p-ADIC LATTICES ARE NOT KÄHLER GROUPS

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Abstract. We show that any lattice in a simple p-adic Lie group is not the fundamental group of a compact Kähler manifold, as well as some variants of this result.

1. Results

1.1. A group is said to be a Kähler group if it is isomorphic to the fundamental group of a connected compact Kähler manifold. In particular such a group is finitely presented. The most elementary necessary condition for a finitely presented group to be Kähler is that every of its finite index subgroups has even rank abelianization. A classical question, due to Serre and still largely open, is to characterize Kähler groups among finitely presented groups. A standard reference for Kähler groups is [ABCKT96].

1.2. In this note we consider the Kähler problem for lattices in simple groups over local fields. Recall that if \( G \) is a locally compact topological group, a subgroup \( \Gamma \subset G \) is called a lattice if it is a discrete subgroup of \( G \) with finite covolume (for any \( G \)-invariant measure on the locally compact group \( G \)).

We work in the following setting. Let \( I \) be a finite set of indices. For each \( i \in I \) we fix a local field \( k_i \) and a simple algebraic group \( G_i \) defined and isotropic over \( k_i \). Let \( G = \prod_{i \in I} G_i(k_i) \). The topology of the local fields \( k_i, i \in I \), make \( G \) a locally compact topological group. We define \( \text{rk} G := \sum_{i \in I} \text{rk} G_i \).

We consider \( \Gamma \subset G \) an irreducible lattice: there does not exist a disjoint decomposition \( I = I_1 \sqcup I_2 \) into two non-empty subsets such that, for \( j = 1, 2 \), the subgroup \( \Gamma_j := \Gamma \cap G_{I_j} \) of \( G_{I_j} := \prod_{i \in I_j} G_i(k_i) \) is a lattice in \( G_{I_j} \).

The reference for a detailed study of such lattices is [Mar91]. In Section 2 we recall a few results for the convenience of the reader.

1.3. Most of the lattices \( \Gamma \) as in Section 1.2 are finitely presented (see Section 2.3). The question whether or not \( \Gamma \) is Kähler has been studied by Simpson using his non-abelian Hodge theory when at least one of the \( k_i \)'s is archimedean. He shows that if \( \Gamma \) is Kähler then necessarily for any \( i \in I \) such that \( k_i \) is archimedean the group \( G_i \) has to be of Hodge type (i.e. admits a Cartan involution which is an inner automorphism), see [Si92, Cor. 5.3 and 5.4]. For example \( \text{SL}(n, \mathbb{Z}) \) is not a Kähler group as \( \text{SL}(n, \mathbb{R}) \) is not a group of Hodge type. In this note we prove:

**Theorem 1.1.** Let \( I \) be a finite set of indices and \( G \) be a group of the form \( \prod_{j \in I} G_j(k_j) \), where \( G_j \) is a simple algebraic group defined and isotropic over a local field \( k_j \). Let \( \Gamma \subset G \) be an irreducible lattice.

Suppose there exists an \( i \in I \) such that \( k_i \) is non-archimedean. If \( \text{rk} G > 1 \) and \( \text{char}(k_i) = 0 \), or if \( \text{rk} G = 1 \) then \( \Gamma \) is not a Kähler group.

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Notice that the case \( \text{rk } G = 1 \) is essentially folkloric (I include a proof for the convenience of the reader as I did not find a reference in this generality). On the other hand, to the best of our knowledge not a single case of Theorem 1.1 in the case where \( \text{rk } G > 1 \) and all the \( k_i, i \in I \), are non-archimedean fields of characteristic zero was previously known. The proof in this case is a corollary of Margulis’ superrigidity theorem and the recent integrality result of Esnault and Groechenig [EG17, Theor. 1.3].

1.4. Let us mention some examples of Theorem 1.1:
- Let \( p \) be a prime number, \( I = \{ 1 \} \), \( k_1 = \mathbb{Q}_p \), \( G = \text{SL}(n) \). A lattice in \( \text{SL}(n, \mathbb{Q}_p) \), \( n \geq 2 \), is not a Kähler group. This is new for \( n \geq 3 \).
- \( I = \{ 1; 2 \} \), \( k_1 = \mathbb{R} \) and \( G_1 = \text{SU}(r, s) \) for some \( r \geq s > 0 \), \( k_2 = \mathbb{Q}_p \) and \( G_2 = \text{SL}(r + s) \). Then any irreducible lattice in \( SU(r, s) \times \text{SL}(r + s, \mathbb{Q}_p) \) is not Kähler. In Section 2 we recall how to construct such lattices (they are \( S \)-arithmetic). The analogous result that any irreducible lattice in \( \text{SL}(n, \mathbb{R}) \times \text{SL}(n, \mathbb{Q}_p) \) (for example \( \text{SL}(n, \mathbb{Z}[1/p]) \)) is not a Kähler group already followed from Simpson’s theorem.

1.5. I don’t know anything about the case not covered by Theorem 1.1: can a (finitely presented) irreducible lattice in \( G = \prod_{i \in I} G_i(k_i) \) with \( \text{rk } G > 1 \) and all \( k_i \) of (necessarily the same, see Theorem 2.1) positive characteristic, be a Kähler group? This question already appeared in [BKT13, Remark 0.2 (5)].

2. Reminder on lattices

2.1. Examples of pairs \((G, \Gamma)\) as in Section 1.2 are provided by \( S \)-arithmetic groups: let \( K \) be a global field (i.e a finite extension of \( \mathbb{Q} \) or \( \mathbb{F}_q(t) \)), \( S \) a non-empty set of places of \( K \), \( S_\infty \) the set of archimedean places of \( K \) (or the empty set if \( K \) has positive characteristic), \( \mathcal{O}_S \cup \mathcal{O}_{S_\infty} \) the ring of elements of \( K \) which are integral at all places not belonging to \( S \cup S_\infty \) and \( G \) an absolutely simple \( K \)-algebraic group, anisotropic at all archimedean places not belonging to \( S \). A subgroup \( \Lambda \subset G(K) \) is said \( S \)-arithmetic (or \( S \cup S_\infty \)-arithmetic) if it is commensurable with \( G(\mathcal{O}_S \cup \mathcal{O}_{S_\infty}) \) (this last notation depends on the choice of an affine group scheme flat of finite type over \( \mathcal{O}_S \cup \mathcal{O}_{S_\infty} \), with generic fiber \( G \); but the commensurability class of the group \( G(\mathcal{O}_S \cup \mathcal{O}_{S_\infty}) \) is independent of that choice).

If \( S \) is finite and \( G(K_v) \) is compact for all \( v \in S_\infty - S \), the image \( \Gamma \) in \( \prod_{v \in S} G(K_v) \) of an \( S \)-arithmetic group \( \Lambda \) by the diagonal map is an irreducible lattice (see [B63] in the number field case and [H69] in the function field case). In the situation of Section 1.2, a (necessarily irreducible) lattice \( \Gamma \subset G \) is said \( S \)-arithmetic if there exist \( K, G, S \) as above, a bijection \( i : S \rightarrow I \), isomorphisms \( K_v \rightarrow k_i(v) \) and, via these isomorphisms, \( k_i \)-isomorphisms \( \varphi_i : G \rightarrow G_i \) such that \( \Gamma \) is commensurable with the image via \( \prod_{i \in I} \varphi_i \) of an \( S \)-arithmetic subgroup of \( G(K) \).

2.2. Margulis’ and Venkataramana’s arithmeticity theorem states that as soon as \( \text{rk } G \) is at least 2 then every irreducible lattice in \( G \) is of this type:

**Theorem 2.1** (Margulis, Venkataramana). *In the situation of Section 1.2, suppose that \( \Gamma \subset G \) is an irreducible lattice and that \( \text{rk } G \geq 2 \). Suppose moreover for simplicity that \( G_i, i \in I, \) is absolutely simple. Then:

(a) \( \text{char}(k_i) = \text{char}(k_j) \) for all \( (i, j) \in I \).
(b) \( \Gamma \) is \( S \)-arithmetic.
Remark 2.2. Margulis [Mar84] proved Theorem 2.1 when \( \text{char} (k_i) = 0 \) for all \( i \in I \). Venkatarama [V88] had to overcome many technical difficulties in positive characteristics to extend Margulis’ strategy to the general case.

On the other hand, if \( \text{rk} \, G = 1 \) (hence \( I = \{1\} \)) and \( k := k_1 \) is non-archimedean, there exists non-arithmetic lattices in \( G \), see [L91, Theor.A].

2.3. With the notations of Section 2.1, an \( S \)-arithmetic lattice \( \Gamma \) is always finitely presented except if \( K \) is a function field and \( \text{rk} \, K \, G = \text{rk} \, G = |S| = 1 \) (in which case \( \Gamma \) is not even finitely generated) or \( \text{rk} \, K \, G > 0 \) and \( \text{rk} \, G = 2 \) (in which case \( \Gamma \) is finitely generated but not finitely presented). In the number field case see the result of Raghunathan [R68] in the classical arithmetic case and of Borel-Serre [BS76] in the general \( S \)-arithmetic case; in the function field case see the work of Behr, e.g. [Behr98]. For example the lattice \( \text{SL}_2 (\mathbb{F}_q[t]) \) of \( \text{SL}_2 (\mathbb{F}_q((t))) \) is not finitely generated, while the lattice \( \text{SL}_3 (\mathbb{F}_q[t]) \) of \( \text{SL}_3 (\mathbb{F}_q((t))) \) is finitely generated but not finitely presented.

3. Proof of Theorem 1.1

3.1. The rank 1 case. Let us deal first with the easy case where \( \text{rk} \, G = 1 \) (hence \( I = \{1\} \)) and we write \( k := k_1 \).

If \( \Gamma \) is not cocompact in \( G \) (this is possible only if \( k \) has positive characteristic) then \( \Gamma \) is not finitely generated by [L91, Cor. 7.3], hence not Kähler.

Hence we can assume that \( \Gamma \) is cocompact. In that case it follows from [L91, Theor. 6.1 and Theor. 7.1] that \( \Gamma \) admits a finite index subgroup \( \Gamma' \) which is a (non-trivial) free group.

But a non-trivial free group is never Kähler, as it always admits a finite index subgroup with odd Betti number (see [ABCKT96, Example 1.19 p.7]). Hence \( \Gamma' \) is not Kähler.

As any finite index subgroup of a Kähler group is Kähler (because the class of connected compact Kähler manifolds is stable under taking a connected finite étale cover), it follows that \( \Gamma \) is not a Kähler group.

3.2. The higher rank case. In this case the main tools for proving Theorem 1.1 are the recent result [EG17, Theor. 1.3] of Esnault and Groechenig and Margulis’ super-rigidity theorem.

3.2.1. Recall that a linear representation \( \rho : \Gamma \to \text{GL}(n, k) \) of a group \( \Gamma \) over a field \( k \) is cohomologically rigid if \( H^1 (\Gamma, \text{Ad} \, \rho) = 0 \). A representation \( \rho : \Gamma \to \text{GL}(n, \mathbb{C}) \) is said to be integral if it factorizes through \( \rho : \Gamma \to \text{GL}(n, K) \), \( K \to \mathbb{C} \) a number field, and moreover stabilizes an \( \mathcal{O}_K \)-lattice in \( \mathbb{C}^n \) (equivalently: for any embedding \( v : K \to k \) of \( K \) in a non-archimedean local field \( k \) the composite representation \( \rho_v : \Gamma \to \text{GL}(n, K) \to \text{GL}(n,k) \) has bounded image in \( \text{GL}(n,k) \) ). A group will be said complex projective if is isomorphic to the fundamental group of a connected smooth complex projective variety. This is a special case of a Kähler group (the question whether or not any Kähler group is complex projective is open).

In [EG17, Theor. 1.3] Esnault and Groechenig prove that if \( \Gamma \) is a complex projective group then any irreducible cohomologically rigid representation \( \rho : \Gamma \to \text{GL}(n, \mathbb{C}) \) is integral. This was conjectured by Simpson.
A corollary of [EG17, Theor. 1.3] is the following:

**Corollary 3.1.** Let $\Gamma$ be a complex projective group. Let $k$ be a non-archimedean local field of characteristic zero and let $\rho : \pi_1(X) \to \text{GL}(n, k)$ be an absolutely irreducible cohomologically rigid representation. Then $\rho$ has bounded image in $\text{GL}(n, k)$.

**Proof.** Let $\overline{k}$ be an algebraic closure of $k$. As $\rho$ is absolutely irreducible and cohomologically rigid there exists $g \in \text{GL}(n, \overline{k})$ and a number field $K \subseteq \overline{k}$ such that the representation $\rho'' := g \cdot \rho \cdot g^{-1} : \Gamma \to \text{GL}(n, \overline{k})$ takes value in $\text{GL}(n, K)$.

Let $k'$ be the finite extension of $k$ generated by $k, K$, and the matrix coefficients of $g$ and $g^{-1}$. This is still a non-archimedean local field of characteristic zero, and both $\rho$ and $\rho''$ takes value in $\text{GL}(n, k')$. As $\rho : \Gamma \to \text{GL}(n, k) \subseteq \text{GL}(n, k')$ has bounded image in $\text{GL}(n, k)$ if and only if $\rho'' : \Gamma \to \text{GL}(n, k')$ has bounded image in $\text{GL}(n, k')$, we can assume, replacing $\rho$ by $\rho''$ and $k$ by $k'$ if necessary, that $\rho$ takes value in $\text{GL}(n, K)$ with $K \subset k$ a number field.

Let $\sigma : K \to \mathbb{C}$ be an infinite place of $K$ and consider $\sigma^* : \Gamma \xrightarrow{\rho^*} \text{GL}(n, K) \xrightarrow{\sigma} \text{GL}(n, \mathbb{C})$ the associated representation. As $\rho$ is absolutely irreducible, the representation $\rho^*$ is irreducible. As $H^1(\Gamma, \text{Ad} \circ \rho^*) = H^1(\Gamma, \text{Ad} \circ \rho) \otimes_{K, \sigma} \mathbb{C} = 0$

the representation $\rho^*$ is cohomologically rigid.

It follows from [EG17, Theor. 1.3] that $\rho^*$ is integral. In particular, considering the embedding $K \subset k$, it follows that the representation $\rho : \Gamma \to \text{GL}(n, k)$ has bounded image in $\text{GL}(n, k)$.

3.2.3. Notice that we can upgrade Corollary 3.1 to the Kähler world if we restrict ourselves to faithful representations:

**Corollary 3.2.** The conclusion of Corollary 3.1 also holds for $\Gamma$ a Kähler group and $\rho : \pi_1(X) \to \text{GL}(n, k)$ a faithful representation.

**Proof.** As the representation $\rho$ is faithful, the group $\Gamma$ is a linear group in characteristic zero. It then follows from [CCE14] and [C17] that the Kähler group $\Gamma$ is a complex projective group. The result now follows from Corollary 3.1.

3.2.4. Let us now apply Corollary 3.1 to the case of Theorem 1.1 where $\text{rk} G > 1$. Renaming the indices of $I$ if necessary, we will assume that $I = \{1, \cdots, r\}$ and $k_1$ is non-archimedean of characteristic zero. Let us choose an absolutely irreducible $k_1$-representation $\rho_{G_1} : G_1 \to \text{GL}(V)$. Let

$$\rho : \Gamma \to G \xrightarrow{p_1} G_1(k_1) \to \text{GL}(V)$$

be the representation of $\Gamma$ deduced from $\rho_{G_1}$ (where $p_1 : G \to G_1(k_1)$ denotes the projection of $G$ onto its first factor). As $p_1(\Gamma)$ is Zariski-dense in $G_1$ it follows that $\rho$ is absolutely irreducible.

As $\text{rk} G > 1$, Margulis’ superrigidity theorem applies to the lattice $\Gamma$ of $G$: it implies in particular that $H^1(\Gamma, \text{Ad} \circ \rho) = 0$ (see [Mar91, Theor. (3)(iii) p.3]). Hence the representation $\rho : \Gamma \to \text{GL}(V)$ is cohomologically rigid.

Suppose by contradiction that $\Gamma$ is a Kähler group. By Theorem 2.1(a) and the assumption that $k_1$ has characteristic zero it follows that $\Gamma$ is linear in characteristic
zero. As in the proof of Corollary 3.2 we deduce that \( \Gamma \) is a complex projective group. It then follows from Corollary 3.1 that \( \rho \) has bounded image in \( \text{GL}(V) \), hence that \( p_1(\Gamma) \) is relatively compact in \( G(k_1) \). This contradicts the fact that \( \Gamma \) is a lattice in \( G = G(k_1) \times \prod_{j \in I \setminus \{1\}} G(k_j) \).

\( \square \)

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