p-ADIC LATTICES ARE NOT KÄHLER GROUPS

B. KLINGLER

ABSTRACT. We show that any lattice in a simple p-adic Lie group is not the fundamental group of a compact Kähler manifold, as well as some variants of this result.

1. Results

1.1. A group is said to be a Kähler group if it is isomorphic to the fundamental group of a connected compact Kähler manifold. In particular such a group is finitely presented. The most elementary necessary condition for a finitely presented group to be Kähler is that every of its finite index subgroups has even rank abelianization. A classical question, due to Serre and still largely open, is to characterize Kähler groups among finitely presented groups. A standard reference for Kähler groups is [ABCKT96].

1.2. In this note we consider the Kähler problem for lattices in simple groups over local fields. Recall that if G is a locally compact topological group, a subgroup $\Gamma \subset G$ is called a *lattice* if it is a discrete subgroup of G with finite covolume (for any G-invariant measure on the locally compact group G).

We work in the following setting. Let I be a finite set of indices. For each $i \in I$ we fix a local field k_i and a simple algebraic group $\mathbf{G}_i$ defined and isotropic over k_i. Let $G = \prod_{i \in I} \mathbf{G}_i(k_i)$. The topology of the local fields k_i, $i \in I$, make G a locally compact topological group. We define $\mathrm{rk}\, G := \sum_{i \in I} \mathrm{rk}_{k_i} \mathbf{G}_i$.

We consider $\Gamma \subset G$ an *irreducible* lattice: there does not exist a disjoint decomposition $I = I_1 \coprod I_2$ into two non-empty subsets such that, for $j = 1, 2$, the subgroup $\Gamma_j := \Gamma \cap G_{I_j}$ of $G_{I_j} := \prod_{i \in I_j} \mathbf{G}_i(k_i)$ is a lattice in G_{I_j}.

The reference for a detailed study of such lattices is [Mar91]. In Section 2 we recall a few results for the convenience of the reader.

1.3. Most of the lattices Γ as in Section 1.2 are finitely presented (see Section 2.3). The question whether or not Γ is Kähler has been studied by Simpson using his non-abelian Hodge theory when at least one of the k_i's is archimedean. He shows that if Γ is Kähler then necessarily for any $i \in I$ such that k_i is archimedean the group $\mathbf{G}_i$ has to be of Hodge type (i.e. admits a Cartan involution which is an inner automorphism), see [Si92, Cor. 5.3 and 5.4]. For example $\mathbf{SL}(n, \mathbb{Z})$ is not a Kähler group as $\mathbf{SL}(n, \mathbb{R})$ is not a group of Hodge type. In this note we prove:

Theorem 1.1. *Let I be a finite set of indices and G be a group of the form $\prod_{j \in I} \mathbf{G}_j(k_j)$, where $\mathbf{G}_j$ is a simple algebraic group defined and isotropic over a local field k_j. Let $\Gamma \subset G$ be an irreducible lattice.*

Suppose there exists an $i \in I$ such that k_i is non-archimedean. If $\mathrm{rk}\, G > 1$ and $\mathrm{char}(k_i) = 0$, or if $\mathrm{rk}\, G = 1$ then Γ is not a Kähler group.

Notice that the case $\operatorname{rk} G = 1$ is essentially folkloric (I include a proof for the convenience of the reader as I did not find a reference in this generality). On the other hand, to the best of our knowledge not a single case of Theorem 1.1 in the case where $\operatorname{rk} G > 1$ and all the k_i, $i \in I$, are non-archimedean fields of characteristic zero was previously known. The proof in this case is a corollary of Margulis' superrigidity theorem and the recent integrality result of Esnault and Groechenig [EG17, Theor. 1.3].

1.4. Let us mention some examples of Theorem 1.1:

- Let p be a prime number, $I = \{1\}$, $k_1 = \mathbb{Q}_p$, $\mathbf{G} = \mathbf{SL}(n)$. A lattice in $\mathbf{SL}(n, \mathbb{Q}_p)$, $n \geq 2$, is not a Kähler group. This is new for $n \geq 3$.
- $I = \{1; 2\}$, $k_1 = \mathbb{R}$ and $\mathbf{G}_1 = \mathbf{SU}(r, s)$ for some $r \geq s > 0$, $k_2 = \mathbb{Q}_p$ and $\mathbf{G}_2 = \mathbf{SL}(r + s)$. Then any irreducible lattice in $SU(r, s) \times \mathbf{SL}(r + s, \mathbb{Q}_p)$ is not Kähler. In Section 2 we recall how to construct such lattices (they are S-arithmetic). The analogous result that any irreducible lattice in $\mathbf{SL}(n, \mathbb{R}) \times \mathbf{SL}(n, \mathbb{Q}_p)$ (for example $\mathbf{SL}(n, \mathbb{Z}[1/p])$) is not a Kähler group already followed from Simpson's theorem.

1.5. I don't know anything about the case not covered by Theorem 1.1: can a (finitely presented) irreducible lattice in $G = \prod_{i \in I} \mathbf{G}_i(k_i)$ with $\operatorname{rk} G > 1$ and all k_i of (necessarily the same, see Theorem 2.1) *positive characteristic*, be a Kähler group? This question already appeared in [BKT13, Remark 0.2 (5)].

2. Reminder on lattices

2.1. Examples of pairs (G, Γ) as in Section 1.2 are provided by *S-arithmetic groups*: let K be a global field (i.e a finite extension of $\mathbb{Q}$ or $\mathbf{F}_q(t)$), S a non-empty set of places of K, S_∞ the set of archimedean places of K (or the empty set if K has positive characteristic), $\mathcal{O}^{S \cup S_\infty}$ the ring of elements of K which are integral at all places not belonging to $S \cup S_\infty$ and $\mathbf{G}$ an absolutely simple K-algebraic group, anisotropic at all archimedean places not belonging to S. A subgroup $\Lambda \subset \mathbf{G}(K)$ is said S-arithmetic (or $S \cup S_\infty$-arithmetic) if it is commensurable with $\mathbf{G}(\mathcal{O}^{S \cup S_\infty})$ (this last notation depends on the choice of an affine group scheme flat of finite type over $\mathcal{O}^{S \cup S_\infty}$, with generic fiber $\mathbf{G}$; but the commensurability class of the group $\mathbf{G}(\mathcal{O}^{S \cup S_\infty})$ is independent of that choice).

If S is finite and $\mathbf{G}(K_v)$ is compact for all $v \in S_\infty - S$, the image Γ in $\prod_{v \in S} \mathbf{G}(K_v)$ of an S-arithmetic group Λ by the diagonal map is an irreducible lattice (see [B63] in the number field case and [H69] in the function field case). In the situation of Section 1.2, a (necessarily irreducible) lattice $\Gamma \subset G$ is said S-arithmetic if there exist K, $\mathbf{G}$, S as above, a bijection $i : S \longrightarrow I$, isomorphisms $K_v \longrightarrow k_{i(v)}$ and, via these isomorphisms, k_i-isomorphisms $\varphi_i : \mathbf{G} \longrightarrow \mathbf{G}_i$ such that Γ is commensurable with the image via $\prod_{i \in I} \varphi_i$ of an S-arithmetic subgroup of $\mathbf{G}(K)$.

2.2. Margulis' and Venkataramana's arithmeticity theorem states that as soon as $\operatorname{rk} G$ is at least 2 then every irreducible lattice in G is of this type:

Theorem 2.1 (Margulis, Venkataramana)**.** *In the situation of Section 1.2, suppose that $\Gamma \subset G$ is an irreducible lattice and that $\operatorname{rk} G \geq 2$. Suppose moreover for simplicity that $\mathbf{G}_i$, $i \in I$, is absolutely simple. Then:*

(a) $\operatorname{char}(k_i) = \operatorname{char}(k_j)$ *for all* $(i, j) \in I$.
(b) Γ *is S-arithmetic.*

Remark 2.2. Margulis [Mar84] proved Theorem 2.1 when $\mathrm{char}(k_i) = 0$ for all $i \in I$. Venkatarama [V88] had to overcome many technical difficulties in positive characteristics to extend Margulis' strategy to the general case.

On the other hand, if $\mathrm{rk}\, G = 1$ (hence $I = \{1\}$) and $k := k_1$ is non-archimedean, there exists non-arithmetic lattices in G, see [L91, Theor.A].

2.3. With the notations of Section 2.1, an S-arithmetic lattice Γ is always finitely presented except if K is a function field and $\mathrm{rk}_K \mathbf{G} = \mathrm{rk}\, G = |S| = 1$ (in which case Γ is not even finitely generated) or $\mathrm{rk}_K \mathbf{G} > 0$ and $\mathrm{rk}\, G = 2$ (in which case Γ is finitely generated but not finitely presented). In the number field case see the result of Raghunathan [R68] in the classical arithmetic case and of Borel-Serre [BS76] in the general S-arithmetic case; in the function field case see the work of Behr, e.g. [Behr98]. For example the lattice $\mathbf{SL}_2(\mathbb{F}_q[t])$ of $\mathbf{SL}_2(\mathbb{F}_q((t)))$ is not finitely generated, while the lattice $\mathbf{SL}_3(\mathbb{F}_q[t])$ of $\mathbf{SL}_3(\mathbb{F}_q((t)))$ is finitely generated but not finitely presented.

3. Proof of Theorem 1.1

3.1. The rank 1 case. Let us deal first with the easy case where $\mathrm{rk}\, G = 1$ (hence $I = \{1\}$ and we write $k := k_1$).

If Γ is not cocompact in G (this is possible only if k has positive characteristic) then Γ is not finitely generated by [L91, Cor. 7.3], hence not Kähler.

Hence we can assume that Γ is cocompact. In that case it follows from [L91, Theor. 6.1 and Theor. 7.1] that Γ admits a finite index subgroup Γ' which is a (non-trivial) free group.

But a non-trivial free group is never Kähler, as it always admits a finite index subgroup with odd Betti number (see [ABCKT96, Example 1.19 p.7]). Hence Γ' is not Kähler.

As any finite index subgroup of a Kähler group is Kähler (because the class of connected compact Kähler manifolds is stable under taking a connected finite étale cover), it follows that Γ is not a Kähler group.

3.2. The higher rank case. In this case the main tools for proving Theorem 1.1 are the recent result [EG17, Theor. 1.3] of Esnault and Groechenig and Margulis' superrigidity theorem.

3.2.1. Recall that a linear representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n, k)$ of a group Γ over a field k is cohomologically rigid if $H^1(\Gamma, \mathrm{Ad}\, \rho) = 0$. A representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n, \mathbb{C})$ is said to be integral if it factorizes through $\rho : \Gamma \longrightarrow \mathbf{GL}(n, K)$, $K \hookrightarrow \mathbb{C}$ a number field, and moreover stabilizes an $\mathcal{O}_K$-lattice in $\mathbb{C}^n$ (equivalently: for any embedding $v : K \hookrightarrow k$ of K in a non-archimedean local field k the composite representation $\rho_v : \Gamma \longrightarrow \mathbf{GL}(n, K) \hookrightarrow \mathbf{GL}(n, k)$ has bounded image in $\mathbf{GL}(n, k)$). A group will be said *complex projective* if is isomorphic to the fundamental group of a connected smooth complex projective variety. This is a special case of a Kähler group (the question whether or not any Kähler group is complex projective is open).

In [EG17, Theor. 1.3] Esnault and Groechenig prove that if Γ is a complex projective group then any irreducible cohomologically rigid representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n, \mathbb{C})$ is integral. This was conjectured by Simpson.

3.2.2. A corollary of [EG17, Theor. 1.3] is the following:

Corollary 3.1. *Let Γ be a complex projective group. Let k be a non-archimedean local field of characteristic zero and let $\rho : \pi_1(X) \longrightarrow \mathbf{GL}(n,k)$ be an absolutely irreducible cohomologically rigid representation. Then ρ has bounded image in $\mathbf{GL}(n,k)$.*

Proof. Let $\overline{k}$ be an algebraic closure of k. As ρ is absolutely irreducible and cohomologically rigid there exists $g \in \mathbf{GL}(n,\overline{k})$ and a number field $K \subset \overline{k}$ such that the representation $\rho^g := g \cdot \rho \cdot g^{-1} : \Gamma \longrightarrow \mathbf{GL}(n,\overline{k})$ takes value in $\mathbf{GL}(n,K)$.

Let k' be the finite extension of k generated by k, K, and the matrix coefficients of g and g^{-1}. This is still a non-archimedean local field of characteristic zero, and both ρ and ρ^g takes value in $\mathbf{GL}(n,k')$. As $\rho : \Gamma \longrightarrow \mathbf{GL}(n,k) \subset \mathbf{GL}(n,k')$ has bounded image in $\mathbf{GL}(n,k)$ if and only if $\rho^g : \Gamma \longrightarrow \mathbf{GL}(n,k')$ has bounded image in $\mathbf{GL}(n,k')$, we can assume, replacing ρ by ρ^g and k by k' if necessary, that ρ takes value in $\mathbf{GL}(n,K)$ with $K \subset k$ a number field.

Let $\sigma : K \hookrightarrow \mathbb{C}$ be an infinite place of K and consider $\rho^\sigma : \Gamma \xrightarrow{\rho} \mathbf{GL}(n,K) \overset{\sigma}{\hookrightarrow} \mathbf{GL}(n,\mathbb{C})$ the associated representation. As ρ is absolutely irreducible, the representation ρ^σ is irreducible. As

$$H^1(\Gamma, \mathrm{Ad} \circ \rho^\sigma) = H^1(\Gamma, \mathrm{Ad} \circ \rho) \otimes_{K,\sigma} \mathbb{C} = 0$$

the representation ρ^σ is cohomologically rigid.

It follows from [EG17, Theor. 1.3] that ρ^σ is integral. In particular, considering the embedding $K \subset k$, it follows that the representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n,k)$ has bounded image in $\mathbf{GL}(n,k)$. $\square$

3.2.3. Notice that we can upgrade Corollary 3.1 to the Kähler world if we restrict ourselves to faithful representations:

Corollary 3.2. *The conclusion of Corollary 3.1 also holds for Γ a Kähler group and $\rho : \pi_1(X) \longrightarrow \mathbf{GL}(n,k)$ a* faithful *representation.*

Proof. As the representation ρ is faithful, the group Γ is a linear group in characteristic zero. It then follows from [CCE14] and [C17] that the Kähler group Γ is a complex projective group. The result now follows from Corollary 3.1. $\square$

3.2.4. Let us now apply Corollary 3.1 to the case of Theorem 1.1 where $\mathrm{rk}\, G > 1$. Renaming the indices of I if necessary, we will assume that $I = \{1, \cdots, r\}$ and k_1 is non-archimedean of characteristic zero. Let us choose an absolutely irreducible k_1-representation $\rho_{\mathbf{G}_1} : \mathbf{G}_1 \longrightarrow \mathbf{GL}(V)$. Let

$$\rho : \Gamma \longrightarrow G \xrightarrow{p_1} \mathbf{G}_1(k_1) \longrightarrow \mathbf{GL}(V)$$

be the representation of Γ deduced from $\rho_{\mathbf{G}_1}$ (where $p_1 : G \longrightarrow \mathbf{G}_1(k_1)$ denotes the projection of G onto its first factor). As $p_1(\Gamma)$ is Zariski-dense in $\mathbf{G}_1$ it follows that ρ is absolutely irreducible.

As $\mathrm{rk}\, G > 1$, Margulis' superrigidity theorem applies to the lattice Γ of G: it implies in particular that $H^1(\Gamma, \mathrm{Ad} \circ \rho) = 0$ (see [Mar91, Theor. (3)(iii) p.3]). Hence the representation $\rho : \Gamma \longrightarrow \mathbf{GL}(V)$ is cohomologically rigid.

Suppose by contradiction that Γ is a Kähler group. By Theorem 2.1(a) and the assumption that k_1 has characteristic zero it follows that Γ is linear in characteristic zero. As in the proof of Corollary 3.2 we deduce that Γ is a complex projective group. It then follows from Corollary 3.1 that ρ has bounded image in $\mathbf{GL}(V)$, hence that $p_1(\Gamma)$ is relatively compact in $\mathbf{G}(k_1)$. This contradicts the fact that Γ is a lattice in $G = \mathbf{G}(k_1) \times \prod_{j \in I \setminus \{1\}} \mathbf{G}(k_j)$.

□

References

[ABCKT96] J. Amoros, M. Burger, K. Corlette, D. Kotschick, D. Toledo, *Fundamental groups of compact Kähler manifolds*, Mathematical Surveys and Monographs vol. **44**, (1996)

[Behr98] H. Behr, *Arithmetic groups over function fields*, J. reine and angew. Math. **495** (1998) 79-118

[B63] A. Borel, *Some finiteness properties of adeles groups over number fields*, Publ. Math. IHES **16** (1963), 101-126

[BS76] A. Borel, J.P. Serre, *Cohomologie d'immeubles et de groupes S-arithmétiques*, Topology **15** (1976), 211-232

[BKT13] Y. Brunebarbe, B. Klingler, B. Totaro, *Symmetric differentials and the fundamental group*, Duke Math. J. **162** (2013) no.14, 2797-2813

[CCE14] F. Campana, B.Claudon. P.Eyssidieux, *Représentations linéaires des groupes kählériens et de leurs analogues projectifs*, J. Ec. Polytechn. Math. 1 (2014), 331-342

[C17] B. Claudon, *Smooth families of tori and linear Kähler groups*, Annales de la Faculté des Mathématiques de Toulouse, to appear

[EG17] H.Esnault, M. Groechenig, *Rigid connections, F-isocrystals and integrality*, preprint

[H69] G. Harder, *Minkowskische Reduktionstheorie über Funktionenkörper*, Inventiones Math., **7** (1969), 33-54

[L91] A. Lubotzky, *Lattices in rank one Lie groups over local fields*, GAFA **1** vol.4 (1991), 405-431

[Mar84] G.A. Margulis, *Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than* 1, Inventiones Math. **76** (1984), 93-120

[Mar91] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grengebiete **17**, Springer-Verlag, (1991)

[R68] M.S. Raghunathan, *A note on quotients of real algebraic groups by arithmetic subgroups*, Invent. Math **4** (1968), 318-335

[Si92] C.T. Simpson, *Higgs bundles and local systems*, Publ. Math. IHES **75** (1992), 5-95

[V88] T.N. Venkataramana, *On superrigidity and arithmeticity of lattices in semisimple groups over local fields of arbitrary characteristic*, Inventiones Math. **92** (1988), 255-306

Humboldt-Universität zu Berlin

E-mail address: `bruno.klingler@hu-berlin.de`