \simeq \mathcal{D}_{k,\mathbb{R}} \times (W_{-k-1}/W_{-k-2})(\mathbb{R})$, and under this isomorphism we have $\mathfrak{F}_{k+1,\mathbb{R}} \simeq \mathfrak{F}_{k,\mathbb{R}} \times (-M, M)^{\dim(W_{-k-1}/W_{-k-2})(\mathbb{R})}$ by construction (7.8). Thus up to adjusting $\gamma_{-k_0/-k-2}$ by an element in $\Gamma_{-k-1/-k-2}$, we obtain that $\gamma^{-1}_{-k_0/-k-2}\widetilde{z}_{k+1} \in \mathfrak{F}_{k+1,\mathbb{R}}$. Note that we stil have $\gamma_{-k_0/-k-2} \mapsto \gamma_{-k_0/-k-1}$.

Thus $\widetilde{z}_{k+1} \in \gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}} \cap \widetilde{Z}_{k+1}(T)_{\mathbb{R}}$. So $\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T)$. We are done. □

The following proposition, which is the lifting statement, will be proved in the next two subsections.

Proposition 9.7. *There exists an integer $k'_0 \in \{k_0, \cdots, m-1\}$ with the following property. For each $k > k'_0$, there exist constants $c'_k, \epsilon'_k > 0$, and a sequence $\{T_i \in \mathbb{R}\}_{i\in\mathbb{N}}$ with $T_i \to \infty$, such that*

$$\#\{\gamma_{-k_0/-k-1} \in \Xi_k(T_i) \cap \Gamma_{-k_0/-k-1} : H(\gamma_{-k_0/-k-1}) \le T_i\} \ge c'_k T_i^{\epsilon'_k}.$$

Let us finish the proof of Theorem 9.2 assuming this proposition. Apply Proposition 9.7 to $k = m$. As $W_{-(m+1)} = 0$, the conclusion of the proposition becomes: there exist constants $c' = c'_m, \epsilon' = \epsilon'_m > 0$ and a sequence $\{T_i \in \mathbb{R}\}_{i\in\mathbb{N}}$ with $T_i \to \infty$, such that

$$\#\{\gamma \in \Gamma_{-k} : H(\gamma) \le T_i,\ \gamma\mathfrak{F} \cap \widetilde{Z}_m(T_i) \neq \emptyset\} \ge c' T_i^{\epsilon'}.$$

But $\Gamma_{-k_0} \subseteq \Gamma_P$ and $\widetilde{Z}_m(T_i) \subseteq \widetilde{Z}$, and so

$$\#\{\gamma \in \Gamma_P : H(\gamma) \le T_i,\ \gamma\mathfrak{F} \cap \widetilde{Z} \neq \emptyset\} \ge c' T_i^{\epsilon'}.$$

Thus by (9.4), we have

$$\#\{\gamma \in \Theta \cap \Gamma_P : H(\gamma) \le T_i\} \ge c' T_i^{\epsilon'}.$$

So Theorem 9.2 follows from Pila–Wilkie, because Θ is a definable set in $\mathbb{R}_{\mathrm{an,exp}}$.

9.6. **Proof of Proposition 9.7: lifting process.** In this subsection, we will give the lifting process. This lifting will be used in the next subsection to prove the full Proposition 9.7.

Recall the notation $k_0 \in \{0, \cdots, m\}$ be such that $\dim \widetilde{Z}_{k_0} > 0$, smallest for this property.

Let $k \in \{k_0, \ldots, m\}$.

The semi-algebraic isomorphism $\mathcal{D}_{k+1,\mathbb{R}} \simeq \mathcal{D}_0 \times (W_{-1}/W_{-2})(\mathbb{R}) \times \cdots \times (W_{-k-1}/W_{-k-2})(\mathbb{R})$ given by (9.7) induces

$$\mathcal{D}_{k+1,\mathbb{R}} \simeq \mathcal{D}_{k,\mathbb{R}} \times (W_{-k-1}/W_{-k-2})(\mathbb{R}). \tag{9.13}$$

Thus we have a natural projection

$$\lambda_{k+1} : \mathcal{D}_{k+1,\mathbb{R}} \to (W_{-k-1}/W_{-k-2})(\mathbb{R}). \tag{9.14}$$

Consider the isomorphism of $\mathbb{Q}$-varieties induced by (B.6)

$$P/W_{-k-1} \simeq G \times (W_{-1}/W_{-2}) \times \cdots \times (W_{-k}/W_{-k-1}).$$

The above isomorphism induces

$$P/W_{-k-2} \simeq P/W_{-k-1} \times W_{-k-1/-k-2}. \tag{9.15}$$

The group $(P/W_{-k-2})(\mathbb{R})^+$ acts on $\mathcal{D}_{k+1,\mathbb{R}} = (\mathcal{D}/W_{-k-2})_{\mathbb{R}}$.

Lemma 9.8. *Consider the Euclidean norm* $|\cdot|$ *on* $(W_{-k-1}/W_{-k-2})(\mathbb{R})$*. Then for any* $\gamma_{-k_0/-k-2} \in \Gamma_{-k_0/-k-2}$*, the set* $\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$ *is contained in a* $|\cdot|$*-ball of radius* $\ll H(\gamma_{-k_0/-k-1})^{k+1}$ *in* $(W_{-k-1}/W_{-k-2})(\mathbb{R})$*. Here,* $\gamma_{-k_0/-k-1} \in \Gamma_{-k_0/-k-1}$ *is the projection of* $\gamma_{-k_0/-k-2}$ *under the natural projection* $P/W_{-k-2} \to P/W_{-k-1}$*.*

Moreover, if we denote by $(\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2})$ *the image of* $\gamma_{-k_0/-k-2}$ *under the isomorphism* (9.15)*, then the* $|\cdot|$*-ball mentioned above can be taken to be centered at* $\gamma_{-k-1/-k-2}$*.*

We will postpone the proof of Lemma 9.8 to Section 9.8.

Lemma 9.9. *Let* $\Xi_k(T)$ *be as defined above Lemma 9.6. There exist constants* $\alpha_k > 0$ *and* $\alpha'_k > 0$ *satisfying the following property. Any* $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$ *satisfies* $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T^{\alpha'_k}$*.*

Proof of Lemma 9.9. We prove this lemma by upward induction on $k \in \{k_0, \ldots, m\}$. The base step is $k = k_0$, which is precisely part (i) of Proposition 9.4.

Assume Lemma 9.9 is proved for $k \in \{k_0, \ldots, m-1\}$, namely $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T^{\alpha'_k}$ for each $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$. We wish to prove the property for $k+1$, namely, there exist constants $\alpha_{k+1} > 0$ and $\alpha'_{k+1} > 0$ such that each $\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T) \cap \Gamma_{-k_0/-k-2}$ satisfies $H(\gamma_{-k_0/-k-2}) \leq \alpha_{k+1} T^{\alpha'_{k+1}}$.

Take $\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T) \cap \Gamma_{-k_0/-k-2}$. In particular, $\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1} \cap \mathbf{B}_{k+1}(T) \neq \emptyset$. Applying $r_{k+1} \colon \mathcal{D}_{k+1} \to \mathcal{D}_{k+1,\mathbb{R}}$ to both sides, we get $\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}} \cap \prod_{i=0}^{k+1} B_i(T) \neq \emptyset$.

Denote by $(\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2})$ the image of $\gamma_{-k_0/-k-2}$ under the isomorphism (9.15). In particular, $\gamma_{-k_0/-k-1}$ is the image of $\gamma_{-k_0/-k-2}$ under the projection $p_{k+1,k} \colon P/W_{-k-2} \to P/W_{-k-1}$.

We claim $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$. Indeed, it is clear that $\gamma_{-k_0/-k-1} \in \Gamma_{-k_0/-k-1}$. Moreover, the definition of $\Xi_{k+1}(T)$ implies that $\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1} \cap \widetilde{Z}_{k+1}(T) \neq \emptyset$. Applying $p_{k+1,k}$ and recalling our construction $p_{k+1,k}(\widetilde{Z}_{k+1}(T)) \subseteq \widetilde{Z}_k(T)$, we obtain $\gamma_{-k_0/-k-1}\mathfrak{F}_k \cap \widetilde{Z}_k(T) \neq \emptyset$. Thus $\gamma_{-k_0/-k-1} \in \Xi_k(T)$.

Therefore, by induction hypothesis, we have $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T^{\alpha'_k}$.

Next, as $\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}} \cap \prod_{i=0}^{k+1} B_i(T) \neq \emptyset$, we then have

$$\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}}) \cap B_{k+1}(T) \neq \emptyset. \tag{9.16}$$

By Lemma 9.8, $\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$ is contained in a $|\cdot|$-ball of radius $\ll H(\gamma_{-k_0/-k-1})^{k+1}$ centered at $\gamma_{-k-1/-k-2}$. We have seen that $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T^{\alpha'_k}$ in the last paragraph. So

$$\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}}) \text{ is contained in a } |\cdot|\text{-ball of radius } \ll T^{(k+1)\alpha'_k} \text{ centered at } \gamma_{-k-1/-k-2}. \tag{9.17}$$

Recall that $B_{k+1}(T)$ is defined to be the $|\cdot|$-ball of radius T centered at 0 in the Euclidean space $(W_{-k-1}/W_{-k-2})(\mathbb{R})$. So (9.16) and (9.17) together imply that $H(\gamma_{-k-1/-k-2})$ is bounded above polynomially in T, namely there exist real numbers $\beta_k, \beta'_k > 0$ such that

$$H(\gamma_{-k-1/-k-2}) \leq \beta_k T^{\beta'_k}.$$

Thus the proposition holds true for $k+1$ with $\alpha_{k+1} = \max(\alpha_k, \beta_k)$ and $\alpha'_{k+1} = \max(\alpha'_k, \beta'_k)$. This finishes the induction step. Hence we are done. □

Up to replacing each α'_k by a larger number, we may and do assume that

$$\alpha'_0 \leq \alpha'_1 \leq \cdots \leq \alpha'_m.$$

For each $k \in \{k_0, \ldots, m\}$, fix a number δ_k such that

$$(k+1)\alpha'_k < \delta_k < (k+2)\alpha'_k \tag{9.18}$$

Now we are ready to state and prove the lemmas concerning the lifting process for Proposition 9.7. Recall the projection $\lambda_{k+1} \colon \mathcal{D}_{k+1,\mathbb{R}} \to (W_{-k-1}/W_{-k-2})(\mathbb{R})$ defined in (9.14).

Lemma 9.10. *Assume there exists a sequence $\{T_i \in \mathbb{R}\}_{i \in \mathbb{N}}$, with $T_i \to \infty$, such that*

$$|\lambda_{k+1}(\widetilde{Z}_{k+1}(T_i)_{\mathbb{R}})| \leq T_i^{\delta_k}. \tag{9.19}$$

Assume furthermore that Proposition 9.7 holds true for k and this sequence, namely there exist constants $c'_k, \epsilon'_k > 0$ (independent of T_i) such that

$$\#\{\gamma_{-k_0/-k-1} \in \Xi_k(T_i) \cap \Gamma_{-k_0/-k-1} : H(\gamma_{-k_0/-k-1}) \leq T_i\} \geq c'_k T_i^{\epsilon'_k}. \tag{9.20}$$

Then there exist constant $c'_{k+1} > 0$ and $\epsilon'_{k+1} > 0$, both independent of T_i, such that

$$\#\{\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T_i) \cap \Gamma_{-k_0/-k-2} : H(\gamma_{-k_0/-k-2}) \leq T_i\} \geq c'_{k+1} T_i^{\epsilon'_{k+1}}. \tag{9.21}$$

Lemma 9.11. *Assume*

$$|\lambda_{k+1}(\widetilde{Z}_{k+1}(T)_{\mathbb{R}})| > T^{\delta_k} \qquad \textit{for all } T \gg 1. \tag{9.22}$$

Then there exist constant $c'_{k+1} > 0$ and $\epsilon'_{k+1} > 0$, both independent of T, such that

$$\#\{\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T) \cap \Gamma_{-k_0/-k-2} : H(\gamma_{-k_0/-k-2}) \leq T\} \geq c'_{k+1} T^{\epsilon'_{k+1}}. \tag{9.23}$$

Proof of Lemma 9.10. By the assumption (9.20) and the naive lifting given by Lemma 9.6, it suffices to prove the following claim.

Claim 1. *Each $\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T_i) \cap \Gamma_{-k_0/-k-2}$ satisfies $H(\gamma_{-k_0/-k-2}) \ll T_i^{\delta_k}$.*

Recall the natural projection $p_{k+1,k} \colon P/W_{-k-2} \to P/W_{-k-1}$. Let $\gamma_{-k_0/-k-1} = p_{k+1,k}(\gamma_{-k_0/-k-2})$.

The definition of $\Xi_{k+1}(T_i)$ yields a point $\widetilde{z}'_{k+1} \in \gamma_{-k_0/-k-2}\mathfrak{F}_{k+1} \cap \widetilde{Z}_{k+1}(T)$. Let $\widetilde{z}_{k+1} = r_{k+1}(\widetilde{z}'_{k+1}) \in \mathcal{D}_{k+1,\mathbb{R}}$. Write $\widetilde{z}_{k+1} = (\widetilde{z}_k, \widetilde{z}_{k+1,k}) \in \mathcal{D}_{k,\mathbb{R}} \times (W_{-k-1}/W_{-k-2})(\mathbb{R}) \simeq \mathcal{D}_{k+1,\mathbb{R}}$. Then $\widetilde{z}_{k+1,k} = \lambda_{k+1}(\widetilde{z}_{k+1})$. Moreover $\gamma_{-k_0/-k-1}\mathfrak{F}_{k,\mathbb{R}} \cap \widetilde{Z}_k(T)_{\mathbb{R}}$ is nonempty as it contains $\widetilde{z}_k$. Thus $\gamma_{-k_0/-k-1} \in \Xi_k(T_i)$, so $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T_i^{\alpha'_k}$ by Lemma 9.9.

Let $(\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2})$ the image of $\gamma_{-k_0/-k-2}$ under the isomorphism (9.15) $P/W_{-k-2} \simeq P/W_{-k-1} \times W_{-k-1/-k-2}$.

We claim that $\widetilde{z}_{k+1,k}$ is contained in a $|\cdot|$-ball of radius $\ll T_k^{(k+1)\alpha'_k}$ centered at $\gamma_{-k-1/-k-2}$. Indeed, $\widetilde{z}_{k+1} \in \gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}}$ by choice. So $\widetilde{z}_{k+1,k} \in \lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$. By Lemma 9.8, the set $\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$ is contained in a $|\cdot|$-ball of radius $\ll H(\gamma_{-k_0/-k-1})^{k+1}$, and hence of radius $\ll T_i^{(k+1)\alpha'_k}$, centered at $\gamma_{-k-1/-k-2}$.

On the other hand, our assumption (9.19) implies $|\widetilde{z}_{k+1,k}| \leq T_i^{\delta_k}$. So by the conclusion of the previous paragraph, we have $H(\gamma_{-k-1/-k-2}) \ll T_i^{\delta_k-(k+1)\alpha'_k}$ since $\delta_k > (k+1)\alpha'_k$.

So $H(\gamma_{-k_0/-k-2}) = \max\{H(\gamma_{-k_0/-k-1}), H(\gamma_{-k-1/-k-2})\} \ll T_i^{\delta_k}$. We are done. $\square$

Proof of Lemma 9.11. Recall that $\mathfrak{F}_{k+1,\mathbb{R}} = \mathfrak{F}_{k,\mathbb{R}} \times (-M, M)^{\dim(W_{-k-1}/W_{-k-2})(\mathbb{R})}$, for some fixed real number $M > 0$, by definition (7.8).

Let $\gamma_{-k_0/-k-2} \in \Xi_{k+1}(T) \cap \Gamma_{-k_0/-k-2}$. Denote by $(\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2}) \in \Gamma_{-k_0/-k-1} \times \Gamma_{-k-1/-k-2}$ be the image of $\gamma_{-k_0/-k-2}$ under the isomorphism (9.15). Then $\gamma_{-k_0/-k-1} = p_{k+1,k}(\gamma_{-k_0/-k-2})$ for the natural projection $p_{k+1,k} \colon P/W_{-k-2} \to P/W_{-k-1}$.

We claim that $\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}} \cap \widetilde{Z}_{k+1}(T)_{\mathbb{R}})$, being a subset of $(W_{-k-1}/W_{-k-2})(\mathbb{R})$, is contained in a $|\cdot|$-ball of radius $\ll T^{(k+1)\alpha'_k}$. Indeed, $\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}} \cap \widetilde{Z}_{k+1}(T)_{\mathbb{R}}) \subseteq \lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$, which furthermore is contained in a $|\cdot|$-ball of radius $\ll H(\gamma_{-k_0/-k-1})^{k+1}$ by Lemma 9.8. But $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T^{\alpha'_k}$ by Lemma 9.9.[5] So the claim holds true.

On the other hand, consider the $|\cdot|$-ball in $(W_{-k-1}/W_{-k-2})(\mathbb{R})$ centered at 0 of radius T^{δ_k}. Our hypothesis (9.22) says that $\lambda_{k+1}(\widetilde{Z}_{k+1}(T)_{\mathbb{R}})$ reaches the boundary of this ball.

Since $(W_{-k-1}/W_{-k-2})(\mathbb{R})$ is Euclidean, the previous two paragraphs together imply

$$\begin{aligned}\#\{\gamma_{-k-1/-k-2} \in \Gamma_{-k-1/-k-2} : & H(\gamma_{-k-1/-k-2}) \leq T^{\delta_k},\ (\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2}) \in \Xi_{k+1}(T) \\ & \text{for some } \gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}\} \gg T^{\delta_k - (k+1)\alpha'_k}\end{aligned}$$

for all $T \gg 1$. As each $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$ satisfies $H(\gamma_{-k_0/-k-1}) \leq \alpha_k T^{\alpha'_k}$ by Lemma 9.9, the counting above yields, for all $T \gg 1$,

$$\begin{aligned}\#\{(\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2}) \in \Xi_{k+1}(T) \cap \Gamma_{-k_0/-k-2} : & H(\gamma_{-k-1/-k-2}) \leq T^{\delta_k}, \\ & H(\gamma_{-k_0/-k-2}) \leq \alpha_k T_i^{\alpha'_k}\} \gg T^{\delta_k - (k+1)\alpha'_k}.\end{aligned}$$

But the only assumption on δ_k is given by (9.18) $(k+1)\alpha'_k < \delta_k < (k+2)\alpha'_k$. Hence we have proved (9.23) by choosing appropriately c'_{k+1} and ϵ'_{k+1}. We are done. $\square$

9.7. **Proof of Proposition 9.7.** Now we are ready to finish the proof of Proposition 9.7. Recall the notation $k_0 \in \{0, \cdots, m\}$ be such that $\dim \widetilde{Z}_{k_0} > 0$, smallest for this property.

Recall the following numbers we have introduced in Section 9.6, for each $k \in \{k_0, \cdots, m\}$.

- $\alpha'_k > 0$ from Lemma 9.9. Namely, each $\gamma_{-k_0/-k-1} \in \Xi_k(T) \cap \Gamma_{-k_0/-k-1}$ satisfies $H(\gamma_{-k_0/-k-1}) \ll T^{\alpha'_k}$.
- $\delta_k \in ((k+1)\alpha'_k, (k+2)\alpha'_k)$ from (9.18).

Let $\lambda_{k+1}\colon \mathcal{D}_{k+1,\mathbb{R}} \to (W_{-k-1}/W_{-k-2})(\mathbb{R})$ be from (9.14).

The proof can be divided into two cases.

$\boxed{\text{Case 1}}$ There exists an integer $k'_0 \in \{k_0, \cdots, m-1\}$ such that

$$|\lambda_{k'_0+1}(\widetilde{Z}_{k'_0+1}(T)_{\mathbb{R}})| > T^{\delta_{k'_0}} \qquad \text{for all } T \gg 1.$$

In this case, let furthermore k'_0 be the largest such integer. We will show that this k'_0 is what we desire in Proposition 9.7.

We finish the proof by induction on k. The base step is $k = k'_0 + 1$. In this case, Proposition 9.7 holds true by Lemma 9.11.

Assume that Proposition 9.7 holds true for some $k > k'_0$, namely there exist constants $c'_k, \epsilon'_k > 0$ and a sequence $\{T_i \in \mathbb{R}\}_{i \in \mathbb{N}}$, with $T_i \to \infty$, such that

$$\#\{\gamma_{-k_0/-k-1} \in \Xi_k(T_i) \cap \Gamma_{-k_0/-k-1} : H(\gamma_{-k_0/-k-1}) \leq T_i\} \geq c'_k T_i^{\epsilon'_k}.$$

Notice that this is precisely (9.20).

We wish to prove it for $k+1$.

By the maximality of k'_0 and because $k > k'_0$, the following assertion holds true. Up to replacing $\{T_i\}$ by a subsequence, we have $|\lambda_{k+1}(\widetilde{Z}_{k+1}(T_i)_{\mathbb{R}})| \leq T_i^{\delta_k}$. This is precisely (9.19).

Thus we can invoke Lemma 9.10 to conclude Proposition 9.7 for $k+1$. This finishes the induction step, and we are done.

[5]Applying the natural projection $p_{k+1,k}\colon \mathcal{D}_{k+1,\mathbb{R}} \to \mathcal{D}_{k,\mathbb{R}}$ to $\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}} \cap \widetilde{Z}_{k+1}(T)_{\mathbb{R}} \neq \emptyset$, we obtain $\gamma_{-k_0/-k-1}\mathfrak{F}_{k,\mathbb{R}} \cap \widetilde{Z}_k(T)_{\mathbb{R}} \neq \emptyset$.

$\boxed{\text{Case 2}}$ For each $k \in \{k_0, \cdots, m-1\}$, there exists a sequence $\{T_i \in \mathbb{R}\}_{i\in\mathbb{N}}$, with $T_i \to \infty$, such that

$$|\lambda_{k+1}(\widetilde{Z}_{k+1}(T_i)_{\mathbb{R}})| \leq T_i^{\delta_k}. \tag{9.24}$$

Up to replacing the sequence $\{T_i\}$ by a subsequence, we may assume that we are taking the same sequence $\{T_i\}$ for all k.

If $k_0 \geq 1$, then Proposition 9.7 holds true for k_0 by part (ii) of Proposition 9.4. Hence by applying Lemma 9.10 successively to $k = k_0, \ldots, m-1$, we can prove that Proposition 9.7 holds true for each $k \in \{k_0, \cdots, m\}$. Hence we are done.

So it remains to handle the case where $k_0 = 0$, namely $\dim \widetilde{Z}_0 > 0$.

From now on, we assume $\dim \widetilde{Z}_0 > 0$. We start from the following inclusion of sets, where $\widetilde{Z}(T) = \widetilde{Z}_m(T)$ is a complex analytic irreducible component of $\widetilde{Z} \cap \mathbf{B}_m(T)$,

$$\widetilde{Z}(T) = \bigcup_{\gamma\in\Gamma_P,\ \gamma\mathfrak{F}\cap\widetilde{Z}(T)} \gamma\mathfrak{F} \cap \widetilde{Z}(T) \subseteq \bigcup_{\gamma\in\Gamma_P,\ \gamma\mathfrak{F}\cap\widetilde{Z}(T)} \gamma\mathfrak{F} \cap \widetilde{Z}.$$

Applying p_0 to both sides and noticing that $p_0(\widetilde{Z}(T)) \to \widetilde{Z}_0(T)$ when $T \to \infty$,[6] we have

$$\mathrm{Vol}(\widetilde{Z}_0(T)) \leq \#\{\gamma \in \Gamma_P,\ \gamma\mathfrak{F} \cap \widetilde{Z}(T)\} \cdot \mathrm{Vol}(p_0(\gamma\mathfrak{F} \cap \widetilde{Z})).$$

By [BT19, Prop. 2.7], we have $\mathrm{Vol}(\widetilde{Z}_0(T)) \geq \beta_1 T^{\beta_2}$. By [BT19, Prop. 3.2] and because $\mathcal{Z}$ is a component of $\mathcal{Z}^{\mathrm{Zar}} \cap \Delta$, we have $\mathrm{Vol}(p_0(\gamma\mathfrak{F} \cap \widetilde{Z})) \leq \beta_3$. So we get

$$\#\{\gamma \in \Gamma_P,\ \gamma\mathfrak{F} \cap \widetilde{Z}(T)\} \geq (\beta_1/\beta_3) T^{\beta_2}. \tag{9.25}$$

Thus Proposition 9.7 for $k = m$ follows from the following lemma. Hence we are done by letting $k_0' = m-1$ for this case.

Lemma 9.12. *Let $\gamma \in \Gamma_P$, and let $\{T_i\}$ be the sequence from (9.24). If $\gamma\mathfrak{F} \cap \widetilde{Z}(T_i) \neq \emptyset$, then $H(\gamma) \ll T_i^{\alpha_m'}$.*

Proof. Write $\gamma = (\gamma_{0/-1}, \gamma_{-1/-2}, \ldots, \gamma_{-m/-m-1})$ under the identification $P \simeq G \times (W_{-1}/W_{-2}) \times \cdots \times W_{-m}$ given by (B.6). Then $(\gamma_{0/-1}, \ldots, \gamma_{-k/-k-1}) \in G \times (W_{-1}/W_{-2}) \times \cdots \times (W_{-k}/W_{-k-1}) \simeq P/W_{-k-1}$, which we denote by $\gamma_{0/-k-1}$ for simplicity. Then $\gamma_{0/-k-1} = p_k(\gamma)$ for the morphism $p_k \colon P \to P/W_{-k-1}$.

Recall our assumption $\alpha_0' \leq \cdots \leq \alpha_m'$. We will prove by induction on k that $H(\gamma_{0/-k-1}) \ll T_i^{\alpha_k'}$.

The base step is $k = 0$. By applying p_0 to $\gamma\mathfrak{F} \cap \widetilde{Z}(T_i) \neq \emptyset$, we get that $\gamma_{0/-1}\mathfrak{F}_0 \cap \widetilde{Z}_0(T_i) \neq \emptyset$. Thus $\gamma_{0/-1} \in \Xi_0(T_i) \cap \Gamma_{0/-1}$ by definition of $\Xi_0(T_i)$. Thus we have $H(\gamma_{0/-1}) \leq \alpha_0 T_i^{\alpha_0'}$ by part (i) of Proposition 9.4. Thus we have proved the base step.

Assume that we have proved for k. Let us prove for $k+1$. By Lemma 9.8 and noticing that $H(\gamma_{0/-k-1}) \ll T_i^{\alpha_k'}$ by induction hypothesis, $\lambda_{k+1}(\gamma_{0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$ is contained in a $|\cdot|$-ball of radius $\ll T_i^{(k+1)\alpha_k'}$ in $(W_{-k-1}/W_{-k-2})(\mathbb{R})$, which moreover is centered at $\gamma_{-k-1/-k-2}$. Thus by (9.24), we have $H(\gamma_{-k-1/-k-2}) \ll T_i^{\delta_k-(k+1)\alpha_k'}$ because $\delta_k > (k+1)\alpha_k'$. As δ_k is chosen to satisfy $\delta_k < (k+2)\alpha_k'$, we are done for the induction step. Hence we are done. $\square$

[6]Recall that $\widetilde{Z}(T)$ is a complex analytic irreducible component of $\widetilde{Z} \cap \mathbf{B}_m(T)$. Thus $\widetilde{Z}(T) \to \widetilde{Z}$ when $T \to \infty$. But $p_0(\widetilde{Z}(T)) \subseteq \widetilde{Z}_0(T)$ by definition of $\widetilde{Z}_0(T)$. So we have $p_0(\widetilde{Z}(T)) \to \widetilde{Z}_0(T)$ when $T \to \infty$.

9.8. **Proof of Lemma 9.8.** Let $\gamma_{-k_0/-k-2} \in \Gamma_{-k_0/-k-2}$. Use the notation as in the lemma, namely $\gamma_{-k_0/-k-2} \mapsto (\gamma_{-k_0/-k-1}, \gamma_{-k-1/-k-2})$ under the isomorphism of $\mathbb{Q}$-varieties $P/W_{-k-2} \simeq P/W_{-k-1} \times (W_{-k-1}/W_{-k-2})$ (9.15).

Next, consider the isomorphism of $\mathbb{Q}$-varieties induced by (B.6)

$$P/W_{-k-1} \simeq G \times (W_{-1}/W_{-2}) \times \cdots \times (W_{-k}/W_{-k-1}),$$

and suppose $\gamma_{-k_0/-k-1} \mapsto (\gamma_{0/-1}, \gamma_{-1/-2}, \ldots, \gamma_{-k/-k-1})$ under this isomorphism. Then $H(\gamma_{-k_0/-k-1}) = \max\{H(\gamma_{0/-1}), \ldots, H(\gamma_{-k/-k-1})\}$.

Thus we have

$$\begin{array}{rcl} P/W_{-k-2} & \simeq & G \times (W_{-1}/W_{-2}) \times \cdots \times (W_{-k}/W_{-k-1}) \times (W_{-k-1}/W_{-k-2}) \\ \gamma_{-k_0/-k-2} & \mapsto & (\gamma_{0/-1}, \gamma_{-1/-2}, \ldots, \gamma_{-k/-k-1}, \gamma_{-k-1/-k-2}). \end{array}$$

where the isomorphism of $\mathbb{Q}$-varieties is induced by (B.6).

On the other hand, for the real-algebraic morphism induced by (9.7)

$$\mathcal{D}_{k+1,\mathbb{R}} \simeq \mathcal{D}_0 \times (W_{-1}/W_{-2})(\mathbb{R}) \times \cdots \times (W_{-k}/W_{-k-1})(\mathbb{R}) \times (W_{-k-1}/W_{-k-2})(\mathbb{R}),$$

we have defined, in (7.8), $\mathfrak{F}_{k+1,\mathbb{R}}$ to be the inverse image of $\mathfrak{F}_0 \times \mathfrak{F}'_1 \times \cdots \mathfrak{F}'_k \times \mathfrak{F}'_{k+1}$, where M is a fixed real number and

$$\mathfrak{F}'_i = (-M, M)^{\dim (W_{-i}/W_{-i-1})(\mathbb{R})} \subseteq (W_{-i}/W_{-i-1})(\mathbb{R}).$$

The formula for the action of the group $(P/W_{-k-2})(\mathbb{R})^+$ on $\mathcal{D}_{k+1,\mathbb{R}}$ is given by Proposition 7.2, or more precisely (7.7). Thus

$$\begin{aligned} &\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}}) \\ &=\{\gamma_{-k-1/-k-2} + \gamma_{0/-1}\widetilde{x} + \mathrm{calb}_k(\gamma_{-k_0/-k-1}, \gamma_0 \widetilde{x}') : \widetilde{x} \in M\mathfrak{F}'_{k+1},\ \widetilde{x}' \in \mathfrak{F}_0 \times \mathfrak{F}'_1 \times \cdots \times \mathfrak{F}'_k\} \\ &=\gamma_{-k-1/-k-2} + \gamma_{0/-1} \cdot \mathfrak{F}'_{k+1} + \mathrm{calb}_k(\gamma_{-k_0/-k-1}, \gamma_{0/-1}\mathfrak{F}_0 \times \gamma_{0/-1}\mathfrak{F}'_1 \times \cdots \times \gamma_{0/-1}\mathfrak{F}'_k), \end{aligned} \tag{9.26}$$

where calb_k is a polynomial of degree at most $k-1$. Notice that M, $\mathfrak{F}_0$ and the $\mathfrak{F}'_i$'s are fixed, and that $H(\gamma_{0/-1}) \leq H(\gamma_{-k_0/-k-1})$. So

$$|\gamma_{0/-1} \cdot \mathfrak{F}'_{k+1} + \mathrm{calb}_k(\gamma_{-k_0/-k-1}, \gamma_{0/-1}(\mathfrak{F}_0 \times \mathfrak{F}'_1 \times \cdots \mathfrak{F}'_k))| \ll H(\gamma_{0/-1}) + H(\gamma_{-k_0/-k-1})^{k+1} \ll H(\gamma_{-k_0/-k-1})^{k+1}.$$

Therefore, by (9.26), $\lambda_{k+1}(\gamma_{-k_0/-k-2}\mathfrak{F}_{k+1,\mathbb{R}})$ is contained in the $|\cdot|$-ball of radius $\ll H(\gamma_{-k_0/-k-1})^{k+1}$ centered at $\gamma_{-k-1/-k-2}$. Hence we are done.

10. Normality of the $\mathbb{Q}$-stabilizer: Part 1

10.1. **Family associated with $\mathcal{Z}$.** Let $\mathbf{H}$ be the component of the Hilbert scheme of $S \times \mathcal{D}^\vee$ which contains $[\mathcal{Z}^{\mathrm{Zar}}]$, the point representing $\mathcal{Z}^{\mathrm{Zar}}$. Then $\mathbf{H}$ is proper. Consider the (modified) universal family

$$\boldsymbol{\mathcal{B}} = \{(x, \widetilde{m}, [\mathcal{B}]) \in (S \times \mathcal{D}) \times \mathbf{H} : (x, \widetilde{m}) \in \mathcal{B}\}.$$

The projection

$$\psi \colon \boldsymbol{\mathcal{B}} \to S \times \mathcal{D} \tag{10.1}$$

is a proper map since $\mathbf{H}$ is proper.

Define

$$\boldsymbol{\mathcal{Z}} = \{(\widetilde{\delta}, [\mathcal{B}]) \in (\Delta \times \mathbf{H}) \cap \boldsymbol{\mathcal{B}} : \dim_{\widetilde{\delta}}(\Delta \cap \mathcal{B}) \geq \dim \mathcal{Z}\}.$$

Then $\boldsymbol{\mathcal{Z}}$ is a closed complex analytic subset of $\boldsymbol{\mathcal{B}}$. So $\psi(\boldsymbol{\mathcal{Z}})$ is closed complex analytic in $S \times \mathcal{D}$ as ψ is proper. Note that $\psi(\boldsymbol{\mathcal{Z}}) \subseteq \Delta$.

Let us summarize the notations in the following diagram.

$$\begin{tikzcd}
\boldsymbol{\mathcal{B}} \arrow[d, "\psi"'] & \supseteq & \boldsymbol{\mathcal{Z}} \arrow[d, "\psi|_{\boldsymbol{\mathcal{Z}}}"] & \\
S \times \mathcal{D} & \supseteq & \Delta \arrow[r] \arrow[d, "u_S"'] & \mathcal{D} \arrow[d, "u"] \\
& & S \arrow[r, "{[\Phi]}"] & \Gamma_P \backslash \mathcal{D}
\end{tikzcd}$$

Recall that the arithmetic group Γ_P acts on $S \times \mathcal{D}$ by its action on the second factor. We claim that $\Gamma_P \psi(\boldsymbol{\mathcal{Z}}) = \psi(\boldsymbol{\mathcal{Z}})$. Indeed, this action of Γ_P on $S \times \mathcal{D}$ induces an action of Γ_P on $\boldsymbol{\mathcal{B}}$ by

$$\gamma(x, \widetilde{m}, [\mathcal{B}]) = (x, \gamma\widetilde{m}, [\gamma\mathcal{B}]). \tag{10.2}$$

Thus $\Gamma_P \Delta = \Delta$ implies $\Gamma_P \boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{Z}}$. But ψ is Γ_P-invariant. So $\Gamma_P \psi(\boldsymbol{\mathcal{Z}}) = \psi(\boldsymbol{\mathcal{Z}})$.

As the map $u_S \colon \Delta \to S$ is Γ_P-invariant (for the trivial action of Γ_P on S), we have that $T := u_S(\psi(\boldsymbol{\mathcal{Z}}))$ is closed complex analytic in S.

Proposition 10.1. *T is an algebraic subvariety of S.*

Proof. By definable Chow ([PS09, Thm. 4.5] or [MPT19, Thm. 2.2]), it suffices to prove that T is definable in $\mathbb{R}_{\mathrm{an,exp}}$. In the rest of the proof, when we say "definable" we mean *definable in* $\mathbb{R}_{\mathrm{an,exp}}$.

Let $\mathfrak{F}_{\mathbb{R}}$ and $\mathfrak{F} = r^{-1}(\mathfrak{F}_{\mathbb{R}})$ be as in Theorem 7.3.

Note that u_S is the restriction of the natural projection $p_S \colon S \times \mathcal{D} \to \mathcal{D}$ to Δ. So $T = u_S(\psi(\boldsymbol{\mathcal{Z}})) = p_S(\psi(\boldsymbol{\mathcal{Z}})) = p_S(\psi(\boldsymbol{\mathcal{Z}}) \cap (S \times \mathfrak{F}))$. Thus it suffices to prove that $\psi(\boldsymbol{\mathcal{Z}}) \cap (S \times \mathfrak{F})$ is definable.

But $\psi(\boldsymbol{\mathcal{Z}}) \cap (S \times \mathfrak{F}) = \psi(\boldsymbol{\mathcal{Z}} \cap (S \times \mathfrak{F} \times \mathbf{H}))$. So it suffices to prove that $\boldsymbol{\mathcal{Z}} \cap (S \times \mathfrak{F} \times \mathbf{H})$ is definable.

By property (ii) of Theorem 7.3, the period map $[\Phi]$ is definable if we endow $\Gamma_P \backslash \mathcal{D}$ with the definable structure given by $u|_{\mathfrak{F}}$. So

$$\Delta \cap (S \times \mathfrak{F}) = \{(x, \widetilde{m}) \in S \times \mathfrak{F} : u(\widetilde{m}) = [\Phi](x)\}$$

is a definable subset of $S \times \mathcal{D}$. So

$$\Big((\Delta \cap (S \times \mathfrak{F})) \times \mathbf{H} \Big) \cap \boldsymbol{\mathcal{B}}$$

is a definable subset of $S \times \mathcal{D} \times \mathbf{H}$. So

$$\boldsymbol{\mathcal{Z}} \cap (S \times \mathfrak{F} \times \mathbf{H}) = \{(\widetilde{\delta}, [\mathcal{B}]) \in \Big((\Delta \cap (S \times \mathfrak{F})) \times \mathbf{H} \Big) \cap \boldsymbol{\mathcal{B}} : \dim_{\widetilde{\delta}}(\Delta \cap (S \times \mathfrak{F}) \cap \mathcal{B}) \geq \dim \mathcal{Z}\}$$

is definable. Hence we are done. $\square$

10.2. **Monodromy.** Let N be the connected algebraic monodromy group of the admissible VMHS $(\mathbb{V}_{\mathbb{Z}}, W_\bullet, \mathcal{F}^\bullet)$ on S. Then $N \lhd P$ by Theorem 3.4.

Lemma 10.2. *$H_{\mathcal{Z}^{\mathrm{Zar}}} \lhd N$.*

Proof. Recall that $\Gamma_P \boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{Z}}$. So $\Gamma_P \backslash \boldsymbol{\mathcal{Z}}$ is a complex analytic space. The proper map ψ (10.1) induces

$$\overline{\psi} \colon \Gamma_P \backslash \boldsymbol{\mathcal{Z}} \to \Gamma_P \backslash \psi(\boldsymbol{\mathcal{Z}}) = u_S(\psi(\boldsymbol{\mathcal{Z}})) = T,$$

which is also proper.

Let $\mathfrak{Z}_0$ be an irreducible component of $\mathfrak{Z}$ which contains $\mathcal{Z} \times [\mathcal{Z}^{\mathrm{Zar}}]$. Then $\mathcal{Z} = \psi(\mathcal{Z} \times [\mathcal{Z}^{\mathrm{Zar}}]) \subseteq \psi(\mathfrak{Z}_0)$. So

$$u_S(\mathcal{Z}) \subseteq u_S(\psi(\mathfrak{Z}_0)) = \overline{\psi}(\Gamma_P \backslash \mathfrak{Z}_0).$$

The right hand side is T because T is irreducible and $\mathfrak{Z}_0$ is an irreducible component of $\mathfrak{Z}$. Recall the assumption $S = u_S(\mathcal{Z})^{\mathrm{Zar}}$. So taking the Zariski closures of both sides, we get $T = S$ by Proposition 10.1.

Thus $\overline{\psi}$ induces a map $\overline{\psi}_* \colon \pi_1(\Gamma_P \backslash \mathfrak{Z}_0) \to \pi_1(S)$, and so a subgroup Γ_0 of $N(\mathbb{Q})$. But $\mathrm{Im}(\overline{\psi}_*)$ has finite index in $\pi_1(S)$ (since $\overline{\psi}$ is proper), so $\Gamma_0^{\mathrm{Zar}} = N$.

Next denote by $\theta \colon \mathfrak{B} \subseteq (S \times \mathcal{D}) \times \mathbf{H} \to \mathbf{H}$ the restriction of the natural projection. Let $\mathfrak{F} = \theta^{-1}(\theta(\mathfrak{Z}_0)) = \{(x, \widetilde{m}, [\mathcal{B}]) : [\mathcal{B}] \in \theta(\mathfrak{Z}_0),\ (x, \widetilde{m}) \in \mathcal{B}\}$. Then $\mathfrak{F} \subseteq \mathfrak{B}$ is the family of algebraic varieties parametrized by $\theta(\mathfrak{Z}_0) \subseteq \mathbf{H}$, with the fiber over each $[\mathcal{B}] \in \theta(\mathfrak{Z}_0)$ being $\mathcal{B}$. Then we have

$$\Gamma_0 \mathfrak{F} \subseteq \mathfrak{F}$$

for the action of Γ_P on $\mathfrak{B}$ defined by (10.2). Thus every $\gamma \in \Gamma_0$ sends a very general fiber of $\mathfrak{F}$ to a very general fiber of $\mathfrak{F}$.

Define

$$\Gamma_{\mathfrak{F}} = \{\gamma \in \Gamma_P : \gamma\mathcal{B} \subseteq \mathcal{B},\ \text{for all } [\mathcal{B}] \in \theta(\mathfrak{Z})\}.$$

Then for a very general $[\mathcal{B}] \in \theta(\mathfrak{Z}_0)$, we have

$$\mathrm{Stab}_{\Gamma_P}(\mathcal{B}) = \Gamma_{\mathfrak{F}}. \tag{10.3}$$

By construction of $\mathfrak{F}$, without loss of generality we may assume that $\mathcal{Z}^{\mathrm{Zar}}$ is a very general fiber of $\mathfrak{F}$. The conclusion of the last paragraph implies that any $\gamma \in \Gamma_0$ sends $\mathcal{Z}^{\mathrm{Zar}}$ to a very general fiber of $\mathfrak{F}$. By taking the stabilizers of the two fibers in consideration, we get $\Gamma_{\mathfrak{F}} = \gamma \Gamma_{\mathfrak{F}} \gamma^{-1}$ for all $\gamma \in \Gamma_0$. By taking the Zariski closures, we get

$$(\Gamma_{\mathfrak{F}}^{\mathrm{Zar}})^\circ \lhd N.$$

On the other hand (10.3) implies $(\Gamma_{\mathfrak{F}}^{\mathrm{Zar}})^\circ = H_{\mathcal{Z}^{\mathrm{Zar}}}$. Hence we are done. $\square$

11. Normality of the $\mathbb{Q}$-stabilizer: Part 2

Proposition 11.1. *$H_{\mathcal{Z}^{\mathrm{Zar}}} \lhd P$.*

Proof. For simplicity we write H for the subgroup $H_{\mathcal{Z}^{\mathrm{Zar}}}$ of N. We have $H \lhd N \lhd P$ and we want to show $H \lhd P$. Let $\mathfrak{h} \subset \mathfrak{n} \subset \mathfrak{p}$ be the corresponding inclusions of Lie algebra. As H and P are connected, we are reduced to showing that $\mathfrak{h}$ is an ideal of $\mathfrak{p}$, equivalently that $\mathfrak{h} \subset \mathfrak{p}$ is a P-submodule for the adjoint representation $\mathrm{Ad}_P \colon P \to \mathrm{GL}(\mathfrak{p})$ of P.

The normality $N \lhd P$ implies that $\mathfrak{n}$ is a P-submodule of $\mathfrak{p}$. Let $(\mathbb{V}_{\mathfrak{n}}, W_\bullet, \mathcal{F}^\bullet)$ be the admissible $\mathbb{Q}$VMHS on S associated to the P-module $\mathfrak{n}$. The underlying local system $\mathbb{V}_{\mathfrak{n}}$ is defined by the N-module structure on $\mathfrak{n}$ given by the monodromy representation $\mathrm{Ad}_N \colon N \hookrightarrow P \to GL(\mathfrak{n})$.

The normality $H \lhd N$ implies that $\mathfrak{h}$ is an N-submodule of $\mathfrak{n}$, hence defines a sub-local system $\mathbb{V}_{\mathfrak{h}} \subset V_{\mathfrak{n}}$. We are reduced to proving that this sub-local system carries the structure of a sub-VMHS of $(\mathbb{V}_{\mathfrak{n}}, W_\bullet, \mathcal{F}^\bullet)$: this exactly means that the N-submodule $\mathfrak{h}$ of $\mathfrak{n}$ is in fact a P-submodule.

This follows immediately from Proposition 11.2 below, which generalizes a result of Deligne in the pure case. $\square$

Proposition 11.2. *Let $\mathbb{V}^H := (\mathbb{V}, W_\bullet, \mathcal{F}^\bullet)$ be an admissible $\mathbb{Q}$VMHS on a complex smooth quasi-projective variety S. Let $\mathbb{L} \subset \mathbb{V}$ be a sub-local system. Then the restriction of $W_\bullet$ to $\mathbb{L}$ and of $\mathcal{F}^\bullet$ to $\mathcal{L} := \mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{L} \subset \mathcal{V}$ makes $\mathbb{L}^H := (\mathbb{L}, W_\bullet, \mathcal{F}^\bullet)$ an admissible sub-$\mathbb{Q}$VMHS of $\mathbb{V}^H$.*

Proof. Let us prove the result by induction on the length $l := \max_{i,j\in I}(j-i)$ of the weight filtration $W_\bullet$ on $\mathbb{V}$, where I denotes the finite set of $n \in \mathbb{Z}$ so that $\mathrm{Gr}_n^W \mathbb{V} \neq 0$.

When $l = 0$, then $\mathbb{V}^H$ is pure and the result is [Del87, Prop.1.13] (Deligne deals with the case of a complex variation of Hodge structure but his proof adapts easily to the rational case).

Let l be a non-negative integer and let us suppose by induction that the result holds true for all admissible $\mathbb{Q}$VMHS on S of length at most l. Let $\mathbb{V}^H := (\mathbb{V}, W_\bullet, \mathcal{F}^\bullet)$ be an admissible $\mathbb{Q}$VMHS of length $l+1 = j-i > 0$ with $\mathrm{Gr}_i^W \mathbb{V} \neq 0$ and $\mathrm{Gr}_j^W \mathbb{V} \neq 0$.

One has an exact sequence in the abelian category $\mathrm{VMHS}(S)_{\mathrm{adm}}$ of admissible $\mathbb{Q}$VMHS on S (cf. Lemma 11.3):

$$0 \to W_i\mathbb{V}^H \to \mathbb{V}^H \xrightarrow{\pi} \mathbb{V}^H/W_i\mathbb{V}^H \to 0 \ . \tag{11.1}$$

Forgetting the Hodge structure we obtain an exact sequence in the abelian category $\mathrm{Loc}(S)$ of $\mathbb{Q}$-local systems on S:

$$0 \to W_i\mathbb{V} \to \mathbb{V} \to \mathbb{V}/W_i\mathbb{V} \to 0 \ . \tag{11.2}$$

The sub-local system $\mathbb{L}$ of $\mathbb{V}$ is thus an extension in $\mathrm{Loc}(S)$:

$$0 \to \mathbb{L} \cap W_i\mathbb{V} \to \mathbb{L} \to \mathbb{L}/(\mathbb{L} \cap W_i\mathbb{V}) \to 0 \ . \tag{11.3}$$

As $\mathbb{V}^H/W_i\mathbb{V}^H \in \mathrm{VMHS}(S)_{\mathrm{adm}}$ has length at most l, it follows from the induction hypothesis that there exists $\mathbb{E}^H \subset \mathbb{V}^H/W_i\mathbb{V}^H \in \mathrm{VMHS}(S)_{\mathrm{adm}}$ with underlying local system $\mathbb{E} = \mathbb{L}/(\mathbb{L} \cap W_i\mathbb{V})$. The pull-back of (11.1) under $\mathbb{E}^H \hookrightarrow \mathbb{V}^H/W_i\mathbb{V}^H$ provides an extension in $\mathrm{VMHS}(S)_{\mathrm{adm}}$

$$0 \to W_i\mathbb{V}^H \to \pi^{-1}(\mathbb{E}^H) \to \mathbb{E}^H \to 0 \ , \tag{11.4}$$

and $\mathbb{L}$ is a sub-local system of the local system $\pi^{-1}(E)$ underlying $\pi^{-1}(\mathbb{E}^H)$. Without loss of generality (replacing $\mathbb{V}^H$ by $\pi^{-1}(\mathbb{E}^H)$ if necessary), we can thus assume that the local system $\mathbb{L}$ surjects on $\mathbb{V}/W_i\mathbb{V}$.

Now $\mathbb{L} \cap W_i\mathbb{V}$ is a sub-local system of the local system $W_i\mathbb{V}$ associated to the admissible VMHS $W_i\mathbb{V}^H$ of length 0. By induction hypothesis there exists $\mathbb{A}^H \subset W_i\mathbb{V}^H$ with underlying local system $\mathbb{L} \cap W_i\mathbb{V}^H$. Moreover, as $W_i\mathbb{V}^H$ is pure, it splits into a direct sum $W_i\mathbb{V}^H = \mathbb{A}^H \oplus (\mathbb{A}^H)^\perp$.

The extension $\mathbb{V}^H$ is given by a class

$$\begin{aligned} \beta_{\mathbb{V}^H} &\in \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathrm{adm}}}(\mathbb{V}^H/W_i\mathbb{V}^H, W_i\mathbb{V}^H) \\ &= \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathrm{adm}}}(\mathbb{V}^H/W_i\mathbb{V}^H, \mathbb{A}^H) \oplus \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathrm{adm}}}(\mathbb{V}^H/W_i\mathbb{V}^H, (\mathbb{A}^H)^\perp) \end{aligned} \tag{11.5}$$

hence decomposes uniquely as $\beta_{\mathbb{V}^H} = \alpha \oplus \alpha^\perp$ for $\alpha \in \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathrm{adm}}}(\mathbb{V}^H/W_i\mathbb{V}^H, \mathbb{A}^H)$ and $\alpha^\perp \in \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathrm{adm}}}(\mathbb{V}^H/W_i\mathbb{V}^H, (\mathbb{A}^H)^\perp)$.

The class α defines an extension in $\mathrm{VMHS}(S)_{\mathrm{adm}}$

$$0 \to \mathbb{A}^H \to \mathbb{L}^H \to \mathbb{V}^H/W_i\mathbb{V}^H \to 0 \tag{11.6}$$

endowed with a natural embedding $\mathbb{L}^H \hookrightarrow \mathbb{V}^H$, with underlying local system $\mathbb{L} \hookrightarrow \mathbb{V}$. $\square$

As the reader will have noticed, Proposition 11.2 follows entirely from Deligne's result in the pure case and the classical following

Lemma 11.3. *The category* $\mathrm{VMHS}(S)_{\mathrm{adm}}$ *is abelian.*

Proof. Surprisingly enough the result does not seem explicitly stated in the literature, although it is certainly used.

It is a direct consequence of M.Saito's theory of mixed Hodge modules. Indeed the category $\mathrm{MHM}(S)$ of mixed Hodge modules is abelian. As admissible variations of mixed Hodge structure on S coincide with mixed Hodge modules on S whose underlying perverse sheaf is a (shifted) local system, the result follows immediately.

Alternatively, and as mentioned to us by C.Schnell, we can avoid the use of mixed Hodge module and appeal to Kashiwara's results in [Kas86]. Indeed, the category of graded-polarizable VMHS on S is clearly polarizable. Hence one only need to check the admissibility of the kernel (resp. the image) of a morphism with target (resp. source) an admissible VMHS. One is reduced to this question near the origin on a disk. The extension of the Hodge filtration and the existence of the relative weight filtration can both be checked on the associated infinitesimal mixed Hodge module (IMHM), see [Kas86, Section 4.5]. Hence they hold true by [Kas86, Prop.5.2.6]. □

12. End of the proof

In this section, we prove Theorem 8.2, which finishes the proof of Theorem 1.1.

Let $\mathcal{Z}$ as in Theorem 8.2. If $\dim H_{\mathcal{Z}^{\mathrm{Zar}}} = 0$ then we are done by Proposition 9.1. Thus we may assume $\dim H_{\mathcal{Z}^{\mathrm{Zar}}} > 0$. For simplicity we write $H := H_{\mathcal{Z}^{\mathrm{Zar}}}$.

Proposition 11.1 says that $H \lhd P$. Thus we can take the quotient $\mathcal{D}/H$ and obtain

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[r, "p_H"] \arrow[d, "u"'] & \mathcal{D}/H \arrow[d, "u_{/H}"] \\
S \arrow[r, "{[\Phi]}"] \arrow[rr, bend right, "{[\Phi_{/H}]}"'] & \Gamma_P\backslash\mathcal{D} \arrow[r, "{[p_H]}"] & \Gamma_{P/H}\backslash(\mathcal{D}/H)
\end{tikzcd} \quad . \tag{12.1}
$$

We can apply Proposition 9.1 to the new period map $[\Phi_{/H}]\colon S \to \Gamma_{P/H}\backslash(\mathcal{D}/H)$ and

$$\mathcal{Z}_{/H} := (\mathrm{id}_S, p_H)(\mathcal{Z}) \subseteq S \times_{\Gamma_{P/H}\backslash(\mathcal{D}/H)} (\mathcal{D}/H).$$

But $H = H_{\mathcal{Z}^{\mathrm{Zar}}}$ is the $\mathbb{Q}$-stabilizer of $\mathcal{Z}^{\mathrm{Zar}}$, so the $\mathbb{Q}$-stabilizer of $\mathcal{Z}_{/H}^{\mathrm{Zar}}$ must be 1. Thus Proposition 9.1 implies

$$\dim \mathcal{Z}_{/H}^{\mathrm{Zar}} - \dim \mathcal{Z}_{/H} \geq \dim p_{\mathcal{D}/H}(\mathcal{Z}_{/H})^{\mathrm{ws}}, \tag{12.2}$$

where $p_{\mathcal{D}/H}\colon S \times \mathcal{D}/H \to \mathcal{D}/H$ is the natural projection.

Let $\mathcal{R}_u(H)$ be the unipotent radical of H. As $H(\mathbb{R})^+\mathcal{R}_u(H)(\mathbb{C})\mathcal{Z}^{\mathrm{Zar}} = \mathcal{Z}^{\mathrm{Zar}}$, we have (for any $\widetilde{s} \in \widetilde{S}$)

$$\dim \mathcal{Z}^{\mathrm{Zar}} = \dim \mathcal{Z}_{/H}^{\mathrm{Zar}} + \dim H(\mathbb{R})^+\mathcal{R}_u(H)(\mathbb{C})\widetilde{s} \tag{12.3}$$

and

$$\dim p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{ws}} = \dim p_{\mathcal{D}/H}(\mathcal{Z}_{/H})^{\mathrm{ws}} + \dim H(\mathbb{R})^+\mathcal{R}_u(H)(\mathbb{C})\widetilde{s}. \tag{12.4}$$

By (12.2), (12.3) and (12.4), we then have

$$\dim \mathcal{Z}^{\mathrm{Zar}} - \dim \mathcal{Z}_{/H} \geq \dim p_{\mathcal{D}}(\mathcal{Z})^{\mathrm{ws}}. \tag{12.5}$$

So it remains to prove $\dim \mathcal{Z} = \dim \mathcal{Z}_{/H}$. Hence it remains to prove that each fiber of

$$(\mathrm{id}_S, p_H)\colon S \times_{\Gamma_P\backslash\mathcal{D}} \mathcal{D} \to S \times_{\Gamma_{P/H}\backslash(\mathcal{D}/H)} (\mathcal{D}/H)$$

is at most a countable set. This is true: Suppose $(s_1, \widetilde{x}_1)$ and $(s_2, \widetilde{x}_2)$ are in the same fiber, then $s_1 = s_2$. But any point $(s, \widetilde{x}) \in S \times_{\Gamma_P\backslash\mathcal{D}} \mathcal{D}$ satisfies $[\Phi](s) = u(\widetilde{x})$. So we have $u(\widetilde{x}_1) = u(\widetilde{x}_2)$, and hence $\widetilde{x}_1 \in \Gamma_P \widetilde{x}_2$. So each fiber of the map (id_S, p_H) above is contained in a Γ_P-orbit, and thus is at most a countable set.

Appendix A. Basic knowledge on Mumford–Tate domains

A.1. **Some fundamental properties of Mumford–Tate domains.** The goal of this subsection is to prove Proposition 2.6 and Corollary 2.8.

Let V be a finite-dimensional $\mathbb{Q}$-vector space, and let $\mathcal{M}$ be the classifying space of $\mathbb{Q}$-mixed Hodge structures constructed in Section 2.2. We have seen that $\mathcal{M}$ is a homogeneous space under $P^{\mathcal{M}}(\mathbb{R})^+ W_{-1}^{\mathcal{M}}(\mathbb{C})$ for the $\mathbb{Q}$-algebraic group $P^{\mathcal{M}}$ constructed in (2.3) and $W_{-1}^{\mathcal{M}} = \mathcal{R}_u(P^{\mathcal{M}})$.

Let $h \in \mathcal{M}$. Recall that the adjoint Hodge structure on $\mathrm{Lie}\, P^{\mathcal{M}}$ defined by h has weight ≤ 0 by part (iii) of Proposition 2.3. The following lemma is a rephrase of [Pea00, Thm. 3.13].

Lemma A.1. *The tangent space $T_h\mathcal{M}$ can be canonically identified with*

$$\bigoplus_{r<0,\ r+s\leq 0} (\mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}})^{r,s} = \bigoplus_{r<0} (\mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}})^{r,s}.$$

With this lemma, we are ready to prove Proposition 2.6.

Proof of Proposition 2.6. Let $\mathcal{D} = P(\mathbb{R})^+ W_{-1}(\mathbb{C}) h$ be a Mumford–Tate domain contained in $\mathcal{M}$, where $P = \mathrm{MT}(h)$ and $W_{-1} = \mathcal{R}_u(P)$.

Because $\mathcal{D}$ and $\mathcal{M}$ are homogeneous spaces, to prove that $\mathcal{D}$ is a complex submanifold of $\mathcal{M}$ it suffices to prove that $T_h\mathcal{D}$ is a complex subspace of $T_h\mathcal{M}$.

$\mathrm{Lie}\, P$ is a sub-Hodge structure of $\mathrm{Lie}\, P^{\mathcal{M}}$ for the adjoint Hodge structure on $\mathrm{Lie}\, P^{\mathcal{M}}$ induced by h. So $F^0 \mathrm{Lie}\, P_{\mathbb{C}} = F^0 \mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}} \cap \mathrm{Lie}\, P_{\mathbb{C}}$. By Lemma A.1, the complex structure on $T_h\mathcal{M}$ is given by

$$\mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}} / F^0 \mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}} = \bigoplus_{r<0} (\mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}})^{r,s}.$$

Thus $T_h\mathcal{D} = \mathrm{Lie}\, P_{\mathbb{C}} / \left(F^0 \mathrm{Lie}\, P^{\mathcal{M}}_{\mathbb{C}} \cap \mathrm{Lie}\, P_{\mathbb{C}}\right) = \mathrm{Lie}\, P_{\mathbb{C}} / F^0 \mathrm{Lie}\, P_{\mathbb{C}}$ is a complex subspace of $T_h\mathcal{M}$. Thus we can conclude that $\mathcal{D}$ is a complex submanifold of $\mathcal{M}$. Moreover we have shown that

$$T_h\mathcal{D} = \bigoplus_{r<0} (\mathrm{Lie}\, P_{\mathbb{C}})^{r,s}. \tag{A.1}$$

The proof for weak Mumford–Tate domains is the similar. The only new input is to prove that $\mathrm{Lie}\, N$ is a sub-Hodge structure of $\mathrm{Lie}\, P^{\mathcal{M}}$ for the normal subgroup N of $P := \mathrm{MT}(h)$ from Definition 2.5.(2). This is true because the adjoint action of P on $\mathrm{Lie}\, P$ leaves $\mathrm{Lie}\, N$ stable (since $N \lhd P$), and the adjoint action $\mathrm{Ad}\colon P \to \mathrm{GL}(\mathrm{Lie}\, P)$ is precisely the restriction of $\mathrm{Ad}^{\mathcal{M}}\colon P^{\mathcal{M}} \to \mathrm{GL}(\mathrm{Lie}\, P^{\mathcal{M}})$ restricted to P (which leaves $\mathrm{Lie}\, P$ stable). $\square$

Next we turn to the Mumford–Tate group $\mathrm{MT}(h)$. For $m, n \in \mathbb{Z}_{\geq 0}$, denote by $T^{m,n}V := V^{\otimes m} \otimes (V^\vee)^{\otimes n}$. Then h induces a $\mathbb{Q}$-mixed Hodge structure on $T^{m,n}V$, whose weight filtration we denote by $W_\bullet$ and Hodge filtration we denote by $F^\bullet$.

The elements of $(T^{m,n}V_{\mathbb{C}})^{0,0} \cap T^{m,n}V = F^0(T^{m,n}V_{\mathbb{C}}) \cap W_0(T^{m,n}V)$, with m and n running over all non-negative integers, are called the *Hodge tensors* for h. Denote by Hdg_h the set of all Hodge tensors for h.

The following result is proved by André [And92, Lem. 2.(a)], with pure case by Deligne.

Lemma A.2. *We have*

(i) *Any element in some* $T^{m,n}V$ *fixed by* $\mathrm{MT}(h)(\mathbb{Q})$ *is a Hodge tensor for* h*;*
(ii) $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_h)$.

By dimension reasons, Lemma A.2.(ii) has the following consequence.

Corollary A.3. *There exists a finite set* $\mathfrak{I} \subseteq \mathrm{Hdg}_h$ *such that* $\mathrm{MT}(h) = Z_{\mathrm{GL}(V)}(\mathfrak{I})$.

Now we are ready to characterize Mumford–Tate domains contained in $\mathcal{M}$ as irreducible components of *Hodge loci*.

Definition A.4. For each $h \in \mathcal{M}$, the **Hodge locus** at h is defined as

$$\mathrm{HL}(h) = \{h' \in \mathcal{M} : \mathrm{Hdg}_h \subseteq \mathrm{Hdg}_{h'}\}. \tag{A.2}$$

Lemma A.5. *We have*

(i) $\mathrm{HL}(h) = \{h' \in \mathcal{M} : \mathrm{MT}(h') < \mathrm{MT}(h)\}$.
(ii) $\mathrm{HL}(h) = \{h' \in \mathcal{M} : \mathfrak{I} \subseteq \mathrm{Hdg}_{h'}\}$ *where* $\mathfrak{I}$ *is the finite set from Corollary A.3.*

Proof. (i) The inclusion $\subseteq$ is clear by Lemma A.2.(ii). Conversely suppose $\mathrm{MT}(h') < \mathrm{MT}(h)$. Then any $t \in \mathrm{Hdg}_h$ is fixed by $\mathrm{MT}(h)$ by Lemma A.2.(ii), and so is also fixed by $\mathrm{MT}(h')$, and thus is a Hodge tensor for h' by Lemma A.2.(i). Therefore $\mathrm{Hdg}_h \subseteq \mathrm{Hdg}_{h'}$.

(ii) We first prove the inclusion $\subseteq$. Let $h' \in \mathrm{HL}(h)$. By Corollary A.3 and (i), each $t \in \mathfrak{I}$ is fixed by $\mathrm{MT}(h')(\mathbb{Q})$, and hence is a Hodge tensor for h' by Lemma A.2.(i). So $\mathfrak{I} \subseteq \mathrm{Hdg}_{h'}$. This proves the desired inclusion.

Conversely suppose that $h' \in \mathcal{M}$ satisfies $\mathfrak{I} \subseteq \mathrm{Hdg}_{h'}$. Then $Z_{\mathrm{GL}(V)}(\mathrm{Hdg}_{h'}) \subseteq Z_{\mathrm{GL}(V)}(\mathfrak{I})$. Thus $\mathrm{MT}(h') < \mathrm{MT}(h)$ by Lemma A.2.(ii) and Corollary A.3. So $h' \in \mathrm{HL}(h)$ by part (i) of the current lemma. This proves the inclusion $\supseteq$. Now we are done. □

By Lemma A.5.(ii), $\mathrm{HL}(h)$ is the complex analytic subvariety of $\mathcal{M}$ which parametrizes $\mathbb{Q}$-mixed Hodge structures (satisfying the properties (1)-(3) in Section 2.2) together with the Hodge tensors in the finite set $\mathfrak{I}$.

Proposition A.6. *Let* $h \in \mathcal{M}$*, with* $P = \mathrm{MT}(h)$ *and* $W_{-1} = \mathcal{R}_u(P)$*. Then* $P(\mathbb{R})^+W_{-1}(\mathbb{C})h$ *is the complex analytic irreducible component of* $\mathrm{HL}(h)$ *passing through* h*.*

Proof. The proof is simply [CMSP17, Prop. 17.1.2] adapted to the mixed case. For completeness we include it here.

Denote by $\mathcal{D} = P(\mathbb{R})^+W_{-1}(\mathbb{C})h$. Each $h' \in \mathcal{D}$ equals $g \cdot h$ for some $g \in P(\mathbb{R})^+W_{-1}(\mathbb{C})$, and hence the homomorphism $h' \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $gP_{\mathbb{C}}g^{-1} = P_{\mathbb{C}}$. Thus $\mathrm{MT}(h') < P$. So Lemma A.5.(i) implies $h' \in \mathrm{HL}(h)$ for each $h' \in \mathcal{D}$. Therefore

$$\mathcal{D} \subseteq \mathrm{HL}(h). \tag{A.3}$$

Next we study $T_h(\mathrm{HL}(h)) \subseteq T_h\mathcal{M} = \bigoplus_{r<0}(\mathrm{Lie}\,P_{\mathbb{C}}^{\mathcal{M}})^{r,s}$; see Lemma A.1 for the last equality. By (A.3) and (A.1), to prove the proposition it suffices to prove

$$T_h(\mathrm{HL}(h)) \subseteq \bigoplus_{r<0}(\mathrm{Lie}\,P_{\mathbb{C}})^{r,s}. \tag{A.4}$$

Indeed the action of $P^{\mathcal{M}}$ on $T^{m,n}V$ induces an action of $T_h\mathcal{M}$ on $T^{m,n}V$ in the following way: $\xi \cdot t = \frac{\mathrm{d}}{\mathrm{d}u}(e^{u\xi} \cdot t)|_{u=0}$, for $\xi \in T_h\mathcal{M} = \bigoplus_{r<0}(\mathrm{Lie}\,P_{\mathbb{C}}^{\mathcal{M}})^{r,s}$ and $t \in T^{m,n}V$. Then for any vector $\xi \in T_h\mathcal{M} = \bigoplus_{r<0}(\mathrm{Lie}\,P_{\mathbb{C}}^{\mathcal{M}})^{r,s}$, we have

$$\xi \in T_h(\mathrm{HL}(h)) \Leftrightarrow \xi \cdot t \in \mathrm{Hdg}_h \text{ for each } t \in \mathrm{Hdg}_h. \tag{A.5}$$

Now take $\xi \in T_h(\mathrm{HL}(h))$. Suppose $t \in T := T^{m,n}V$ is a Hodge tensor, namely $t \in F^0T_{\mathbb{C}} \cap W_0T \subseteq T_{\mathbb{C}}^{0,0}$.[7] Then (A.5) implies $\xi \cdot t \in F^0T_{\mathbb{C}} \cap W_0T \subseteq T_{\mathbb{C}}^{0,0}$. On the other hand $\xi \in \bigoplus_{r<0}(\mathrm{Lie}\, P_{\mathbb{C}}^{\mathcal{M}})^{r,s}$. Write $\xi = \sum_{r<0} \xi^{r,s}$. Then $\xi \cdot t = \sum_{r<0} \xi^{r,s} \cdot t \in \bigoplus_{r<0} T_{\mathbb{C}}^{r,s}$. Thus $\xi \cdot t \in T^{0,0} \cap \bigoplus_{r<0} T_{\mathbb{C}}^{r,s} = 0$. In summary

$$\xi \in T_h(\mathrm{HL}(h)) \Rightarrow \xi \cdot t = 0 \text{ for each } t \in \mathrm{Hdg}_h. \tag{A.6}$$

But part (ii) of Lemma A.2 implies that $\{\xi \in \mathrm{Lie}\, P_{\mathbb{C}}^{\mathcal{M}} : \xi \cdot t = 0 \text{ for each } t \in \mathrm{Hdg}_h\} \subseteq \mathrm{Lie}\, P_{\mathbb{C}}$ with $P = \mathrm{MT}(h)$. Thus $T_h(\mathrm{HL}(h)) \subseteq \mathrm{Lie}\, P_{\mathbb{C}}$. So

$$T_h(\mathrm{HL}(h)) \subseteq \mathrm{Lie}\, P_{\mathbb{C}} \cap \oplus_{r>0}(\mathrm{Lie}\, P_{\mathbb{C}}^{\mathcal{M}})^{r,s} = \oplus_{r>0}(\mathrm{Lie}\, P_{\mathbb{C}})^{r,s}.$$

This is precisely (A.4). Hence we are done. $\square$

Now by Proposition A.6 and Lemma A.5.(ii), the Mumford–Tate domains contained in $\mathcal{M}$ are precisely the complex irreducible components of the moduli spaces parametrizing $\mathbb{Q}$-mixed Hodge structures (satisfying the properties (1)-(3) in Section 2.2) together with a finite number of Hodge tensors.

Proof of Lemma 2.7. This is an immediate consequence of the moduli interpretation of Mumford–Tate domains above. $\square$

Another application is as follows.

Corollary A.7. *There are at most countably many Mumford–Tate domains in $\mathcal{M}$.*

Proof. We have the moduli interpretation of Mumford–Tate domains above. On the other hand, every complex analytic variety has at most countably many irreducible components, and by definition there are countably many Hodge tensors. Hence there are at most countably many Mumford–Tate domains contained in $\mathcal{M}$. $\square$

This allows to prove a stronger version of Corollary 2.8.

Lemma A.8. *Let $\mathcal{Z}$ be a complex analytic irreducible subvariety of $\mathcal{M}$. Let $P = \mathrm{MT}(\mathcal{Z})$ be the generic Mumford–Tate group of $\mathcal{Z}$. Then $\mathcal{Z}^{\mathrm{sp}}$, the smallest Mumford–Tate domain which contains $\mathcal{Z}$, is precisely $P(\mathbb{R})^+W_{-1}(\mathbb{C})h$ for some $h \in \mathcal{Z}$, where $W_{-1} = \mathcal{R}_u(P)$.*

Proof. Denote by $\mathcal{Z}^o$ the set of Hodge generic points in $\mathcal{Z}$. Then $\mathcal{Z}^o$ is the complement of the union of countably many proper complex analytic irreducible subvarieties of $\mathcal{Z}$. In particular, $\mathcal{Z}^o$ is irreducible since $\mathcal{Z}$ is.

It is clearly true that $\mathcal{Z}^o \subseteq \bigcup_{h \in \mathcal{Z}^o} P(\mathbb{R})^+W_{-1}(\mathbb{C})h$. Each member in the union is by definition a Mumford–Tate domain, and hence the union is at most a countable union by Corollary A.7. Moreover two $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-orbits either coincide or are disjoint. So $\mathcal{Z}^o$ is contained in a countable disjoint union of some $P(\mathbb{R})^+W_{-1}(\mathbb{C})$-orbits. But $\mathcal{Z}^o$ is irreducible, so it is contained some member in the union. Thus $\mathcal{Z}^o \subseteq P(\mathbb{R})^+W_{-1}(\mathbb{C})h$ for some $h \in \mathcal{Z}^o$. But then $\mathcal{Z} \subseteq P(\mathbb{R})^+W_{-1}(\mathbb{C})h$. Hence we are done. $\square$

Now we are ready to prove Lemma 2.10.

Proof of Lemma 2.10. By assumption $\mathcal{D} = P(\mathbb{R})^+W_{-1}(\mathbb{C})h$. From now on we fix $h' \in \mathcal{D}$ Hodge generic, namely $\mathrm{MT}(h') = \mathrm{MT}(\mathcal{D})$.

By Lemma A.8 we have

$$P(\mathbb{R})^+W_{-1}(\mathbb{C})h' = \mathcal{D} \subseteq \mathrm{MT}(\mathcal{D})(\mathbb{R})^+\mathcal{R}_u(\mathrm{MT}(\mathcal{D}))(\mathbb{C})h'. \tag{A.7}$$

[7]Here the notation $T_{\mathbb{C}}^{0,0}$ means the $(0,0)$-constituent for the bi-grading of T given by Proposition 2.2.

Let us prove $\mathrm{MT}(\mathcal{D}) < P$. Indeed, each point $h' \in \mathcal{D}$ is of the form $g \cdot h$ for some $g \in P(\mathbb{R})^+ W_{-1}(\mathbb{C})$. The homomorphism $h' = g \cdot h \colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ factors through $gh(\mathbb{S}_{\mathbb{C}})g^{-1} \subseteq gP_{\mathbb{C}}g^{-1} = P_{\mathbb{C}}$. Hence $\mathrm{MT}(h') < P$ for all $h' \in \mathcal{D}$. So $\mathrm{MT}(\mathcal{D}) < P$.

Next we show that $\mathrm{MT}(\mathcal{D})$ is normal in P. Indeed for any $g \in P(\mathbb{Q})$, we have

$$\mathrm{MT}(\mathcal{D}) \supseteq \mathrm{MT}(g \cdot h') = g\mathrm{MT}(h')g^{-1} = g\mathrm{MT}(\mathcal{D})g^{-1}.$$

By comparing dimensions, we have $\mathrm{MT}(\mathcal{D}) = g\mathrm{MT}(\mathcal{D})g^{-1}$. By letting g run over elements in $P(\mathbb{Q})$, we get $\mathrm{MT}(\mathcal{D}) \lhd P$. In particular $\mathcal{R}_u(\mathrm{MT}(\mathcal{D})) = W_{-1} \cap \mathrm{MT}(\mathcal{D})$.

This implies

$$\mathrm{MT}(\mathcal{D})(\mathbb{R})^+ \mathcal{R}_u(\mathrm{MT}(\mathcal{D}))(\mathbb{C})h' \subseteq P(\mathbb{R})^+ W_{-1}(\mathbb{C})h'. \tag{A.8}$$

Thus $\mathcal{D} = \mathrm{MT}(h')(\mathbb{R})^+ \mathcal{R}_u(\mathrm{MT}(h'))(\mathbb{C})h'$ by (A.7) and (A.8). So $\mathcal{D}$ is a Mumford–Tate domain. □

Appendix B. Underlying group

Let $\mathcal{D}$ be a Mumford–Tate domain in some classifying space $\mathcal{M}$ with $P = \mathrm{MT}(\mathcal{D})$. Each $h \in \mathcal{D}$ defines an adjoint Hodge structure on $\mathrm{Lie}\, P$. Write $W_\bullet$ for the weight filtration. By property (iii) of Proposition 2.3 $W_\bullet$ does not depend on the choice of $h \in \mathcal{D}$ and satisfies $W_0(\mathrm{Lie}\, P) = \mathrm{Lie}\, P$ and $W_{-1} = \mathcal{R}_u(P)$.

The weight filtration $0 = W_{-m-1}(\mathrm{Lie}\, P) \subseteq W_{-m}(\mathrm{Lie}\, P) \subseteq \cdots \subseteq W_{-1}(\mathrm{Lie}\, P)$ defines a sequence of connected subgroups

$$0 = W_{-(m+1)} \subseteq W_{-m} \subseteq \cdots \subseteq W_{-1} \tag{B.1}$$

of P. Each W_{-k}, $k \in \{1, \ldots, m\}$, is a normal unipotent subgroup of P.

Write as before $G = P/W_{-1}$ the reductive part of P. We wish to reconstruct P from G and the W_{-k}'s.

Let us start with the unipotent radical W_{-1}.

Lemma B.1. (a) *For each $k \in \{1, \ldots, m\}$, $W_{-k}/W_{-(k+1)}$ is a vector group.*

(b) *There is an isomorphism of $\mathbb{Q}$-algebraic varieties*

$$\begin{array}{rcl} W_{-1} & \to & (W_{-1}/W_{-2}) \times \cdots \times (W_{-(m-1)}/W_{-m}) \times W_{-m} \\ \mathbf{w} & \mapsto & (w_1, \cdots, w_{m-1}, w_m) \end{array} . \tag{B.2}$$

Proof. We first prove (a). For each $k \in \{1, \ldots, m\}$, the algebraic group $W_{-k}/W_{-(k+1)}$ is unipotent since W_{-k} is unipotent. On the other hand $[\mathrm{Lie}\, W_{-k}, \mathrm{Lie}\, W_{-k}] \subseteq W_{-2k}$ by reason of weight, and $W_{-2k} \subseteq W_{-(k+1)}$ as $k \geq 1$. Thus $\mathrm{Lie}\, W_{-k}/W_{-(k+1)}$ is a commutative Lie algebra, hence $W_{-k}/W_{-(k+1)}$ is an abelian algebraic group. Finally the algebraic group $W_{-k}/W_{-(k+1)}$ is a vector group as it is abelian and unipotent.

We now turn to the description of the isomorphism (B.2). As W_{-1} is unipotent, the exponential map $\exp\colon \mathrm{Lie}\, W_{-1} \to W_{-1}$ is an isomorphism of $\mathbb{Q}$-algebraic varieties.

Fix an isomorphism of $\mathbb{Q}$-vector spaces $\mathrm{Lie}\, W_{-1} \simeq \bigoplus_{j=1}^m \mathrm{Lie}\, W_{-j}/W_{-(j+1)}$. As part (a) provides a canonical identification of $\mathbb{Q}$-algebraic varieties $\mathrm{Lie}\, W_{-k}/W_{-(k+1)} = W_{-k}/W_{-(k+1)}$ between a vector group and its Lie algebra, we get the desired isomorphism (B.2) by

$$W_{-1} \xleftarrow[\sim]{\exp} \mathrm{Lie}\, W_{-1} = \bigoplus_{j=1}^m \mathrm{Lie}(W_{-j}/W_{-(j+1)}) = \prod_{j=1}^m W_{-j}/W_{-(j+1)}. \qquad \square$$

Notice that this isomorphism (B.2) is not canonical. In this paper, we fix such an isomorphism once and for all.

Next we give the formula for the group law on W_{-1} under this identification given by (B.2).

Definition B.2. For $k \in \{1, \cdots, m\}$ we define the k-truncation $\mathbf{w}_k \in W_{-1}/W_{-k-1} \simeq \prod_{j=1}^{k} W_{-j}/W_{-(j+1)}$ of an element $\mathbf{w} \in W_{-1}$ as follows. If $\mathbf{w} = (w_1, \cdots, w_{m-1}, w_m)$ under the identification (B.2), then $\mathbf{w}_k = (w_1, \cdots, w_k)$.

Lemma B.3. *For each $k \geq 2$, there exists a polynomial map*

$$\mathrm{calb}_k \colon W_{-1}/W_{-k-1} \times W_{-1}/W_{-k-1} \to W_{-k}/W_{-k-1}$$

of degree at most $k-1$ and constant term 0 such that for any $\mathbf{w}, \mathbf{w}' \in W_{-1}$, their product is given under the identification (B.2) by

$$\text{(B.3)} \quad \mathbf{w} \cdot \mathbf{w}' = (w_1 + w_1', w_2 + w_2' + \mathrm{calb}_2(\mathbf{w}_1, \mathbf{w}_1'), \ldots, w_m + w_m' + \mathrm{calb}_m(\mathbf{w}_{m-1}, \mathbf{w}_{m-1}')).$$

Proof. Let $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{w}' = (w_1', \ldots, w_m')$ under (B.2). The Baker–Campbell–Hausdorff formula gives:
(B.4)

$$\mathbf{w} \cdot \mathbf{w}' = \exp\left((w_1, \ldots, w_m) + (w_1', \ldots, w_m') + \frac{1}{2}[(w_1, \ldots, w_m), (w_1', \ldots, w_m')] + \ldots \right),$$

where all operations in the exponential are taken in $\mathrm{Lie}\, W_{-1}$, and the sum is finite as $\mathrm{Lie}\, W_{-1}$ is nilpotent. Noticing that

$$[W_{-k}/W_{-(k+1)}, W_{-k'}/W_{-(k'+1)}] \subseteq W_{-(k+k')}/W_{-(k+k'+1)},$$

one can rewrite (B.4) as

$$\mathbf{w} \cdot \mathbf{w}' = \exp\left((w_1 + w_1', w_2 + w_2' + \mathrm{calb}_2(\mathbf{w}_1, \mathbf{w}_1'), \ldots, w_m + w_m' + \mathrm{calb}_m(\mathbf{w}_{m-1}, \mathbf{w}_{m-1}')) \right),$$

with polynomials calb_k for each $k \geq 2$ as required by the lemma. □

The next lemma explains how $G = P/W_{-1}$ acts on $W_{-1} = \mathcal{R}_u(P)$ under the identification (B.2).

Lemma B.4. *For each $k \geq 1$, $W_{-k}/W_{-(k+1)}$ is a G-module. Moreover this G-module structure is induced by the action of G on W_{-1}.*

As a consequence, for each $g_0 \in G$ and $\mathbf{w} = (w_1, \ldots, w_m) \in W_{-1}$ under (B.2), we have

$$\text{(B.5)} \qquad g_0 \cdot \mathbf{w} = (g_0 w_1, \ldots, g_0 w_m).$$

Proof. As $\mathrm{Gr}_0^{W_\bullet}(\mathrm{Lie}\, P) = \mathrm{Lie}\, G$ and $W_{-k}(\mathrm{Lie}\, P) = \mathrm{Lie}\, W_{-k}$ for each $k \geq 1$, we have $[\mathrm{Lie}\, G, \mathrm{Lie}\, W_{-k}] \subseteq \mathrm{Lie}\, W_{-k}$. Hence the action of G on W_{-1} preserves W_{-k} for each $k \geq 1$, and hence furthermore induces an action on $W_{-k}/W_{-(k+1)}$ which is a $\mathbb{Q}$-vector space. This concludes the lemma. □

We are now ready to state the result to reconstruct P from G and the W_{-k}'s. First let us fix a Levi decomposition $P = W_{-1} \rtimes G$.

Proposition B.5. *The fixed Levi decomposition $P = W_{-1} \rtimes G$ and the fixed isomorphism (B.2) together induce an isomorphism as algebraic varieties defined over $\mathbb{Q}$*

$$\text{(B.6)} \qquad P \simeq G \times (W_{-1}/W_{-2}) \times \cdots (W_{-(m-1)}/W_{-m}) \times W_{-m}.$$

The group law on the right hand side of (B.6) is given as follows. Let $(g_0, w_1, \ldots, w_m)$ and $(g_0', w_1', \ldots, w_m')$ be two elements in P under the identification (B.6). Denote by $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{w}' = (w_1', \ldots, w_m')$. Then

$$\text{(B.7)} \quad (g_0, \mathbf{w}) \cdot (g_0', \mathbf{w}') = (g_0 g_0', w_1 + g_0 w_1', w_2 + g_0 w_2' + \mathrm{calb}_2(w_1, g_0 w_1'), \ldots, w_m + g_0 w_m' + \mathrm{calb}_m(\mathbf{w}_{m-1}, g_0 \mathbf{w}_{m-1}'))$$

where $\mathrm{calb}_2, \ldots, \mathrm{calb}_m$ are the $\mathbb{Q}$-polynomials from Lemma B.3, $\mathbf{w}_k$ (resp. $\mathbf{w}_k'$) is the k-th truncation as in Lemma B.3, and $g_0 \mathbf{w}_k' = (g_0 w_1', \ldots, g_0 w_k')$ for each $k \geq 1$.

Proof. (B.6) follows directly from the fixed Levi decomposition and (B.2).

To prove (B.7), first note that $(g_0, \mathbf{w}) = (1, \mathbf{w}) \cdot (g_0, 0)$ for $P = W_{-1} \rtimes G$. Similarly $(g_0', \mathbf{w}') = (1, \mathbf{w}') \cdot (g_0', 0)$. So

$$\begin{aligned}
(g_0, \mathbf{w}) \cdot (g_0', \mathbf{w}') &= (1, \mathbf{w}) \cdot \big((g_0, 0) \cdot (1, \mathbf{w}')\big) \cdot (g_0', 0) \\
&= (1, \mathbf{w}) \cdot (g_0, g_0 \cdot \mathbf{w}') \cdot (g_0', 0) \\
&= (1, \mathbf{w}) \cdot \big((1, g_0 \cdot \mathbf{w}') \cdot (g_0, 0)\big) \cdot (g_0', 0) \\
&= (1, w_1, \ldots, w_m) \cdot (1, g_0 w_1', \ldots, g_0 w_m') \cdot (g_0, 0) \cdot (g_0', 0) \quad \text{by (B.5)} \\
&= (1, w_1 + g_0 w_1', w_2 + g_0 w_2' + \mathrm{calb}_2(w_1, g_0 w_1'), \ldots, \\
&\qquad w_m + g_0 w_m' + \mathrm{calb}_m(\mathbf{w}_{m_1}, g_0 \mathbf{w}_{m-1}')) \cdot (g_0 g_0', 0) \quad \text{by (B.3)} \\
&= (g_0 g_0', w_1 + g_0 w_1', w_2 + g_0 w_2' + \mathrm{calb}_2(w_1, g_0 w_1'), \ldots, \\
&\qquad\qquad w_m + g_0 w_m' + \mathrm{calb}_m(\mathbf{w}_{m_1}, g_0 \mathbf{w}_{m-1}')). \qquad \square
\end{aligned}$$

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