ON THE GEOMETRIC ZILBER–PINK THEOREM AND THE LAWRENCE–VENKATESH METHOD

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Abstract. Using our recent results on the algebraicity of the Hodge locus for variations of Hodge structures of level at least 3, we improve the results of Lawrence-Venkatesh in direction of the refined Bombieri–Lang conjecture.

The aim of this short note is to explain how the *Geometric Zilber–Pink conjecture*, recently established by the authors [2], can be used to improve the main result of Lawrence and Venkatesh [7], giving a special case of the *refined Bombieri–Lang conjecture.* We prove that there is a closed strict algebraic subvariety E defined over $\mathbb{Z}$ of $U_{n,d}$, the parameter space of smooth hypersurfaces of degree d in $\mathbf{P}^{n+1}$, such that

$$(U_{n,d} - E)\,(\mathcal{O}_{K,S})$$

is finite, for every finitely generated $\mathbb{Q}$-field K and every finite set of places S of K, as soon as n and d are "big enough" (see the condition (2.0.1)).

We first recall in Section 1 (a special case of) the Geometric Zilber–Pink conjecture mentioned above, which is a purely geometric result; and then in Section 2 the Lawrence-Venkatesh method, which is of arithmetic nature. In Section 3 we explain what can be obtained by combining the two results.

1. The geometry of the Hodge locus

Let $f : X \to S$ be a smooth projective morphism of smooth irreducible complex quasi-projective varieties, of relative dimension n. The primitive Betti cohomology $H^n(X_s^{\mathrm{an}}, \mathbb{Z})_{\mathrm{prim}}$ of the fibres X_s, $s \in S(\mathbb{C})$, form a polarized $\mathbb{Z}$-variation of Hodge structures $\mathbb{V}$ on the complex manifold S^{an}, described by a complex analytic period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ (we refer for instance to [2] for more details on period maps). Motivated by the study of the Hodge conjecture for the fibres of f, one defines the Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ as the locus of points $s \in S^{\mathrm{an}}$ for which the Hodge structure $H^n(X_s^{\mathrm{an}}, \mathbb{Q})_{\mathrm{prim}}$ admits more *Hodge tensors* than the primitive cohomology of the very general fibre. Here a Hodge class of a pure $\mathbb{Z}$-Hodge structure $V = (V_{\mathbb{Z}}, F^{\bullet})$ is a class in $V_{\mathbb{Q}}$ whose image in $V_{\mathbb{C}}$ lies in the zeroth piece $F^0V_{\mathbb{C}}$ of the Hodge filtration, or equivalently a morphism of Hodge structures $\mathbb{Q}(0) \to V_{\mathbb{Q}}$; and a Hodge tensor for V is a Hodge class in $V^{\otimes} := \bigoplus_{a,b\in\mathbb{N}} V^{\otimes a} \otimes (V^{\vee})^{\otimes b}$, where $V^{\vee}$ denotes the Hodge structure dual to V. Cattani, Deligne and Kaplan [3, Theorem 1.1] proved in particular that the Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})$ is a *countable* union of irreducible algebraic subvarieties of S, called the special subvarieties of S for $\mathbb{V}$ (or f). We denote by $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ the Hodge locus *of positive period dimension*, that is the union of the special subvarieties whose image under Φ has positive dimension in $\Gamma\backslash D$.

Let $\mathbf{P}_{\mathbb{Q}}^{N(n,d)} := \mathbf{P}(H^0(\mathbf{P}_{\mathbb{Q}}^{n+1}, \mathcal{O}_{\mathbf{P}_{\mathbb{Q}}^{n+1}}(d)))$ be the parameter space of hypersurfaces X of $\mathbf{P}_{\mathbb{Q}}^{n+1}$ of degree d (where $N(n,d) = \binom{n+d+1}{d} - 1$). Let $U_{n,d} \subset \mathbf{P}_{\mathbb{Q}}^{N(n,d)}$ be the Zariski-open subset parametrising the smooth hypersurfaces X and consider

$$f_{n,d} : X_{n,d} \to U_{n,d},$$

the universal family of smooth degree d hypersurfaces in $\mathbf{P}_{\mathbb{Q}}^{n+1}$. We denote by $\mathbb{V}$ the polarized $\mathbb{Z}$-variation of Hodge structure $(R^n f_{n,d,\mathbb{C}_*}^{\mathrm{an}} \mathbb{Z})_{\mathrm{prim}}$ on $U_{n,d,\mathbb{C}}$. We write $\mathrm{HL}(U_{n,d,\mathbb{C}}, \mathbb{V}^{\otimes})$ for its Hodge locus and $\mathrm{HL}(U_{n,d,\mathbb{C}}, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ for its Hodge locus of positive period dimension.

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In our previous paper we have established the following as a particular case of our main result:

Theorem 1.1 ([2, Corollary 2.7]). *If $n \geq 3, d \geq 5$ and $(n,d) \neq (4,5)$ then the Hodge locus $\mathrm{HL}(U_{n,d,\mathbb{C}}, \mathbb{V})_{\mathrm{pos}}$ of positive period dimension is a closed (not necessarily irreducible) algebraic subvariety of $U_{n,d,\mathbb{C}}$. That is, there are only finitely many (rather than countably many) maximal strict special subvarieties of $U_{n,d,\mathbb{C}}$ for $\mathbb{V}$ of positive period dimension.*

Remark 1.2. The complement of $U_{n,d}$ in $\mathbf{P}_{\mathbb{Q}}^{N(n,d)}$ is a hypersurface. Hence $U_{n,d}$ is an open affine subvariety, stable under the natural $\mathbf{SL}(n+2)$-action on $\mathbf{P}_{\mathbb{Q}}^{N(n,d)}$. For $d \geq 3$ this action is regular in the sense of GIT (the dimensions of the stabilizers are locally constant) hence closed (the orbits are closed). Since $\mathcal{O}_{\mathbf{P}_{\mathbb{Q}}^{n+1}}(1)$ admits an $\mathbf{SL}(n+2)$-linearization, $U_{n,d}$ is contained in the open set of stable points for this action and the geometric quotient $\mathcal{M}_{n,d} := U_{n,d}/\mathbf{SL}(n+2)$ is the moduli space of smooth hypersurfaces of degree d in $\mathbf{P}^{n+1}$. The period map $\Phi : U_{n,d,\mathbb{C}}^{\mathrm{an}} \to \Gamma\backslash D$ factorizes through $\mathcal{M}_{n,d,\mathbb{C}}^{\mathrm{an}}$.

Remark 1.3. For what follows, the easier [2, Theorem 5.1] would actually be enough (that is the *Geometric Part of Zilber-Pink*, for weakly-special subvarieties).

2. Non-denseness of integral points and the Lawrence–Venkatesh method

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. In a recent breakthrough, Lawrence and Venkatesh proved the following:

Theorem 2.1 ([7, Theorem 10.1, Proposition 10.2]). *There exist $n_0 \in \mathbb{N}_{\geq 3}$ and a function $d_0 : \mathbb{N} \to \mathbb{N}$ such that,*

$$\text{for every } n \geq n_0 \text{ and } d \geq d_0(n), \tag{2.0.1}$$

the set $U_{n,d}(\mathcal{O}_{K,S})$ is not Zariski dense in $U_{n,d,\mathbb{C}}$, for every number field K and every finite set of places S of K.

Remark 2.2. Being $\mathcal{M}_{n,d,\mathbb{C}}^{\mathrm{an}}$ *hyperbolic*, a famous conjecture of Lang (see for instance [4, Chapter F.5.2]) predicts that $\mathcal{M}_{n,d}(\mathcal{O}_{K,S})$ is finite, hence $U_{n,d}(\mathcal{O}_{K,S})$ should be a finite union of $\mathbf{PGL}_{n+2}(\mathcal{O}_{K,S})$-orbits, as soon as $d \geq 3$ and all n. We remark here that the finiteness of the integral points of $\mathcal{M}_{n,d}$ is independent on the choice of an integral model of $\mathcal{M}_{n,d}$, hence we do not have to consider finer questions about geometric invariant theory over the integers.

Remark 2.3. Thanks to the main theorem of [5], the same conclusion of Theorem 2.1 holds true for over every finitely generated field K of characteristic zero (not necessarily a number field). This is indeed the level of generality we employ from now on.

The following elucidation of the Lawrence-Venkatesh method for proving Theorem 2.1 will be crucial for us. Lawrence and Venkatesh actually prove that (quoting the third paragraph of [7, Section 1.1]) *the monodromy for the universal family of hypersurfaces must drop over each component of the Zariski closure of the integral points* (see also the last three lines of [7, Theorem 10.1]): for any K, S, there exists a closed subvariety $V_{K,S}$ of $U_{n,d}/\mathcal{O}_{K,S}$ whose irreducible components are of positive period dimension and not *monodromy generic*, such that $(U_{n,d} - V_{K,S})(\mathcal{O}_{K,S})$ is finite. By the Deligne-André monodromy theorem (see for example [2, Section 3 and 4]) and the fact that the $\mathbb{Z}$VHS $\mathbb{V}$ is irreducible, it follows that each $V_{K,S}$ lies in the Hodge locus of positive period dimension $\mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}}$. [1]

Remark 2.4. The Lawrence-Venkatesh method requires the choice of an auxiliary prime number p, and the choice of an identification between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. Indeed, to prove that the $\mathcal{O}_{K,S}$-points of $U_{n,d}$ are not Zariski dense, Lawrence and Venkatesh prove that some *p-adic period map*

[1] In fact, and to justify Remark 1.3, their proof actually shows that each $V_{K,S}$ is contained in the *atypical Hodge locus* of positive period dimension. Such subspace of the Hodge locus is proven to be non-Zariski dense in $U_{n,d}$ in [2] as a first step towards Theorem 1.1, but it holds true for any variety supporting any variation of Hodge structures.

sending $x \in U_{n,d}(\mathcal{O}_{K,S})$ to some p-adic representation of the absolute Galois group of K_v (where v denotes some place above the rational prime p) has fibers that are not Zariski dense in $U_{n,d}$. This is done by working on a residue disk in $U_{n,d}(K_v)$ and the p-adic and complex period maps are then related by a study of the Gauss-Manin connection [7, Lemma 3.2]. What their proof actually shows, with respect to our fixed embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$ is that for each K, S, there exists an automorphism ι_p of $\mathbb{C}$ such that $V_{K,S}$ is contained in $\mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\otimes}^{\iota_p} \subset U_{n,d,\mathbb{C}}$. What allows us to say that $V_{K,S}$ lies in $\mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ is the fact that $\mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ is actually defined over $\mathbb{Q}$, as one sees by combining Theorem 1.1 and [6, Theorem 1.10].

Remark 2.5. Let us emphasize that both [7, Theorem 10.1] and Theorem 1.1 build on the Ax-Schanuel theorem [1], a deep and general theorem establishing strong functional transcendence properties of period maps. Actually, in Lawrence-Venkatesh, a p-adic version of such a result is used, see indeed [7, Lemma 9.3].

3. Proof of the Main result

We are finally ready to state and prove the main result of the paper. The following is a consequence of the *refined form of the Bombieri–Lang conjecture* for quasi-projective[2] varieties of general type [4, Chapter F.5.2].

Theorem 3.1. *As long as* (2.0.1) *is satisfied, there exists a closed strict subvariety $E \subset U_{n,d}$ such that, for all K and all S, we have*

$$\overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathrm{pos}} \subset E,$$

where $\overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathrm{pos}}$ denotes the positive dimensional components of the Zariski closure of $U_{n,d}(\mathcal{O}_{K,S})$ in $U_{n,d}$. That is $U_{n,d} - E$ has only finitely many $\mathcal{O}_{K,S}$-points.

Even if K is fixed, the above is still a non-trivial improvement of Theorem 2.1. Indeed

$$\bigcup_{S} \overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathrm{pos}},$$

where the union ranges along the finite set of K-places, could be, a priori, Zariski dense in $U_{n,d}$.

Remark 3.2. The same improvement applies also to [7, Theorem 10.1], since, as recalled in Remark 1.3, Theorem 1.1 is just a special case of a much more general theorem in variational Hodge theory.

3.1. Proof of Theorem 3.1.

As explained above the proof is essentially a combination of Theorem 2.1 and Theorem 1.1. We present here the details needed.

It follows from Theorem 1.1 that $\mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ is a (closed, strict) algebraic subvariety of $U_{n,d}$ and, thanks to the elucidation of Theorem 2.1, we have

$$\bigcup_{K,S} \overline{U_{n,d}(\mathcal{O}_{K,S})}_{\mathrm{pos}} = \bigcup_{K,S} V_{K,S} \subset \mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}},$$

where the union ranges over all $\mathbb{Q}$-finitely generated fields K and all finite set of places S. It follows from Theorem 1.1 that

$$E' := \overline{\bigcup_{K,S} V_{K,S}}^{\mathrm{Zar}} \subset \mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}}.$$

We remark here that the above inclusion may happen to be strict. Therefore we obtained a closed $\overline{\mathbb{Q}}$-subvariety $E' \subset \mathrm{HL}(U_{n,d}, \mathbb{V}^{\otimes})_{\mathrm{pos}}$ containing all $V_{K,S}$ (seen as $\overline{\mathbb{Q}}$-varieties). The Zariski closure E in $\mathbf{P}^N_{\mathbb{Z}}$ of E' enjoys the desired property: $U_{n,d} - E$ has only finitely many $\mathcal{O}_{K,S}$-points (for all K, S as in the statement). The proof of the Theorem is eventually concluded.

[2]Bombieri–Lang and Lang conjectures are often stated with projective varieties and rational points. Here we mean smooth quasi-projective varieties V of *log-general type* and hyperbolic in the sense of Brody (that is to say that the only holomorphic maps from $\mathbb{C}$ to V are the constants).

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