# LOCAL QUATERNIONIC RIGIDITY FOR COMPLEX HYPERBOLIC LATTICES

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Abstract. Let $\Gamma \overset{i}{\hookrightarrow} L$ be a lattice in the real simple Lie group $L$. If $L$ is of rank at least 2 (respectively locally isomorphic to $Sp(n,1)$) any unbounded morphism $\rho : \Gamma \longrightarrow G$ into a simple real Lie group $G$ essentially extends to a Lie morphism $\rho_L : L \longrightarrow G$ (Margulis's superrigidity theorem, respectively Corlette's theorem). In particular any such morphism is infinitesimally, thus locally, rigid.

On the other hand for $L = SU(n,1)$ even morphisms of the form $\rho : \Gamma \overset{i}{\hookrightarrow} L \longrightarrow G$ are not infinitesimally rigid in general. Almost nothing is known about their local rigidity. In this paper we prove that any *cocompact* lattice $\Gamma$ in $SU(n,1)$ is essentially locally rigid (while in general not infinitesimally rigid) in the quaternionic groups $Sp(n,1)$, $SU(2n,2)$ or $SO(4n,4)$ (for the natural sequence of embeddings $SU(n,1) \subset Sp(n,1) \subset SU(2n,2) \subset SO(4n,4)$).

Contents



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## 1. Introduction

1.1. **Complex hyperbolic lattices and rigidity.** The main open question concerning lattices of Lie groups is certainly the study of complex hyperbolic lattices and their finite dimensional representations. Indeed, Margulis's super-rigidity theorem states that any irreducible complex finite-dimensional representation of a lattice $\Gamma$ of a simple real Lie group $L$ of real rank $r > 1$ either has bounded image, or is the restriction to $\Gamma$ of an irreducible finite-dimensional representation of $L$. The remaining case of simple real Lie groups of rank 1 contains 3 families: the real hyperbolic group $SO(n, 1)$, the complex hyperbolic group $SU(n, 1)$ and the quaternionic hyperbolic group $Sp(n, 1)$, plus one exceptional group $F_4^{-20}$. Margulis's description has been extended to lattices of $Sp(n, 1)$ and $F_4^{-20}$ by Corlette [7] and Gromov-Schoen [12]. On the other hand one knows that $SO(n, 1)$ admits lattices with unbounded representations not coming from $SO(n, 1)$. Examples have been constructed by Makarov [17] and Vinberg [26] for small $n$ and by Johnson-Millson [13] and Gromov-Piatetski-Shapiro [11] for any $n \in \mathbb{N}$. Concerning $SU(n, 1)$, Mostow [20] exhibited a striking counterexample to superrigidity for $n = 2$: namely two cocompact (arithmetic) lattices $\Gamma$ and $\Gamma'$ in $SU(2, 1)$ and a *surjective* morphism $\rho : \Gamma \longrightarrow \Gamma'$ with *infinite kernel*. Essentially nothing is known for $n > 3$.

In this paper, we restrict ourselves to the deformation theory of complex hyperbolic cocompact lattices. Let $n > 1$ be an integer and consider the complex hyperbolic group $L = SU(n, 1)$: this is the group of real point of $\mathbf{L} = \mathbf{SU}(n, 1) = \mathbf{SU}(V_{\mathbb{C}}, h_{\mathbb{C}})$, the special unitary algebraic $\mathbb{R}$-group of linear isometries of $(V_{\mathbb{C}}, h_{\mathbb{C}})$ where $V_{\mathbb{C}}$ denotes the $(n+1)$-dimensional $\mathbb{C}$-vector space endowed with the Hermitian form $h_{\mathbb{C}}(\mathbf{z}, \mathbf{w}) = -z_0\overline{w_0} + z_1\overline{w_1} \cdots + z_n\overline{w_n}$. Let $i : \Gamma \hookrightarrow SU(n, 1)$ be a cocompact complex hyperbolic lattice. Let $j : \mathbf{SU}(n, 1) \hookrightarrow \mathbf{G}$ be an injective $\mathbb{R}$-morphism of $\mathbb{R}$-algebraic groups. Does there exist any *non-trivial* deformation of $\rho = j \circ i : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$, i.e. a continuous family of morphisms $\rho_t : \Gamma \longrightarrow G$, $t \in I = [0,1]$, with $\rho_0 = \rho$ not of the form $\rho_t = g_t \cdot \rho \cdot g_t^{-1}$ for some continuous family $g_t \in G$, $t \in I$ ?

1.2. **First order deformations.** Let $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R}) = (\mathrm{Hom}(\mathbf{\Gamma}, \mathbf{G})//\mathbf{G})(\mathbb{R})$ be the moduli space of representations of $\Gamma$ in $\mathbf{G}(\mathbb{R})$ up to conjugacy. The space of first-order deformations of $\rho$, i.e. the real Zariski tangent space at $[\rho]$ to $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$, naturally identifies with the first cohomology group $H^1(\Gamma, \mathrm{Ad}\ \rho)$, where $\mathrm{Ad}\ \rho : \Gamma \overset{\rho}{\hookrightarrow} G \overset{\mathrm{Ad}}{\to} \mathrm{Aut}(\mathfrak{g})$ is the natural representation deduced from $\rho$ and the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. Thus the non-vanishing of $H^1(\Gamma, \mathrm{Ad}\ \rho)$ is a necessary condition for $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ not being trivial at the point $[\rho]$. Raghunathan [21] gave the list of irreducible finite-dimensional $\mathbf{SU}(n,1)$-modules which may have non-vanishing $\Gamma$-cohomology in degree 1:

**Theorem 1.2.1** (Raghunathan)**.** *Let $\lambda : \mathbf{SU}(n,1) \longrightarrow \mathbf{GL}(W)$ be a real finite dimensional irreducible representation of $\mathbf{SU}(n,1) = \mathbf{SU}(V_\mathbb{C}, h_\mathbb{C})$. Let $\Gamma$ be a cocompact lattice in $SU(n,1)$. Then $H^1(\Gamma, W) = 0$ except if $W \simeq S^j V_\mathbb{C}$ for some $j \geq 0$, where $S^j$ denotes the $j-$th symmetric power.*

*Remark* 1.2.2. In this theorem $V_\mathbb{C}$ is seen as a *real* representation. In particular $S^j V_\mathbb{C}^* \simeq S^j V_\mathbb{C}$ as a real $SU(n,1)$-module.

As a corollary, $[\rho] \in \mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ is isolated except maybe if $\mathrm{Ad}\ j : \mathbf{SU}(n,1) \longrightarrow \mathbf{Aut}(\mathfrak{g})$ contains an $\mathbf{SU}(n,1)$-direct factor isomorphic to $S^j V_\mathbb{C}$ or $S^j V_\mathbb{C}^*$ for some integer $j \geq 0$.

*Remark* 1.2.3. For each $n$ and each $j$ one can, following a method first introduced by Kazhdan, exhibit a cocompact lattice $\Gamma$ of $SU(n,1)$ such that $H^1(\Gamma, S^j V_\mathbb{C}) \neq 0$, c.f. [2, chap. VIII].

*Example* 1.2.4. Let $\Gamma \overset{i}{\hookrightarrow} SU(n,1)$ be a cocompact lattice. Let $j = \mathrm{Id} : \mathbf{SU}(n,1) \longrightarrow \mathbf{SU}(n,1)$. By Raghunathan's theorem, $H^1(\Gamma, \mathrm{Ad}\ i) = 0$, thus $\Gamma$ cannot be non-trivially deformed in $SU(n,1)$. This was already proved by Weil [27].

*Example* 1.2.5. Let $j : \mathbf{SU}(n,1) = \mathbf{SU}(V_\mathbb{C}, h_\mathbb{C}) \hookrightarrow \mathbf{SO}(2n,2) = \mathbf{SO}((V_\mathbb{C})^\mathbb{R}, \mathrm{Re}\, h_\mathbb{C})$ be the natural embedding. Notice that $j$ factorizes as $\mathbf{SU}(n,1) \hookrightarrow \mathbf{U}(n,1) \hookrightarrow \mathbf{SO}(2n,2)$. One easily checks that the Lie algebra $\mathfrak{so}(2n,2)$ is isomorphic as an $\mathbf{SU}(n,1)$-module to the direct sum of irreducible modules $\mathbb{R} \oplus \mathfrak{su}(n,1) \oplus \Lambda^2 V_\mathbb{C}$, where $\mathbb{R} = \mathrm{Lie}(\mathbf{Z}(\mathbb{R}))$ is the Lie algebra of the centralizer $\mathbf{Z}$ of $\mathbf{SU}(n,1)$ in $\mathbf{U}(n,1)$. Thus $H^1(\Gamma, \mathrm{Ad}\ \rho) = H^1(\Gamma, \mathbb{R})$ and any deformation of $\rho$ in $SO(2n,2)$ is of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow \mathbf{Z}(\mathbb{R}) = S^1$ is a unitary character of $\Gamma$.

1.3. **Local rigidity.**

1.3.1. *Formal completion of* $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ *at* $[\rho]$. Studying first-order deformations is not enough for solving the local rigidity problem stated in the introduction: even if $j : \mathbf{SU}(n,1) \hookrightarrow \mathbf{G}$ is such that a priori $H^1(\Gamma, \mathrm{Ad}\,\rho)$ does not vanish it may happen that very few of these infinitesimal deformations can be integrated. However it is enough to study second order deformations. Let $\mathbf{H}^n_{\mathbb{C}} = SU(n,1)/U(n)$ denote the symmetric space of $SU(n,1)$: this is the complex hyperbolic $n$-space of negative lines in $(V_{\mathbb{C}}, h_{\mathbb{C}})$, it is naturally endowed with an $SU(n,1)$-invariant Kähler form $\omega_{\mathbf{H}^n_{\mathbb{C}}}$. Without loss of generality (passing to a finite index subgroup) one can assume that $\Gamma$ is torsion-free, so that $M = \Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$ is a compact Kähler manifold with fundamental group $\Gamma$. One can then apply the following formality theorem of Goldman-Millson [10] (for the case of complex variations of Hodge structures) and Simpson [25] (in general) :

**Theorem 1.3.1** (Goldman-Millson, Simpson)**.** *Let $M$ be a connected compact Kähler manifold with fundamental group $\Gamma$, $\mathbf{G}$ a real reductive algebraic group and $\rho : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ a reductive representation. Let $C \subset H^1(\Gamma, \mathrm{Ad}\,\rho)$ be the affine cone defined by*

$$C = \{u \in H^1(\Gamma, \mathrm{Ad}\,\rho)\ /[u,u] = 0 \in H^2(\Gamma, \mathrm{Ad}\,\rho)\}\ \ .$$

*Then the formal completion of $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$ at $[\rho]$ is isomorphic to the formal completion of the good quotient $C/H$, where $H$ denotes the centralizer of $\rho(\Gamma)$ in $G$.*

1.3.2. *Goldman-Millson rigidity result.* The first result about non-integrability of some first-order deformations for cocompact complex hyperbolic lattices is due to Goldman-Millson [9]: they consider the embedding

$$j : \mathbf{SU}(n,1) = \mathbf{SU}(V_{\mathbb{C}}, h_{\mathbb{C}}) \hookrightarrow \mathbf{SU}(n+1,1) = \mathbf{SU}(V_{\mathbb{C}} \oplus \mathbb{C}, h_{\mathbb{C}} \oplus 1)\ \ .$$

In this case the space of first-order deformations $H^1(\Gamma, \mathrm{Ad}\,\rho)$ at $\rho = j \circ i$ decomposes as $H^1(\Gamma, \mathbb{R}) \oplus H^1(\Gamma, V_{\mathbb{C}})$. The first summand $H^1(\Gamma, \mathbb{R})$ corresponds once more to the uninteresting deformations obtained by deforming $\Gamma$ in $U(n,1)$ by a curve of homomorphism into the centralizer $\mathbf{Z} = \mathbf{U}(1)$ of $\mathbf{SU}(n,1)$ in $\mathbf{U}(n,1)$. The second summand, which potentially corresponds to Zariski-dense deformations of $\rho$ in $SU(n+1,1)$, is non-zero for general $\Gamma$. However Goldman and Millson prove that none of these deformations can be integrated. Thus any representation $\lambda : \Gamma \longrightarrow SU(n+1,1)$ sufficiently close to $\rho$ is conjugate to a representation of the form $\rho \cdot \chi$, where $\chi : \Gamma \longrightarrow Z = S^1$. A similar result can be obtained by replacing the natural embedding $j : \mathbf{SU}(n,1) \hookrightarrow \mathbf{SU}(n+1,1)$ with the natural embedding $j : \mathbf{SU}(n,1) \hookrightarrow \mathbf{SU}(n+k,1)$ for some integer $k \geq 1$.

1.3.3. *Possible extensions.* One natural generalization of Goldman-Millson's result consists in studying *global rigidity* of representations $\rho : \Gamma \longrightarrow G$ with $G$ simple of Hermitian type, under certain assumptions on $\rho$. Let $X_G$ be the (Kähler) symmetric space associated to $G$, with Kähler form $\omega_G$. Let $\omega_M$ be the natural Kähler form on $M$. Let $f : \tilde{M} = \mathbf{H}^n_{\mathbb{C}} \longrightarrow X_G$ be any smooth $\rho$-equivariant map. The de Rham class $[f^*\omega_G] \in H^2_{dR}(M)$ depends only on

$\rho$, not on $f$, and will be denoted $[\rho^*\omega_G]$. Define the *Toledo invariant* $\tau(\rho)$ of $\rho$ as the number

$$\tau(\rho) = \frac{1}{n!}\int_M \rho^*\omega_G \wedge \omega_M^{n-1} \quad .$$

One easily shows that $\tau$ is a locally constant function on $\mathbf{M}(\Gamma, \mathbf{G})(\mathbb{R})$. Moreover it satisfies a Milnor-Wood inequality: under suitable normalizations of the metrics one has

$$|\tau(\rho)| \leq \mathrm{rk}\, X_G \cdot \mathrm{Vol}(M) \quad .$$

One expects a *global rigidity result* for representations $\rho : \Gamma \longrightarrow G$ with *maximal Toledo invariant*: namely $\rho$ is expected to be faithful, discrete and stabilizing a holomorphic totally geodesic copy of $\mathbf{H}^n_{\mathbb{C}}$ in $X_G$. This has been proven by Corlette [5, theor. 6.1] when $G$ is of rank one and $\Gamma$ cocompact (thus generalizing Goldman-Millson's result), then by Bürger-Iozzi [3] and Koziarz-Maubon [15] for $G$ of rank 1 and any complex hyperbolic lattice $\Gamma$. Recently Koziarz-Maubon [16] proved it when the group $G$ is of real rank 2. In the same kind of direction, we also refer to [6].

1.4. **The main result.** From the point of view of non-abelian Hodge theory, it is natural to enlarge the study of representations of complex hyperbolic lattices into groups of Hermitian type to the study of representations into groups of Hodge type (i.e. simple real Lie groups admitting discrete series). Among groups of Hodge type there is a particularly simple subclass : the groups of *quaternionic type*, that is such that the associated symmetric space $X_G$ is quaternionic-Kähler. The classical families in this class are $Sp(n,1)$, $SU(n,2)$ and $SO(n,4)$, $n \geq 1$. The corresponding 3 families of quaternionic Kähler non-compact irreducible symmetric spaces of dimension $4n$, $n \geq 2$, are: $\mathbf{H}^n_{\mathbb{H}} = Sp(n,1)/Sp(n) \cdot Sp(1)$, $X^n = SU(n,2)/S(U(n) \times U(2))$ and $Y^n = SO(n,4)/S(O(n) \times O(4))$. The only Kähler ones are $X^n$ and $Y^2$.

The main result of this paper concerns quaternionic deformations of cocompact complex hyperbolic lattices. Let $V_{\mathbb{H}} = V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{H}$ be the quaternionic right vector space of dimension $n+1$ (thus of real dimension $4n+4$) endowed with the quaternionic Hermitian form $h_{\mathbb{H}}$ of signature $(n,1)$ deduced from $h_{\mathbb{C}}$. The complex Hermitian part $H$ of $h_{\mathbb{H}}$ is a complex Hermitian form on $V_{\mathbb{C}} \oplus jV_{\mathbb{C}}$ of signature $(2n,2)$. Let $\mathbf{Sp}(n,1) = \mathbf{SU}(V_{\mathbb{H}}, h_{\mathbb{H}})$ be the special unitary algebraic $\mathbb{R}$-group of linear transformation of $(V_{\mathbb{H}}, h_{\mathbb{H}})$, $\mathbf{U}(2n,2)$ the unitary $\mathbb{R}$-group of linear transformations of $(V_{\mathbb{C}} \oplus jV_{\mathbb{C}}, H)$ and $\mathbf{SO}(4n,4)$ the special orthogonal group of linear transformation of $((V_{\mathbb{H}})_{\mathbb{R}}, \mathrm{Re}H)$. One obtains a natural sequence of embeddings

$$\mathbf{SU}(n,1) \overset{j_{\mathbf{U}(n,1)}}{\hookrightarrow} \mathbf{U}(n,1) \overset{j_{\mathbf{Sp}(n,1)}}{\hookrightarrow} \mathbf{Sp}(n,1) \overset{j_{\mathbf{U}(2n,2)}}{\hookrightarrow} \mathbf{U}(2n,2) \overset{j_{\mathbf{SO}(4n,4)}}{\hookrightarrow} \mathbf{SO}(4n,4)$$

corresponding to equivariant totally geodesic embeddings of symmetric spaces

$$\mathbf{H}^n_{\mathbb{C}} \overset{f_{\mathbf{H}^n_{\mathbb{H}}}}{\hookrightarrow} \mathbf{H}^n_{\mathbb{H}} \overset{f_{X^{2n}}}{\hookrightarrow} X^{2n} \overset{f_{Y^{4n}}}{\hookrightarrow} Y^{4n} \quad .$$

*Remark* 1.4.1. Notice that the totally geodesic embedding $\mathbf{H}^n_{\mathbb{C}} \stackrel{f_{X^{2n}} \circ f_{\mathbf{H}^n_{\mathbb{H}}}}{\hookrightarrow} X^{2n}$ between Hermitian symmetric spaces *is not holomorphic*: the pull-back $(f_{X^{2n}} \circ f_{\mathbf{H}^n_{\mathbb{H}}})^* \omega_{X^{2n}}$ is identically zero.

For $i : \Gamma \hookrightarrow SU(n,1)$ a cocompact lattice, and $\mathbf{G} = \mathbf{U}(n,1)$, $\mathbf{Sp}(n,1)$, $\mathbf{U}(2n,2)$ or $\mathbf{SO}(4n,4)$ let $\rho_{\mathbf{G}} : \Gamma \longrightarrow G$ be the composition $j_{\mathbf{G}} \circ \cdots \circ j_{\mathbf{U}(n,1)} \circ i$. The space of first-order deformations $H^1(\Gamma, \mathrm{Ad}\ \rho_{\mathbf{G}})$ at $[\rho_{\mathbf{G}}]$ is non-trivial for general $\Gamma$. As in Goldman-Millson's result we however prove :

**Theorem 1.4.2.** *Let* $\Gamma \stackrel{i}{\hookrightarrow} SU(n,1)$ *be a cocompact lattice and* $\mathbf{G}$ *one of the groups* $\mathbf{Sp}(n,1)$, $\mathbf{U}(2n,2)$ *or* $\mathbf{SO}(4n,4)$. *Then any morphism* $\lambda : \Gamma \longrightarrow G = \mathbf{G}(\mathbb{R})$ *close enough to* $\rho_{\mathbf{G}}$ *is conjugate to a representation of the form* $\rho_{\mathbf{G}} \cdot \chi$, *where* $\chi : \Gamma \longrightarrow Z_G(SU(n,1))$ *(thus* $Z_{Sp(n,1)}(SU(n,1)) = U(1)$ *and* $Z_{U(2n,2)}(SU(n,1)) = Z_{SO(4n,4)}(SU(n,1)) = U(1) \times U(1)$*).*

*Remark* 1.4.3. Following remark 1.4.1 notice that the representation $\rho_{\mathbf{U}(2n,2)} : \Gamma \longrightarrow U(2n,2)$ satisfies $\tau(\rho_{\mathbf{U}(2n,2)}) = 0$, thus has the smallest possible (in absolute value) Toledo invariant. In particular theorem 1.4.2 in this case is not covered by Koziarz-Maubon [16] (nor Corlette [6]). Also the same method applies to prove the case when $G = U(n+k,m)$, more generally when $G = Sp(n+k,m)$.

1.5. **Organization of the paper.** The proof of theorem 1.4.2 essentially reduces to the case $\mathbf{G} = \mathbf{Sp}(n,1)$, with an extra argument for $\mathbf{SO}(4n,4)$ (c.f. section 2). In sections 3, 4 and 5, we give a first proof of the main theorem 1.4.2 using Goldman-Millson's strategy: first, using Matsushima and Murakami's method [19], we show that harmonic 1-forms representing nontrivial classes in $H^1(\Gamma, \mathrm{Ad}\ \rho_{\mathbf{Sp}(n,1)})$ are severely restricted: most of their components vanish, and one can interpret them as $(1,0)$-forms $\alpha$ with values in a certain complex vector bundle. Then we show that the cup-square $[\alpha, \alpha] \in H^2(\Gamma, \mathrm{Ad}\ \rho_{\mathbf{Sp}(n,1)})$ paired with the Kähler form of complex hyperbolic space is proportional to the squared $L^2$-norm of $\alpha$, which implies the result.

In section 6, we indicate a more geometric proof of the main theorem 1.4.2 based on period domains and a result of Carlson-Toledo [4].

## 2. Infinitesimal deformations of lattices of $SU(n,1)$ in $G$

### 2.1. The groups.

**Definition 2.1.1.** *Let* $n > 1$ *be an integer. We denote by* $V_{\mathbb{R}}$ *the* $n+1$*-dimensional* $\mathbb{R}$*-vector space,* $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ *its complexification and* $V_{\mathbb{H}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}$ *its quaternionification (thus* $V_{\mathbb{H}}$ *is a* right *quaternionic vector space). We define* $\mathbf{GL}(n+1, \mathbb{H})$ *as the* $\mathbb{R}$*-group of* $\mathbb{H}$*-linear automorphism of* $V_{\mathbb{H}}$.

**Definition 2.1.2.** *Let* $Q_{\mathbb{R}}$ *be a real quadratic form of signature* $(n,1)$ *on* $V_{\mathbb{R}}$. *We denote by* $Q_{\mathbb{C}}$ *(respectively* $Q_{\mathbb{H}}$*) its complexification (resp. its quaternionification) on* $V_{\mathbb{C}}$ *(resp. on* $V_{\mathbb{H}}$*).*

**Definition 2.1.3.** *We denote by $h_{\mathbb{C}}$ the complex Hermitianization of $Q_{\mathbb{R}}$ on $V_{\mathbb{C}}$ and by $h_{\mathbb{H}}$ the quaternionic Hermitianization of $Q_{\mathbb{R}}$ on $V_{\mathbb{H}}$. Thus $h_{\mathbb{C}}(\mathbf{z},\mathbf{w}) = Q_{\mathbb{C}}(\mathbf{z},\overline{\mathbf{w}}^{\mathbb{C}})$ where $\overline{\mathbf{w}}^{\mathbb{C}}$ denotes the complex conjugate of $w \in V_{\mathbb{C}}$ and $h_{\mathbb{H}}(\mathbf{z},\mathbf{w}) = Q_{\mathbb{H}}(\mathbf{z},\overline{\mathbf{w}}^{\mathbb{H}})$ where $\overline{\mathbf{w}}^{\mathbb{H}}$ denotes the quaternionic conjugate of $w \in V_{\mathbb{H}}$.*

On the complex vector space $V_{\mathbb{H}} = V_{\mathbb{C}} \oplus jV_{\mathbb{C}}$, the quaternionic Hermitian form $h_{\mathbb{H}}(\mathbf{z},\mathbf{w})$ can be written as

$$h_{\mathbb{H}}(\mathbf{z},\mathbf{w}) = H(\mathbf{z},\mathbf{w}) - j\Omega(\mathbf{z},\mathbf{w}) \ ,$$

where $H$ is a complex Hermitian form on $V_{\mathbb{C}} \oplus jV_{\mathbb{C}}$ and $\Omega$ is the skew-symmetric complex bilinear form on $V_{\mathbb{C}} \oplus jV_{\mathbb{C}}$ defined by $\Omega(\mathbf{z},\mathbf{w}) = H(\mathbf{z}\cdot j, \bar{\mathbf{w}})$.

**Definition 2.1.4.** *We define the real algebraic groups:*

- $\mathbf{Sp}(n,1) = \mathbf{Sp}(V_{\mathbb{H}}, h_{\mathbb{H}})$ *as the subgroup of* $\mathbf{GL}(n+1,\mathbb{H})$ *preserving* $h_{\mathbb{H}}$.
- $\mathbf{U}(2n,2)$ *the unitary group* $\mathbf{U}(V_{\mathbb{C}} \oplus jV_{\mathbb{C}}, H)$.
- $\mathbf{SO}(4n,4)$ *the special orthogonal group* $\mathbf{SO}((V_{\mathbb{C}} \oplus jV_{\mathbb{C}})_{\mathbb{R}}, \mathrm{Re}H)$.

*Moreover we denote by $\mathbf{Sp}(2n+2,\mathbb{C})$ the complex symplectic group $\mathbf{Sp}(V_{\mathbb{C}} \oplus jV_{\mathbb{C}}, \Omega)$.*

The previous discussion implies immediately (where we consider $\mathbf{Sp}(2n+2,\mathbb{C})$ as a real algebraic group):

**Lemma 2.1.5.** $\mathbf{Sp}(n,1) = \mathbf{GL}(n+1,\mathbb{H}) \cap \mathbf{U}(2n,2) = \mathbf{Sp}(2n+2,\mathbb{C}) \cap \mathbf{U}(2n,2)$.

Consider the sequence of natural embeddings:

$$\mathbf{SU}(n,1) \overset{j_{\mathbf{U}(n,1)}}{\hookrightarrow} \mathbf{U}(n,1) \overset{j_{\mathbf{Sp}(n,1)}}{\hookrightarrow} \mathbf{Sp}(n,1) \overset{j_{\mathbf{U}(2n,2)}}{\hookrightarrow} \mathbf{U}(2n,2) \overset{j_{\mathbf{SO}(4n,4)}}{\hookrightarrow} \mathbf{SO}(4n,4) \ . \tag{2.1}$$

**Lemma 2.1.6.** *The sequence (2.1) induces an exact sequence of $U(n,1)$-modules (under the adjoint representation):*

(2.2)
$$\begin{aligned}0 \longrightarrow \mathfrak{u}(n,1) \longrightarrow &\mathfrak{sp}(n,1) = \mathfrak{u}(n,1) \oplus S^2V_{\mathbb{C}}^* \longrightarrow \mathfrak{u}(2n,2) = 2\mathfrak{u}(n,1) \oplus \Lambda^2 V_{\mathbb{C}}^* \oplus S^2V_{\mathbb{C}}^* \longrightarrow \\ &\longrightarrow \mathfrak{so}(4n,4) = 2\mathfrak{u}(n,1) \oplus 2\Lambda^2V_{\mathbb{C}}^* \oplus 2\Lambda^2\bar{V}_{\mathbb{C}}^* \oplus S^2V_{\mathbb{C}}^* \oplus S^2\bar{V}_{\mathbb{C}}^* \ .\end{aligned}$$

*Proof.* *Case* $\mathbf{G} = \mathbf{Sp}(n,1)$. Let $M \in \mathfrak{gl}(n+1,\mathbb{H})$ and $M = C + jD$ where $C, D \in \mathfrak{gl}(n+1,\mathbb{C})$. Then $M \in \mathfrak{sp}(n,1)$, if and only if $C \in \mathfrak{u}(n,1)$ and $QD$ is symmetric where $Q$ is the diagonal matrix with entries $1,\cdots,1,-1$. Write $E = QD$. If $A \in U(n,1)$,

$$A^{-1}MA = A^{-1}(C + jQE)A = A^{-1}CA + jQA^tEA.$$

So, under $U(n,1)$, $\mathfrak{sp}(n,1) = \mathfrak{u}(n,1) \oplus S^2V_{\mathbb{C}}^*$.

*Case* $\mathbf{G} = \mathbf{SU}(2n,2)$. Let $q = a + jb$ be a quaternion, with $a$, $b \in \mathbb{C}$. The matrix of left multiplication by $q$ is $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a}\end{smallmatrix}\right)$. Therefore, if $A \subset GL(n+1,\mathbb{C})$, its image under the embeddings $GL(n+1,\mathbb{C}) \to GL(n+1,\mathbb{H}) \to GL(2n+2,\mathbb{C})$ is $\left(\begin{smallmatrix} A & 0 \\ 0 & \bar{A}\end{smallmatrix}\right)$. If $\mathfrak{u}(n,1) \subset \mathfrak{gl}(n+1,\mathbb{C})$ is the subspace of matrices $A$ such that $A^*Q + QA = 0$ with $Q = \left(\begin{smallmatrix} I_n & 0 \\ 0 & -1\end{smallmatrix}\right)$, then $\mathfrak{u}(n,1)$ is mapped to $\mathfrak{u}(2n,2)$ defined as the subspace of matrices $M \in \mathfrak{gl}(2n+2)$ such that $M^*Q' + Q'M = 0$ with $Q' = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$. Thus, under the adjoint action of $U(n,1)$,

$$\mathfrak{u}(2n,2) = \mathfrak{u}(n,1) \oplus \mathfrak{u}(n,1) \oplus Hom_{\mathbb{C}}(\mathbb{C}^{n+1}, \mathbb{C}^{n+1}),$$

where $U(n,1)$ acts on a square matrix $N \in Hom_{\mathbb{C}}(\mathbb{C}^{n+1}, \mathbb{C}^{n+1})$ as follows,

$$(A, N) \mapsto A^{-1} N \bar{A}.$$

Putting $B = NQ$ conjugates this action to

$$(A, B) \mapsto A^{-1} B (A^{-1})^t,$$

i.e. $Hom_{\mathbb{C}}(\mathbb{C}^{n+1}, \mathbb{C}^{n+1}) = V_{\mathbb{C}}^* \otimes_{\mathbb{C}} V_{\mathbb{C}}^*$. Then

$$\mathfrak{u}(2n,2) = 2\mathfrak{u}(n,1) \oplus \Lambda^2 V_{\mathbb{C}}^* \oplus S^2 V_{\mathbb{C}}^*,$$

where $S^2 V_{\mathbb{C}}^*$ corresponds to matrices of the form $\begin{pmatrix} 0 & BQ \\ -B^*Q & 0 \end{pmatrix}$ in $\mathfrak{u}(2n,2)$ with $B$ symmetric.

*Case* $\mathbf{G} = \mathbf{SO}(4n,4)$. We have seen that the embedding $GL(n+1,\mathbb{C}) \to GL(n+1,\mathbb{H}) \to GL(2n+2,\mathbb{C})$ lands into the block diagonal subgroup $GL(n+1,\mathbb{C}) \times GL(n+1,\mathbb{C}) \subset GL(4n+4,\mathbb{R})$. In particular, $U(n,1)$ lands into $U(n,1) \times U(n,1) \subset O(2n,2) \times O(2n,2)$. Under $O(2n,2) \times O(2n,2)$,

$$\mathfrak{so}(4n,4) = \mathfrak{so}(2n,2) \oplus \mathfrak{so}(2n,2) \oplus End_{\mathbb{R}}(\mathbb{R}^{2n+2}).$$

Since $U(n,1)$ preserves a complex structure, $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$, every $\mathbb{R}$-linear map $L$ is the sum of a $\mathbb{C}$-linear and an anti-$\mathbb{C}$-linear one, $L = L_{\mathbb{C}} + L_{\bar{\mathbb{C}}}$, and the action of $A \in U(n,1)$ on $L$ is $A^{-1} L_{\mathbb{C}} \bar{A} + \bar{A}^{-1} L_{\bar{\mathbb{C}}} A$. Thus $End_{\mathbb{R}}(\mathbb{C}^{n+1})$ equals the sum of $End_{\mathbb{C}}(\mathbb{C}^{n+1}) = V_{\mathbb{C}}^* \otimes_{\mathbb{C}} V_{\mathbb{C}}^*$ and its conjugate $\bar{V}_{\mathbb{C}}^* \otimes_{\mathbb{C}} \bar{V}_{\mathbb{C}}^*$.

The map $\mathfrak{so}(2n,2) \to \Lambda^2(\mathbb{R}^{2n+2})^*$, $C \mapsto QC$ conjugates the adjoint $SO(2n,2)$ action with its action on real alternating 2-forms. In presence of the $U(n,1)$-invariant complex structure $J$, alternating 2-forms split into two subspaces $\Lambda_+$ and $\Lambda_-$. Indeed, $\Lambda^2 J$ is an involution. The inverse map $B \mapsto QB$ maps $\Lambda_+$ to $\mathfrak{u}(n,1) \subset \mathfrak{so}(2n,2)$. $J$ also acts as a derivation on alternating 2-forms, yielding a complex structure on $\Lambda_-$. Since

$$\mathfrak{so}(2n,2) \otimes \mathbb{C} = \Lambda^{2,0}(\mathbb{R}^{2n+2})^* \oplus \Lambda^{1,1}(\mathbb{R}^{2n+2})^* \oplus \Lambda^{0,2}(\mathbb{R}^{2n+2})^*,$$

$\Lambda^{1,1}(\mathbb{R}^{2n+2})^* = \Lambda_+ \otimes \mathbb{C}$, $\Lambda^{2,0}(\mathbb{R}^{2n+2})^* \oplus \Lambda^{0,2}(\mathbb{R}^{2n+2})^* = \Lambda_- \otimes \mathbb{C}$, thus, as a complex representation of $U(n,1)$, the $\Lambda_-$ factor in the first diagonal block is isomorphic to $\Lambda^2 V_{\mathbb{C}}^*$, and the $\Lambda_-$ factor in the second diagonal block is isomorphic to $\Lambda^2 \bar{V}_{\mathbb{C}}^*$.

We conclude that

$$\mathfrak{so}(4n,4) = \mathfrak{z} \oplus 2\mathfrak{su}(n,1) \oplus 2\Lambda^2 V_{\mathbb{C}}^* \oplus 2\Lambda^2 \bar{V}_{\mathbb{C}}^* \oplus S^2 V_{\mathbb{C}}^* \oplus S^2 \bar{V}_{\mathbb{C}}^*,$$

where $\mathfrak{z} = \mathbb{R}^2$ is the sum of the centers of the 2 copies of $\mathfrak{u}(n,1)$, generated respectively by $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$. □

Choose $J = \begin{pmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}$ as a complex structure on $\mathbb{R}^{2n+2}$.

**Lemma 2.1.7.** *If $M = A + iB$ is a complex matrix representing an anti-$\mathbb{C}$-linear map, it is mapped to $\left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$ in $GL(2n+2, \mathbb{R})$.*

*Then $Z \in S^2 V_{\mathbb{C}}^* \subset \mathfrak{so}(4n, 4)$, $Z' \in S^2 \bar{V}_{\mathbb{C}}^* \subset \mathfrak{so}(4n, 4)$ can be written*

$$Z = \begin{pmatrix} 0 & BQ' \\ -B^*Q' & 0 \end{pmatrix}, \quad Z' = \begin{pmatrix} 0 & B'Q' \\ -B'^*Q' & 0 \end{pmatrix}$$

*respectively, where $B = \left(\begin{smallmatrix} C & -D \\ D & C \end{smallmatrix}\right)$, $B^* = \left(\begin{smallmatrix} C & D \\ -D & C \end{smallmatrix}\right)$, $B' = B'^* = \left(\begin{smallmatrix} C' & D' \\ D' & -C' \end{smallmatrix}\right)$ and $C$, $D$, $C'$, $D'$ are symmetric real matrices.*

*Proof.* The first statement comes directly from calculation. The second follows from the first statement and the fact that the matrices are symmetric and the fact that they are in $\mathfrak{so}(4n, 4)$. $\square$

## 2.2. Some reductions.

**Lemma 2.2.1.** *The special case of the main theorem 1.4.2 for $\mathbf{G} = \mathbf{Sp}(n, 1)$ implies the main theorem for $\mathbf{G} = \mathbf{SU}(2n, 2)$, but not quite for $\mathbf{G} = \mathbf{SO}(4n, 4)$ .*

*Proof.* One deduces from the sequence (2.1) the following commutative diagram:

$$\begin{array}{ccccc} H^1(\Gamma, \mathrm{Ad}\,\rho_{\mathbf{Sp}(n,1)}) & \xhookrightarrow{j_{\mathbf{U}(2n,2)}} & H^1(\Gamma, \mathrm{Ad}\,\rho_{\mathbf{U}(2n,2)}) & \xhookrightarrow{j_{\mathbf{SO}(4n,4)}} & H^1(\Gamma, \mathrm{Ad}\,\rho_{\mathbf{SO}(4n,4)}) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ H^2(\Gamma, \mathrm{Ad}\,\rho_{\mathbf{Sp}(n,1)}) & \xhookrightarrow[j_{\mathbf{U}(2n,2)}]{} & H^2(\Gamma, \mathrm{Ad}\,\rho_{\mathbf{U}(2n,2)}) & \xhookrightarrow[j_{\mathbf{SO}(4n,4)}]{} & H^2(\Gamma, \mathrm{Ad}\,\rho_{\mathbf{SO}(4n,4)}) \end{array} ,$$

where $q : H^1(\Gamma, \mathrm{Ad}\,\rho_G) \longrightarrow H^2(\Gamma, \mathrm{Ad}\,\rho_G)$ denotes the quadratic map deduced from the symmetric bilinear map

$$[\cdot,\cdot] : H^1(\Gamma, \mathrm{Ad}\,\rho_G) \times H^1(\Gamma, \mathrm{Ad}\,\rho_G) \longrightarrow H^2(\Gamma, \mathrm{Ad}\,\rho_G) \; .$$

As $H^1(\Gamma, \Lambda^2 V_{\mathbb{C}}^*) = H^1(\Gamma, \mathfrak{su}(n, 1)) = 0$ and as the space $H^1(\Gamma, z_{\mathfrak{g}}(\mathfrak{su}(n, 1))$ belongs to the null-space of the quadratic map $q$, the proof of the main theorem for $\mathbf{G} = \mathbf{Sp}(n, 1)$ or $\mathbf{SU}(2n, 2)$ reduces to showing that the quadratic map $q : H^1(\Gamma, S^2 V_{\mathbb{C}}^*) \longrightarrow H^2(\Gamma, \mathfrak{sp}(n, 1)) \subset H^2(\Gamma, \mathfrak{u}(2n, 2))$ is anisotropic. Thus solving the case $\mathbf{G} = \mathbf{Sp}(n, 1)$ simultaneously solves the case $\mathbf{G} = \mathbf{SU}(2n, 2)$. However, the proof of the main theorem for $\mathbf{G} = \mathbf{SO}(4n, 4)$, which amounts to showing that the quadratic map

$$q : H^1(\Gamma, S^2 V_{\mathbb{C}}^*) \oplus H^1(\Gamma, S^2 \bar{V}_{\mathbb{C}}^*) \longrightarrow H^2(\Gamma, \mathfrak{so}(4n, 4))$$

is anisotropic, requires an extra computation. $\square$

**Lemma 2.2.2.** *The main theorem 1.4.2 for torsion-free lattices $\Gamma$ implies the main theorem in full generality.*

*Proof.* Let $\Gamma \overset{i}{\hookrightarrow} SU(n,1)$ be any cocompact lattice. By Selberg's lemma [24] the lattice $\Gamma$ admits a torsion-free normal finite index subgroup $\Gamma'$. By [19, prop.6.1 p.385] the natural pull-back map $H^\bullet(\Gamma, \mathrm{Ad}\,\rho) \longrightarrow H^\bullet(\Gamma', \mathrm{Ad}\,\rho_{|\Gamma'})$ is injective, which implies the result. □

## 3. A classical vanishing theorem

3.1. **Matsushima and Murakami's vanishing theorem.** Let $\mathbf{L}$ be a simple real algebraic group of non-compact type, $L = \mathbf{L}(\mathbb{R})$ its Lie group of real points, $K$ a maximal compact subgroup of $L$, $\theta : \mathfrak{l} \longrightarrow \mathfrak{l}$ the Cartan involution associated to $K$ of the Lie algebra $\mathfrak{l}$ of $L$, $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\theta$, $X = L/K$ the symmetric space of $L$.

Let $i : \Gamma \hookrightarrow L$ be a cocompact lattice. By lemma 2.2.2 we can assume that $\Gamma$ is torsion-free. Let $\pi : \Gamma\backslash L \longrightarrow M = \Gamma\backslash L/K$ the natural principal $K$-bundle on the locally symmetric manifold $M$. Let $\rho : \mathbf{L} \longrightarrow \mathbf{GL}(F)$ be a finite dimensional representation of $\mathbf{L}$. For $p$ a positive integer, the cohomology $H^p(\Gamma, F)$ is canonically isomorphic to the cohomology $H^p(M, F_\rho)$ of the local system $F_\rho$ on $M$ associated to $\rho$, which can be computed using the usual de Rham complex $(C^\bullet(M, F_\rho), d)$.

Fix an admissible inner product $(,)_F$ on $F$, i.e. one which is $\rho(K)$-invariant and for which elements of $\rho(\mathfrak{p})$ are symmetric. This is enough to define a natural Laplacian $\Delta : C^\bullet(M, F_\rho) \longrightarrow C^\bullet(M, F_\rho)$ and prove that $H^p(M, F_\rho)$ is isomorphic to the space

$$\mathcal{H}^p(M, F_\rho) = \{\eta \in C^P(M, F_\rho) \;/\; \Delta\eta = 0\}$$

of harmonic forms [19, section 6].

Following p. 376 of [19], define an $F$-valued differential form $\eta^0$ on $L$ as follows.

$$\eta^0_s = \rho(s^{-1})\pi^*\eta_s, \quad s \in L. \tag{3.1}$$

Fix a Killing-orthonormal basis $X_1, \dots, X_N$ of $\mathfrak{p}$. The induced inner product on $\mathrm{Hom}(\mathfrak{p}, F)$ is given by

$$(\eta, \zeta) = \sum_{h=1}^{N} (\eta(X_h), \zeta(X_h))_F.$$

**Definition 3.1.1.** *Let $p$ be a positive integer. One defines a symmetric operator $T_p$ on* $\mathrm{Hom}(\mathfrak{p}, F)$ *as follows.*

$$\forall\, \eta \in \mathrm{Hom}(\mathfrak{p}, F), \;\; \forall\, Y \in \mathfrak{p}, \;\; T_p\eta(Y) = \frac{1}{p}\sum_{k=1}^{N} \rho(X_k)^2\eta(Y) + \rho([Y, X_k])\eta(X_k) \;\; .$$

**Theorem 3.1.2** (Matsushima-Murakami). [19, theor.7.1] *If $\eta$ is a harmonic $p$-form on $M = \Gamma \setminus L/K$, then*

$$\int_{\Gamma\backslash L} (T\eta^0, \eta^0) \leq 0.$$

*As a consequence, if the symmetric operator $T_p$ on* $\mathrm{Hom}(\mathfrak{p}, F)$ *is positive definite, then the cohomology group* $H^p(\Gamma, F_\rho)$ *vanishes.*

3.2. **Case of 1-forms.**

**Proposition 3.2.1.** *Let* $\eta \in \mathrm{Hom}(\mathfrak{p}, F)$. *Let* $\beta : \mathfrak{p} \otimes \mathfrak{p} \longrightarrow F$ *denote the $F$-valued bilinear form on* $\mathfrak{p}$ *defined by* $\beta(X,Y) = \rho(X)(\eta(Y))$. *Split* $\beta = \sigma + \alpha$ *into its symmetric and skew-symmetric parts. Then* $(T\eta, \eta) = 2|\alpha|^2 + |\mathrm{Trace}(\beta)|^2$. *So* $\alpha = \mathrm{Trace}(\beta) = 0$.

*Proof.* The first term in $(T\eta, \eta)$ is

$$
\begin{aligned}
(T_1\eta, \eta) &:= \sum_{k,\,\ell=1}^{N} (\rho(X_k)^2\eta(X_\ell), \eta(X_\ell))_F = \sum_{k,\,\ell=1}^{N} (\rho(X_k)\eta(X_\ell), \rho(X_k)\eta(X_\ell))_F \\
&= \sum_{k,\,\ell=1}^{N} |\beta(X_k, X_\ell)|_F^2 = |\beta|^2 \quad .
\end{aligned}
$$

The second term in $(T\eta, \eta)$ is

$$
(T_2\eta, \eta) := \sum_{k,\,\ell=1}^{N} (\rho([X_\ell, X_k])\eta(X_k), \eta(X_\ell))_F = (T_3\eta, \eta) - (T_4\eta, \eta) \quad ,
$$

where

$$
\begin{aligned}
(T_3\eta, \eta) &:= \sum_{k,\,\ell=1}^{N} (\rho(X_\ell) \circ \rho(X_k)\eta(X_k), \eta(X_\ell))_F = \sum_{k,\,\ell=1}^{N} (\rho(X_k)\eta(X_k), \rho(X_\ell)\eta(X_\ell))_F \\
&= \sum_{k,\,\ell=1}^{N} (\beta(X_k, X_k), \beta(X_\ell, X_\ell))_F = |\sum_{k=1}^{N} \beta(X_k, X_k)|_F^2 \\
&= |\mathrm{Trace}(\beta)|^2 \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
(T_4\eta, \eta) &:= \sum_{k,\,\ell=1}^{N} (\rho(X_k) \circ \rho(X_\ell)\eta(X_k), \eta(X_\ell))_F = \sum_{k,\,\ell=1}^{N} (\rho(X_\ell)\eta(X_k), \rho(X_k)\eta(X_\ell))_F \\
&= (\beta, \beta \circ \phi) \quad .
\end{aligned}
$$

Here, $\phi \in \mathrm{End}(\mathfrak{p} \otimes \mathfrak{p})$ is defined by $\phi(X,Y) = (Y,X)$. Note that $\phi$ merely permutes vectors in the basis of $\mathfrak{p} \otimes \mathfrak{p}$. Therefore

$$
(\sigma, \alpha) = (\sigma \circ \phi, \alpha \circ \phi) = (\sigma, -\alpha) = -(\sigma, \alpha),
$$

thus $(\sigma, \alpha) = 0$. Hence $|\beta|^2 = |\sigma|^2 + |\alpha|^2$ and

$$
(\beta, \beta \circ \phi) = (\sigma + \alpha, \sigma - \alpha) = |\sigma|^2 - |\alpha|^2 = |\beta|^2 - 2|\alpha|^2 \quad .
$$

Summing up,

$$
(T\eta, \eta) = |\beta|^2 + |\mathrm{Trace}(\beta)|^2 - (|\beta|^2 - 2|\alpha|^2) = 2|\alpha|^2 + |\mathrm{Trace}(\beta)|^2.
$$

The last assertion follows from Theorem 3.1.2. ∎

## 4. Consequences of Matsushima-Murakami's method

4.1. **Restriction on $S^2V_{\mathbb{C}}^*$-harmonic one-forms.** From now on, $\mathbf{L} = \mathbf{SU}(n,1)$, $K = U(n)$ and $F = S^2V_{\mathbb{C}}^*$ is the space of complex quadratic forms on $\mathbb{C}^{n+1}$, with the usual action of $\mathbf{GL}(n+1,\mathbb{C})$, $(X,Q) \mapsto X^tQX$, restricted to $\mathbf{SU}(n,1)$. The admissible inner product on $F$ is the usual $\mathbf{U}(n+1)$-invariant Hermitian form.

Let $\mathfrak{su}(n,1) = \mathfrak{u}(n) \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{su}(n,1)$. Here, $\mathfrak{u}(n) = s(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$ consists of traceless block-diagonal skew-Hermitian complex $(n+1)\times(n+1)$ matrices, and $\mathfrak{p}$ consists of complex matrices of the form $\left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right)$, $x \in \mathbb{C}^n$.

**Definition 4.1.1.** *We denote by $\chi : U(n) \longrightarrow \mathbb{C}^*$ the standard character* det.

The $SU(n,1)$-module $V_{\mathbb{C}}$ decomposes as a $U(n)$-module :

$$V_{\mathbb{C}} = \mathfrak{p} \otimes \chi^{-1} \oplus \chi^{-1} \ ,$$

(notice that $\mathfrak{p} \otimes \chi^{-1}$ is nothing else than the standard $U(n)$-module $\mathbb{C}^n$). Thus $S^2V_{\mathbb{C}}^*$ decomposes as $U(n)$-modules as

$$S^2V_{\mathbb{C}}^* = (S^2\mathfrak{p}^* \oplus \mathfrak{p}^* \oplus \mathbb{C}) \otimes \chi^2$$

(notice that the $U(n)$-module $S^2\mathfrak{p}^*$ is nothing else than $S^2V_{\mathbb{C}}^* \cap \mathfrak{sp}(n)$) and $\mathrm{Hom}(\mathfrak{p}, S^2V_{\mathbb{C}}^*)$ as :

$$\mathrm{Hom}(\mathfrak{p}, S^2V_{\mathbb{C}}^*) = (\mathrm{Hom}(\mathfrak{p}, S^2\mathfrak{p}^*) \oplus \mathrm{End}\,\mathfrak{p}^* \oplus \mathfrak{p}) \otimes \chi^2 \ .$$

As $U(n)$-modules, $\mathfrak{p}$ and $S^2\mathfrak{p}^*$ are $\mathbb{C}$-linear. Thus the $U(n)$-module $\mathrm{Hom}(\mathfrak{p}, S^2\mathfrak{p}^*)$ contains as a direct factor $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{p}, S^2\mathfrak{p}^*)$, which itself contains $S^3\mathfrak{p}^*$ as a direct factor.

**Proposition 4.1.2.** *Let $\Gamma$ be a cocompact lattice in $L = SU(n,1)$. Let $\alpha$ be a $\Gamma$-equivariant harmonic $S^2V_{\mathbb{C}}^*$-valued 1-form on $\mathbf{H}_{\mathbb{C}}^n$. Then, for all $Y \in T_{x_0}\mathbf{H}_{\mathbb{C}}^n = \mathfrak{p}$, $\alpha_{x_0}(Y) \in S^2\mathfrak{p}^* \otimes \chi^2$. Furthermore, $\alpha_{x_0} \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{p}, S^2\mathfrak{p}^*) \otimes \chi^2$ is $\mathbb{C}$-linear and belongs to the summand $S^3\mathfrak{p}^* \otimes \chi^2$ of* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{p}, S^2\mathfrak{p}^*) \otimes \chi^2$.

*Remark* 4.1.3. Since $S^2\bar{V}_{\mathbb{C}}^*$ is the conjugate vector space of $S^2V_{\mathbb{C}}^*$, Proposition 4.1.2 implies that $\Gamma$-equivariant harmonic $S^2\bar{V}_{\mathbb{C}}^*$-valued 1-forms on $H_{\mathbb{C}}^n$ are in fact $S^2\bar{\mathfrak{p}}^* \otimes \chi^2$-valued $(0,1)$-forms. The fact that $\alpha_{x_0} \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{p}, S^2\mathfrak{p}^*) \otimes \chi^2$ will be used to prove the main theorem.

4.2. **Proof of Proposition 4.1.2.** A straightforward calculation yields

**Lemma 4.2.1.** *Let $X = \left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right)$, $x \in \mathbb{C}^n$, be a vector of $\mathfrak{p}$. Let $Z = \left(\begin{smallmatrix} A & B \\ B^t & d \end{smallmatrix}\right) \in S^2V_{\mathbb{C}}^*$. Then*

$$\rho(X)(Z) = X^tZ + ZX = \begin{pmatrix} (Bx^*)^t + Bx^* & Ax + d\bar{x} \\ (Ax + d\bar{x})^t & 2x^tB \end{pmatrix}.$$

Let $\eta \in \mathrm{Hom}(\mathfrak{p}, S^2V_{\mathbb{C}}^*)$ be represented by a matrix $Z = Y \mapsto \begin{pmatrix} A(Y) & B(Y) \\ B(Y)^t & d(Y) \end{pmatrix}$ of $\mathbb{R}$-linear forms on $\mathfrak{p}$. Then the bilinear form $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2^t & \beta_3 \end{pmatrix}$ becomes a triple of matrix valued bilinear forms on $\mathbb{C}^n$,

$$\begin{aligned}\beta_1(x,y) &= (B(y)x^*)^t + B(y)x^*,\\ \beta_2(x,y) &= A(y)x + d(y)\bar{x},\\ \beta_3(x,y) &= 2x^tB(y).\end{aligned}$$

According to Proposition 3.2.1, $(T\eta,\eta)=0$ if and only if the following 6 equations hold.

$$\beta_1,\ \beta_2 \text{ and } \beta_3 \text{ are symmetric},\ \ \mathrm{Trace}(\beta_1)=0,\ \ \mathrm{Trace}(\beta_2)=0,\ \ \mathrm{Trace}(\beta_3)=0.$$

**Lemma 4.2.2.** *If $\beta_3$ and $\beta_1$ are symmetric, then $B=0$.*

*Proof.* Let $B_{\mathbb{C}}$ and $B_{\bar{\mathbb{C}}}$ denote the $\mathbb{C}$-linear (resp. anti $\mathbb{C}$-linear) components of the $\mathbb{R}$-linear map $B$. Matrixwise, each of $B_{\mathbb{C}}$ and $B_{\bar{\mathbb{C}}}$ is given by a $n\times n$ complex matrix $\mathcal{B}_{\mathbb{C}}$ (resp. $\mathcal{B}_{\bar{\mathbb{C}}}$), and $B(y)=\mathcal{B}_{\mathbb{C}}y+\mathcal{B}_{\bar{\mathbb{C}}}\bar{y}$. Thus

$$\beta_3(x,y) = x^t\mathcal{B}_{\mathbb{C}}y + x^t\mathcal{B}_{\bar{\mathbb{C}}}\bar{y}$$

is the sum of a $\mathbb{C}$-bilinear and a sesquilinear form. If $\beta_3$ is symmetric, the sesquilinear part vanishes (i.e. $\mathcal{B}_{\bar{\mathbb{C}}}=0$), and $\mathcal{B}_{\mathbb{C}}$ is symmetric.

Next,

$$\beta_1(x,y) = (\mathcal{B}_{\mathbb{C}}yx^*)^t + \mathcal{B}_{\mathbb{C}}yx^*$$

is sesquilinear. If $\beta_1$ is symmetric, it is identically zero. Since rank one matrices of the form $yx^*$ span all $n\times n$ complex matrices, $(\mathcal{B}_{\mathbb{C}}M)^t+\mathcal{B}_{\mathbb{C}}M=0$ for all $n\times n$ complex matrices $M$. Take $M=\mathcal{B}_{\mathbb{C}}^*$ and take the trace to conclude that $\mathcal{B}_{\mathbb{C}}=0$. □

**Lemma 4.2.3.** *If $\beta_2$ is symmetric and* $\mathrm{Trace}(\beta_2)=0$, *then $d=0$ and $A(y)$ depends $\mathbb{C}$-linearly on $y$. Furthermore, identifying $\mathbb{C}^n$-valued bilinear maps with trilinear forms, $(x,y)\mapsto A(y)x$ is fully symmetric.*

*Proof.* Let $A_{\mathbb{C}}$ and $A_{\bar{\mathbb{C}}}$ denote the $\mathbb{C}$-linear (resp. anti $\mathbb{C}$-linear) components of the $\mathbb{R}$-linear map $A:\mathbb{C}^n\to S^2(\mathbb{C}^n)$. Similarly, let $d_{\mathbb{C}}$ and $d_{\bar{\mathbb{C}}}$ denote the $\mathbb{C}$-linear (resp. anti $\mathbb{C}$-linear) components of the $\mathbb{R}$-linear form $d$. If $\beta_2:(x,y)\mapsto A(y)x+d(y)\bar{x}$ is symmetric, then

$$\begin{cases}\forall x,\, y\in\mathbb{C}^n,\ A_{\bar{\mathbb{C}}}(y)x & = d_{\mathbb{C}}(x)\bar{y},\\ (x,y)\mapsto d_{\bar{\mathbb{C}}}(y)\bar{x} & \text{is symmetric},\\ (x,y)\mapsto A_{\mathbb{C}}(y)x & \text{is symmetric}.\end{cases}$$

The trace of the restriction of $\beta$ to a complex line $\mathbb{C}e$, $|e|=1$, depends only on its sesquilinear part

$$\beta_2^{sq}(x,y) = A_{\bar{\mathbb{C}}}(y)x + d_{\mathbb{C}}(y)\bar{x} = d_{\mathbb{C}}(x)\bar{y} + d_{\mathbb{C}}(y)\bar{x}.$$

and is equal to $2\beta_2^{sq}(e,e)=4d_{\mathbb{C}}(e)\bar{e}$. Let $e_1,\dots,e_n$ be a Hermitian basis of $\mathbb{C}^n$. Then

$$\mathrm{Trace}(\beta_2) = \mathrm{Trace}(\beta_2^{sq}) = 4\sum_{k=1}^{n} d_{\mathbb{C}}(e_k)\bar{e}_k.$$

Since $\mathrm{Trace}(\beta_2) = 0$, we get $d_{\mathbb{C}} = 0$. This implies that $A_{\bar{\mathbb{C}}}(y)x = 0$ for all $x$ and $y$, i.e. $A_{\bar{\mathbb{C}}} = 0$.

Next, pick a nonzero vector $y \in \ker(d_{\bar{\mathbb{C}}})$. Since $(x, y) \mapsto d_{\bar{\mathbb{C}}}(y)\bar{x}$ is symmetric, for all $x \in \mathbb{C}^n$, $d_{\bar{\mathbb{C}}}(x)\bar{y} = 0$, thus $d_{\bar{\mathbb{C}}} = 0$.

Finally, view the components of $A_{\mathbb{C}}(y)x$ in some Hermitian basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$ as bilinear forms on $\mathbb{C}^n$, with respective matrices $\mathcal{A}^1 = A_{\mathbb{C}}(e_1), \ldots, \mathcal{A}^n = A_{\mathbb{C}}(e_n)$. Since the values $A_{\mathbb{C}}(y)$ are symmetric matrices, these matrices are symmetric, $\mathcal{A}^\ell_{jk} = \mathcal{A}^\ell_{kj}$. But for every $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$,

$$(A_{\mathbb{C}}(y)x)_j = y_\ell \mathcal{A}^\ell_{jk} x_k = (A_{\mathbb{C}}(x)y)_j = x_k \mathcal{A}^k_{j\ell} y_\ell.$$

This implies that $\mathcal{A}^\ell_{jk} = \mathcal{A}^k_{j\ell}$. Hence $\mathcal{A}^\ell_{jk}$ is fully symmetric. $\square$

## 5. Second order obstruction

### 5.1. Cup-product, case $\mathbf{G} = \mathbf{Sp}(n, 1)$.

**Definition 5.1.1.** *Let $\lambda : \mathfrak{sp}(n, 1) \longrightarrow \mathbb{R}$ be the $SU(n, 1)$-invariant linear form defined by the Killing inner product with the $SU(n, 1)$-invariant vector $iI_{n+1}$, which generates the centralizer of $SU(n, 1)$ in $Sp(n, 1)$.*

The restriction of the Killing form of $Sp(n, 1)$ to $Sp(n)$ is proportional to the Killing form of $Sp(n)$, which is proportional to $\Re e(\mathrm{Trace}_{\mathbb{H}}(A^* A))$. Therefore, for $A \in \mathfrak{sp}(n) \subset \mathfrak{sp}(n, 1)$,

$$\lambda(A) = A \cdot iI_{n+1} = -\Re e(i \, \mathrm{Trace}_{\mathbb{H}}(A)),$$

**Lemma 5.1.2.** *Let $\alpha$ be an $\mathfrak{sp}(n)$-valued $(1, 0)$-form on $T_{x_0}\mathbf{H}^n_{\mathbb{C}} = \mathfrak{p}$. Assume that $\alpha$ belongs to $Hom_{\mathbb{C}}(\mathfrak{p}, S^2\mathfrak{p}^*)$. Thanks to the Lie bracket of $\mathfrak{sp}(n)$, $[\alpha, \alpha]$ becomes an $\mathfrak{sp}(n)$-valued 2-form on $\mathfrak{p}$. Let $\omega$ denote the Kähler form on $\mathfrak{p}$. There is a nonzero constant $c$ such that*

$$\lambda \circ [\alpha, \alpha] \wedge \omega^{n-1} = c|\alpha|^2 \omega^n.$$

*Proof.* Recall that the embedding of $S^2\mathfrak{p}^*$ to $\mathfrak{sp}(n)$ is defined by $A \mapsto jQA$ where $Q = (I_n, -1)$ a diagonal matrix. Write $\alpha = jQ\delta$ where $\delta$ is a symmetric complex matrix of $(1, 0)$-forms. Then, for all $Y$, $Y' \in \mathfrak{p}$,

$$\alpha \wedge \alpha(Y, Y') = \alpha(Y) \otimes \alpha(Y') - \alpha(Y') \otimes \alpha(Y) \in \mathfrak{sp}(n) \otimes \mathfrak{sp}(n) \ ,$$

$$[\alpha, \alpha](Y, Y') = [\alpha(Y), \alpha(Y')] - [\alpha(Y'), \alpha(Y)] = 2[\alpha(Y), \alpha(Y')] \in \mathfrak{sp}(n).$$

Let $A$, $B$ be two symmetric complex matrices. The Lie bracket of their images in $\mathfrak{sp}(n)$ is

$$[jQA, jQB] = jQAjQB - jQBjQA = -\bar{A}B + \bar{B}A,$$

(note it belongs to $\mathfrak{u}(n)$), thus

$$[\alpha, \alpha](Y, Y') = -2(\overline{\delta(Y)}\delta(Y') - \overline{\delta(Y')}\delta(Y)),$$

showing that $[\alpha,\alpha]$ is a matrix of $(1,1)$-forms. Up to a nonzero constant,

$$\lambda\circ[\alpha,\alpha](Y,Y')=\Im m(\mathrm{Trace}_{\mathbb{C}}(\delta(Y)^*\delta(Y'))).$$

Note that $\lambda\circ[\alpha,\alpha](Y,iY)=\mathrm{Trace}(\delta(Y)^*\delta(Y))=|\delta(Y)|^2>0$.

If $\phi$ is a $(1,1)$-form on $\mathbb{C}^n$, then

$$\frac{\phi\wedge\omega^{n-1}}{\omega^n}=\frac{2}{n}\sum_{k=1}^{n}\phi(E_k,iE_k),$$

where $E_1,\dots,E_n$ is a unitary basis of $\mathbb{C}^n$ (i.e. $(E_1,iE_1,\dots,E_n,iE_n)$ is an orthonormal basis of the underlying real Euclidean vectorspace). Therefore

$$\frac{\lambda\circ[\alpha,\alpha]\wedge\omega^{n-1}}{\omega^n}=\frac{2}{n}\sum_{k=1}^{n}|\delta(E_k)|^2$$

is a nonzero multiple of $|\alpha|^2$. □

The following proposition finishes the proof of theorem 1.4.2, in case $\mathbf{G}=\mathbf{Sp}(n,1)$:

**Proposition 5.1.3.** *Let $\alpha$ be a nonzero harmonic $\mathfrak{sp}(n,1)$-valued 1-form on $\Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$. Assume that the component of $\alpha$ on the centralizer of $\mathfrak{su}(n,1)$ in $\mathfrak{sp}(n,1)$ vanishes. Then $[\alpha,\alpha]\neq 0$ in $H^2(\Gamma,\mathfrak{sp}(n,1))$. In particular, $\alpha$ does not integrate into a nontrivial deformation of the conjugacy class of the embedding $\Gamma\hookrightarrow Sp(n,1)$.*

*Proof.* By contradiction. According to Weil's vanishing theorem, the $\mathfrak{su}(n,1)$-component of $\alpha$ vanishes, thus $\alpha$ is $S^2V^*_{\mathbb{C}}$-valued. According to Proposition 4.1.2, $\alpha$ can be viewed as a smooth section of the homogeneous bundle over $\Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$ whose fiber is the subspace $S^3\mathfrak{p}^*$ of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{p},S^2\mathfrak{p}^*)$. In particular, $\alpha$ can be viewed pointwise as a $\mathfrak{sp}(n)$-valued $(1,0)$-form. Assume that $[\alpha,\alpha]=0$, i.e., that there exists a $\mathfrak{sp}(n,1)$-valued 1-form $\eta$ on $M$ such that $d\eta=[\alpha,\alpha]$. Then, with Lemma 5.1.2,

$$\begin{aligned}2c\parallel\alpha\parallel^2_{L^2(M)}&=\int_M c|\alpha|^2\omega^n=\int_M\lambda\circ[\alpha,\alpha]\wedge\omega^{n-1}\\&=\int_M\lambda\circ(d\eta)\wedge\omega^{n-1}=\int_M d(\lambda\circ\eta\wedge\omega^{n-1})=0\quad,\end{aligned}$$

thus $\alpha=0$, contradiction. □

5.2. **Cup-product, case $\mathbf{G}=\mathbf{SO}(4n,4)$.** Choose $J=\begin{pmatrix}0&-I_{n+1}\\I_{n+1}&0\end{pmatrix}$ as a complex structure on $\mathbb{R}^{2n+2}$.

**Definition 5.2.1.** *On $\mathfrak{so}(4n,4)$, there are $SU(n,1)$-invariant linear forms $\lambda'$ and $\lambda''$, given by Killing inner product with the $SU(n,1)$-invariant vectors $J'=\begin{pmatrix}J&0\\0&-J\end{pmatrix}$ and $J''=\begin{pmatrix}J&0\\0&J\end{pmatrix}$.*

**Proposition 5.2.2.** *$\lambda'$ vanishes on $[S^2\bar V^*_{\mathbb{C}},S^2\bar V^*_{\mathbb{C}}]$, $\lambda''$ vanishes on $[S^2V^*_{\mathbb{C}},S^2V^*_{\mathbb{C}}]$, and they both vanish on $[S^2V^*_{\mathbb{C}},S^2\bar V^*_{\mathbb{C}}]$.*

*Proof.* Let $Z = \begin{pmatrix} 0 & BQ' \\ -B^*Q' & 0 \end{pmatrix}$, $Z' = \begin{pmatrix} 0 & B'Q' \\ -B'^*Q' & 0 \end{pmatrix} \in S^2V^*_{\mathbb{C}} \oplus S^2\bar{V}^*_{\mathbb{C}}$. View $Z$ and $Z'$ as vectors in $\mathfrak{so}(4n,4)$. Then using the fact that $Q'$ commutes with $B, B', B^*, B'^*$,

$$\begin{aligned}\lambda'([Z,Z']) &= \mathrm{Trace}(J[-BB'^* + B'B^* + B^*B' - B'^*B])\\ \lambda''([Z,Z']) &= \mathrm{Trace}(J[-BB'^* + B'B^* - B^*B' + B'^*B]).\end{aligned}$$

If both $Z$ and $Z'$ are in $S^2V^*_{\mathbb{C}}$,

$$\lambda'([Z,Z']) = 2\mathrm{Trace}(J[B'B^* - BB'^*])$$

and vanishes if both are in $S^2\bar{V}^*_{\mathbb{C}}$, using the fact that $J$ anti-commutes with $B'$ and $B'^*$.

If both $Z$ and $Z'$ are in $S^2\bar{V}^*_{\mathbb{C}}$,

$$\lambda''([Z,Z']) = 2\mathrm{Trace}([B'B^* - BB'^*]J)$$

and vanishes if both are in $S^2V^*_{\mathbb{C}}$.

If $Z \in S^2V^*_{\mathbb{C}}$ and $Z' \in S^2\bar{V}^*_{\mathbb{C}}$, both $\lambda'$, $\lambda''$ vanish. Indeed, since $BJ = JB$ and $B'J = -JB'$, for example, $\mathrm{Trace}(JB'B^*) = \mathrm{Trace}(B'B^*J) = \mathrm{Trace}(-B'JB^*) = \mathrm{Trace}(-B'B^*J) = 0$. □

If $Z$ and $Z' \in S^2\mathfrak{p}^* \subset S^2V^*_{\mathbb{C}}$, write $B = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$ and $B' = \begin{pmatrix} C' & -D' \\ D' & C' \end{pmatrix}$ where $C$, $C'$, $D$, $D'$ are symmetric.

$$\lambda'([Z,Z']) = 8\mathrm{Trace}(DC' - CD').$$

The complex structure on $S^2V^*_{\mathbb{C}}$ is $B \mapsto \mathcal{J}(B) = JB$, i.e. $(C,D) \mapsto (-D,C)$. Thus

$$\begin{aligned}\lambda'([Z,\mathcal{J}(Z)]) &= 8\mathrm{Trace}(-D^2 - C^2)\\ &= -8\mathrm{Trace}(D^\top D + C^\top C) = -4|B|^2 = -2|Z|^2.\end{aligned}$$

If $Z$ and $Z' \in S^2\bar{\mathfrak{p}}^* \subset S^2\bar{V}^*_{\mathbb{C}}$, write $B = B^* = \begin{pmatrix} C & D \\ D & -C \end{pmatrix}$ and $B' = B'^* = \begin{pmatrix} C' & D' \\ D' & -C' \end{pmatrix}$ where $C$, $C'$, $D$ and $D'$ are symmetric. Then

$$\lambda'([Z,Z']) = 8\mathrm{Trace}(DC' - CD').$$

The complex structure on $S^2\bar{V}^*_{\mathbb{C}}$ is $B \mapsto \mathcal{J}(B) = -JB$, i.e. $(C,D) \mapsto (D,-C)$. Thus

$$\begin{aligned}\lambda'([Z,\mathcal{J}(Z)]) &= 8\mathrm{Trace}(D^2 + C^2)\\ &= 8\mathrm{Trace}(D^\top D + C^\top C) = 4|B|^2 = 2|Z|^2.\end{aligned}$$

Let $\eta$ be an equivariant harmonic $\mathfrak{so}(4n,4)$-valued 1-form. According to [21] and Proposition 4.1.2, $\eta = \tau + \alpha + \alpha'$ where $\tau$ is $\mathfrak{z}$-valued, $\alpha$ is a $S^2\mathfrak{p}^*$-valued $(1,0)$-form and $\alpha'$ a $S^2\bar{\mathfrak{p}}^*$-valued $(0,1)$-form.

We have seen that

$$\lambda' \circ ([\alpha \wedge \alpha']) = 0,$$

$$\lambda'' \circ ([\alpha \wedge \alpha']) = 0.$$

Therefore

$$(\lambda' + \lambda'') \circ ([(\alpha + \alpha') \wedge (\alpha + \alpha')]) = \lambda' \circ ([\alpha \wedge \alpha]) + \lambda'' \circ ([\alpha' \wedge \alpha']).$$

If $Y \in \mathfrak{p}$, since $\alpha$ has type $(1,0)$, $\alpha(iY) = \mathcal{J}(\alpha(Y))$,

$$\lambda' \circ ([\alpha \wedge \alpha])(Y, iY) = 2|\alpha(Y)|^2.$$

Since $\alpha'$ has type $(0,1)$, $\alpha'(iY) = -\mathcal{J}(\alpha'(Y))$,

$$\lambda'' \circ ([\alpha' \wedge \alpha'])(Y, iY) = 2|\alpha'(Y)|^2.$$

It follows that

$$\frac{(\lambda' + \lambda'') \circ [(\alpha + \alpha') \wedge (\alpha + \alpha')] \wedge \omega^{n-1}}{\omega^n} = \frac{2}{n}(|\alpha|^2 + |\alpha'|^2).$$

Again, if the cohomology class of $[(\alpha + \alpha') \wedge (\alpha + \alpha')]$ vanishes, then the $L^2$ norm of $\alpha$ and $\alpha'$ vanishes. This shows that the quadratic map induced by bracket-cup product on $H^1(\Gamma, \mathfrak{so}(4n,4))/H^1(\Gamma, \mathfrak{z})$ is anisotropic.

## 6. A more geometric proof

In this section, we sketch a second proof of Proposition 4.1.2 for $\mathbf{G} = \mathbf{Sp}(n,1)$, using a theorem of Carlson-Toledo [4] and some non-Abelian Hodge theory.

6.1. **Reminder on quaternionic Kähler manifolds.** For the convenience of the reader we recall some general facts on quaternionic Kähler manifolds. We refer to [23] for a panorama.

**Definition 6.1.1.** *A Riemannian manifold $M$ of dimension $4n$ is quaternionic Kähler if its holonomy group is contained in the subgroup $Sp(n)Sp(1) := Sp(n) \times_{\mathbb{Z}/2\mathbb{Z}} Sp(1)$ of $SO(4n)$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by $(-I, -1)$.*

It is well known that such a Riemannian manifold $M$ is always Einstein. Even if $M$ is not necessarily Kähler, its geometry can be essentially understood from the point of view of complex geometry.

**Definition 6.1.2.** *We denote by $\mathcal{P}_M$ the canonical $Sp(n)Sp(1)$-reduction of the principal bundle of orthogonal frames of $M$, and by $\mathcal{E}_M$ the canonical 3-dimensional parallel sub-bundle $\mathcal{P}_M \times_{Sp(n)Sp(1)} \mathbb{R}^3$ of $\mathrm{End}(TM)$.*

**Definition 6.1.3.** *Let $p : Z \longrightarrow M$ be the $S^2$-fiber bundle on $M$ associated to the action of $Sp(1)/\mathbb{Z}_2 \simeq SO(3)$ on $S^2$: $Z = \mathcal{P}_M \times_{Sp(n)Sp(1)} S^2$. The space $Z$ is called the twistor space of $M$.*

In other words, $Z$ is the unit sphere of $\mathcal{E}_M$.

**Theorem 6.1.4.** [22, theor. 4.1] *Let $M$ be a quaternionic Kähler manifold. Then its twistor space $Z$ admits a canonical complex structure, for which the fibers of $p : Z \longrightarrow M$ are complex rational curves.*

As $\mathcal{E}_M$ is parallel, it inherits from the Levi-Civita connection on $TM$ a linear connection compatible with the metric. It follows that the corresponding horizontal distribution induces a horizontal distribution $T_h^{\mathbb{R}}(Z) \subset T^{\mathbb{R}}(Z)$. In the case where the scalar curvature of $M$ is non-zero, one can show that $T_h^{\mathbb{R}}(Z)$ naturally defines an horizontal holomorphic distribution $T_h Z \subset TZ$ making $Z$ a holomorphic contact manifold [23, prop. 5.2].

Another ingredient of some importance for us is the following:

**Lemma 6.1.5.** *Any quaternionic Kähler manifold $M$ admits a non-zero closed 4-form $\Omega_M$, canonical up to homothety.*

*Proof.* Just notice that the $Sp(n)Sp(1)$-module $\bigwedge^4(\mathbb{R}^{4n})^*$ admits a unique trivial submodule of rank 1. □

**Lemma 6.1.6.** *The form $\Omega_M$ (conveniently normalized) is the Chern-Weil form of the first Pontryagin class $p_1(\mathcal{E}_M) \in H^4(M, \mathbb{Z})$.*

*Proof.* This is proved in [22, p.148-151]. □

6.2. **Quaternionic Kähler symmetric spaces.** The description of quaternionic Kähler symmetric spaces and their twistor spaces is due to Wolf [28], following Boothby [1]. There exists 3 families of quaternionic Kähler non-compact irreducible symmetric spaces of dimension $4n$, $n \geq 2$: $\mathbf{H}^n_{\mathbb{H}} = Sp(n,1)/Sp(n) \cdot Sp(1)$, $X^n = SU(n,2)/S(U(n) \times U(2))$ and $Y^n = SO(n,4)/S(O(n) \times O(4))$. The only Kähler one is $X^n$. In each case the isotropy group is of the form $K \cdot Sp(1)$ and the twistor space is obtained by replacing the $Sp(1)$-factor by $U(1)$. Notice that the twistor map for $X^n$ is *not holomorphic.*

By functoriality of the twistor construction, we associate to the sequence of totally geodesic quaternionic Kähler embeddings $\mathbf{H}^n_{\mathbb{H}} \hookrightarrow X^{2n} \longrightarrow Y^{4n}$ the commutative diagram :
(6.1)

$$\begin{array}{ccccc} D^{\mathbb{H}}_{2n+1} = \frac{Sp(n,1)}{Sp(n)\times U(1)} & \hookrightarrow & D^{\mathbb{C}}_{4n+1} = \frac{SU(2n,2)}{S(U(1)\times U(2n)\times U(1))} & \hookrightarrow & D^{\mathbb{R}}_{8n+1} = \frac{SO(4n,4)}{S(O(4n)\times U(2))} \\ \downarrow p & & \downarrow p & & \downarrow p \\ \mathbf{H}^n_{\mathbb{H}} & \hookrightarrow & X^{2n} & \hookrightarrow & Y^{4n} \end{array},$$

where the vertical maps are twistor fibrations and the horizontal maps on the top line are holomorphic closed *horizontal* (i.e. preserving the contact structure) immersions.

6.3. **An invariant for quaternionic representations.** Let $X$ a smooth manifold and $\rho : \Gamma = \pi_1(X) \longrightarrow Sp(n,1)$ a representation. Choose any $\rho$-equivariant smooth map $\phi : \tilde{X} \longrightarrow \mathbf{H}^n_{\mathbb{H}}$. The pull-back $\phi^*\mathcal{E}_{\mathbf{H}^n_{\mathbb{H}}}$ is a $\Gamma$-equivariant rank 3 real bundle on $\tilde{X}$. Thus it descends to a bundle on $X$, still denoted $\phi^*\mathcal{E}_{\mathbf{H}^n_{\mathbb{H}}}$. By lemma 6.1.6 and the functoriality of characteristic classes, the 4-form $\phi^*\Omega_{\mathbf{H}^n_{\mathbb{H}}}$ represents the Pontryagin class $p_1(\phi^*\mathcal{E}_{\mathbf{H}^n_{\mathbb{H}}}) \in H^4(X, \mathbb{Z})$. As $\mathbf{H}^n_{\mathbb{H}}$ is a contractible space, any two $\rho$-equivariant maps $\phi, \phi' : \tilde{X} \longrightarrow \mathbf{H}^n_{\mathbb{H}}$ are $\rho$-equivariantly homotopic. Finally the class $[\phi^*\Omega_{\mathbf{H}^n_{\mathbb{H}}}] \in H^4(X, \mathbb{Z})$ depends only on $\rho$.

**Definition 6.3.1.** *Let $X$ a smooth manifold and $\rho : \Gamma = \pi_1(X) \longrightarrow Sp(n,1)$ a representation. We denote by $c_\rho \in H^4(X, \mathbb{Z})$ the class $[\phi^* \Omega_{\mathbf{H}^n_{\mathbb{H}}}] \in H^4(X, \mathbb{Z})$.*

*Remark* 6.3.2. The invariant $c_\rho$ is a quaternionic version of the (Hermitian) Toledo invariant.

**Lemma 6.3.3.** *Let $M$ be a smooth manifold and $\Gamma = \pi_1(M)$. The function*

$$c : \mathbf{M}(\Gamma, \mathbf{Sp}(n,1))(\mathbb{R}) \longrightarrow H^4(X, \mathbb{Z})$$

*which to $[\rho]$ associates $c_\rho$ is constant on connected components of $\mathbf{M}(\Gamma, \mathbf{Sp}(n,1))(\mathbb{R})$.*

*Proof.* This follows immediately from the integrality of $c_\rho$. □

6.4. **Link with Hodge theory.** Twistor spaces of quaternionic Kähler symmetric spaces are the simplest examples of Griffiths's period domains for variations of Hodge structures. One easily proves the following lemma (a proof for the global embedding $\mathbf{H}^n_{\mathbb{H}} \hookrightarrow Y^{4n}$ is provided in [4, p.192-193], the proof for the other maps is similar):

**Lemma 6.4.1.** *Let $K$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $r_K$ be 4, 2 or 1 respectively.*

- *Each twistor space $D^K_{2r_K \cdot (n+1)}$ is the Griffiths's period domain for polarized weight 2 pure Hodge structures with Hodge numbers $(2, 4n, 2)$ on $\mathbb{R}^{4n+4}$, stable under $K$-multiplication (when we identify $\mathbb{R}^{4n+4}$ with $K^{r_K \cdot (n+1)}$).*
- *The inclusions in the sequence $D^{\mathbb{H}}_{2n+1} \hookrightarrow D^{\mathbb{C}}_{4n+1} \hookrightarrow D^{\mathbb{R}}_{8n+1}$ correspond to the functors partially forgetting the $K$-stability condition, for the inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.*

6.5. **Any deformation is a complex variation of Hodge structures.** Let $n > 1$, $\mathbf{SU}(n,1) = \mathbf{SU}(V_{\mathbb{C}}, h_{\mathbb{C}})$ and let $j : \mathbf{SU}(n,1) \hookrightarrow \mathbf{U}(n,1) \hookrightarrow \mathbf{Sp}(n,1) = \mathbf{SU}(V_{\mathbb{H}}, h_{\mathbb{H}})$ be the natural embedding. Let $f : \mathbf{H}^n_{\mathbb{C}} \hookrightarrow \mathbf{H}^n_{\mathbb{H}}$ be the corresponding totally geodesic $U(n,1)$-equivariant embedding. Notice that it canonically lifts to a holomorphic $U(n,1)$-equivariant embedding $\tilde{f} : \mathbf{H}^n_{\mathbb{C}} \hookrightarrow D^{\mathbb{H}}_{2n+1}$ making the $U(n,1)$-equivariant diagram

$$\begin{tikzcd} \mathbf{H}^n_{\mathbb{C}} \arrow[r, hook, "\tilde{f}"] \arrow[dr, hook, "f"'] & D^{\mathbb{H}}_{2n+1} \arrow[d, "p"] \\ & \mathbf{H}^n_{\mathbb{H}} \end{tikzcd} ,$$

commutative.

Let $i : \Gamma \hookrightarrow SU(n,1) = \mathbf{SU}(n,1)(\mathbb{R})$ be a cocompact torsion-free lattice and $\rho = j \circ i : \Gamma \longrightarrow Sp(n,1)$ the corresponding representation. Let $M$ be the compact Kähler manifold $\Gamma \backslash \mathbf{H}^n_{\mathbb{C}}$.

**Lemma 6.5.1.** *$c_\rho \neq 0 \in H^4(M, \mathbb{Z})/torsion$.*

*Proof.* By lemma 6.1.6 the (descent to $M$ of the) curvature form $f^* \Omega_{\mathbf{H}^n_{\mathbb{H}}}$ is nothing else than the Chern-Weil form of (the descent to $M$ of) the bundle $p_1(f^* \mathcal{E}_{\mathbf{H}^n_{\mathbb{H}}})$. As the standard representation $\mathbb{R}^3$ of $SO(3)$ decomposes as $\mathbb{R} \oplus \mathbb{C}$ as an $U(1) \subset SO(3)$-module, where $\mathbb{R}$

is the trivial representation and $\mathbb{C}$ is the standard $U(1)$-module, we obtain that $f^*\mathcal{E}_{\mathbf{H}^n_{\mathbb{H}}}$ is the direct sum of the trivial bundle and (the descent to $M$ of) the holomorphic line bundle $\mathcal{L} = U(n,1) \times_{U(n)\times U(1)} \mathbb{C} \longrightarrow \mathbf{H}^n_{\mathbb{C}}$. Thus

$$p_1(f^*\mathcal{E}_{\mathbf{H}^n_{\mathbb{H}}}) = -c_2(\mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(\mathcal{L} \oplus \overline{\mathcal{L}}) = c_1^2(\mathcal{L}) \ .$$

Finally $f^*\Omega_{\mathbf{H}^n_{\mathbb{H}}} = \omega_M^2$, where $\omega_M$ is the standard Kähler form on the quotient $M$ of $\mathbf{H}^n_{\mathbb{C}}$. Thus $c_\rho = [\omega_M]^2 \neq 0 \in H^4(M, \mathbb{Z})/\text{torsion}$ (as $n > 1$). □

Let $\lambda : \Gamma \longrightarrow Sp(n,1)$ be a reductive representation in the same connected component of $M(\Gamma, \mathbf{Sp}(n,1))(\mathbb{R})$ as $\rho$ (recall that the quotient $M(\Gamma, \mathbf{Sp}(n,1))$ of $\mathrm{Hom}(\Gamma, \mathbf{Sp}(n,1))$ only knows about reductive representations). By lemma 6.3.3 and 6.5.1, $c_\lambda = c_\rho \neq 0 \in H^4(M, \mathbb{Z})/\text{torsion}$. Let $f_\lambda : \tilde{M} \longrightarrow \mathbf{H}^n_{\mathbb{H}}$ be the $\lambda$-equivariant harmonic map. As $c_\lambda = [f_\lambda^*\Omega_{\mathbf{H}^n_{\mathbb{H}}}] \in H^4(M, \mathbb{R})$, the harmonic map $f_\lambda$ is of rank at least 4 on some open subset of $\tilde{M}$. Thus we can apply the following result of Carlson-Toledo:

**Theorem 6.5.2.** [4, theor. 6.1.] *Let $X$ be a compact Kähler manifold, $\lambda : \Gamma = \pi_1(X) \longrightarrow \mathbf{Sp}(n,1)(\mathbb{R})$ a reductive representation and $f_\lambda : \tilde{X} \longrightarrow \mathbf{H}^n_{\mathbb{H}}$ the $\rho$-equivariant harmonic map, where $\tilde{X}$ denotes the universal covering of $X$.* Assume that the rank of the differential $df : T\tilde{X} \longrightarrow T\mathbf{H}^n_{\mathbb{H}}$ is larger than 2 at some point $x$ of $X$. *Then there exists a horizontal holomorphic $\lambda$-equivariant period map $\tilde{f}_\lambda : \tilde{X} \longrightarrow D^{\mathbb{H}}_{2n+1}$ making the following diagram commute:*

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "\tilde{f}_\lambda"] \arrow[dr, hook, "f_\lambda"'] & D^{\mathbb{H}}_{2n+1} \arrow[d, "p"] \\ & \mathbf{H}^n_{\mathbb{H}} \end{tikzcd} \ .$$

Thus we obtain that any deformation $\lambda$ of $\rho$ is still the monodromy of a variation of Hodge structure $\tilde{f}_\lambda : \tilde{M} \longrightarrow D^{\mathbb{H}}_{2n+1}$. To prove proposition 4.1.2 is thus equivalent to the following:

**Proposition 6.5.3.** *The tangent space at $(f, \rho)$ to the space of $\mathbf{Sp}(n,1)$-variations of Hodge structures identifies (as a real vector space) with:*

$$H^1(M, \mathfrak{z}_{\mathfrak{u}(n,1)}\mathfrak{su}(n,1)) \oplus H^0(M, S^3T^*M \otimes L_{\chi^2}) \ ,$$

*where $\mathfrak{z}_{\mathfrak{u}(n,1)}\mathfrak{su}(n,1) \simeq \mathbb{R}$ denotes the Lie algebra of the centralizer $S^1$ of $SU(n,1)$ in $U(n,1)$ and $L_{\chi^2} = \Gamma\backslash SU(n,1) \times_{U(n),\chi^2} \mathbb{C}$ denotes the automorphic line bundle on $M$ associated to the character $\chi^2 : U(n) \longrightarrow S^1$.*

## 6.6. **Proof of proposition 6.5.3.**

6.6.1. *Explicit notations.* We fix a basis $(e_0, \cdots, e_n)$ of $V_{\mathbb{R}}$ over $\mathbb{R}$. This is also a basis of $V_{\mathbb{C}}$ over $\mathbb{C}$. As a $(2n+2)$-$\mathbb{C}$-vector space, $V_{\mathbb{H}}$ is isomorphic to $V_{\mathbb{C}} \oplus jV_{\mathbb{C}}$ and we choose $(f_0 = e_o, \cdots, f_n = e_n, f_{n+1} = -e_n \cdot j, \cdots, f_{2n} = -e_1 \cdot j, f_{2n+1} = e_0 \cdot j)$. The right multiplication by $j \in \mathbb{H}$ on a column vector $\mathbf{v} = (v_0, \cdots, v_{2n})^t$ in the basis $(f_i)_{0 \leq i \leq 2n+1}$ is given by $\mathbf{v} \cdot j = \begin{pmatrix} & -J_1 \\ J_1^t & \end{pmatrix} \cdot \overline{\mathbf{v}}$, where $J_1$ denotes the $(n+1) \times (n+1)$-matrix

$$J_1 = \begin{pmatrix} & & -1 \\ & \cdot^{\cdot^{\cdot}} 1 & \\ 1 & & \end{pmatrix} \quad .$$

This realizes the group $\mathbf{GL}(n+1, \mathbb{H})$ of $\mathbb{H}$-linear automorphism of $V_{\mathbb{H}}$ as the matrix group

$$\mathbf{GL}(n+1, \mathbb{H}) = \{X \in \mathbf{GL}(2n+2, \mathbb{C}) \;/\; X \cdot \begin{pmatrix} & J_1 \\ -J_1^t & \end{pmatrix} = \begin{pmatrix} & J_1 \\ -J_1^t & \end{pmatrix} \cdot \overline{X}\} \quad .$$

The real orthogonal form $Q_{\mathbb{R}}$ of signature $(n,1)$ on $V_{\mathbb{R}}$ is defined by: $Q_{\mathbb{R}}(x_0 \cdot e_0 + \cdots x_n e_n) = -x_0^2 + x_1^2 + \cdots x_n^2$. We define the matrix

$$J_0 = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \quad .$$

Let $\Lambda_0$ be the $(n+1) \times (n+1)$ matrix $\mathrm{diag}(-1, 1, \cdots, 1)$ in the basis $(e_i)_{0 \leq i \leq n}$. Let $Q = J_0 \cdot \Lambda_0 \cdot J_0 = \mathrm{diag}(1, \cdots 1, -1)$. As a complex Hermitian form, $H$ is of signature $(2, 2n)$ with matrix $\Lambda = \mathrm{diag}(-1, 1, \cdots, 1, -1) = \mathrm{diag}(\Lambda_0, Q)$ in the basis $(f_i)_{0 \leq i \leq 2n+1}$. Thus

$$\mathbf{Sp}(n,1) = \{X \in \mathbf{GL}(2n+2, \mathbb{C}) \;/\; X \cdot \begin{pmatrix} & J_1 \\ -J_1^t & \end{pmatrix} = \begin{pmatrix} & J_1 \\ -J_1^t & \end{pmatrix} \cdot \overline{X} \text{ and } X^* \cdot \Lambda \cdot X = \Lambda \;\} \;,$$

where $X^*$ denotes the complex trans-conjugate of $X$.

**Definition 6.6.1.** *We denote by* $\tilde{J} = \begin{pmatrix} & J_0 \\ -J_0 & \end{pmatrix}$ *the product* $\Lambda \cdot \begin{pmatrix} & J_1 \\ -J_1^t & \end{pmatrix}$.

The complex symplectic form $\Omega$ on $V_{\mathbb{C}} \oplus jV_{\mathbb{C}}$ has matrix $\tilde{J}$ in the basis $(f_i)_{0 \leq i \leq 2n+1}$. One can rewrite

$$\mathbf{Sp}(n,1) = \{X \in \mathbf{GL}(2n+2, \mathbb{C}) \;/\; X^* \cdot \Lambda \cdot X = \Lambda \text{ and } X^t \cdot \tilde{J} \cdot X = \tilde{J} \;\} \quad .$$

We thus recover the isomorphism $\mathbf{Sp}(n,1) = \mathbf{U}(2n,2) \cap \mathbf{Sp}(2n+2, \mathbb{C})$, where $\mathbf{Sp}(2n+2, \mathbb{C}) = \mathbf{Sp}(V_{\mathbb{C}} \oplus jV_{\mathbb{C}}, \Omega)$.

From the previous descriptions we obtain:

$$\mathfrak{sp}(2n+2, \mathbb{C}) = \{\begin{pmatrix} A & B \\ C & -J_0 \cdot A^t \cdot J_0 \end{pmatrix} \;/\; J_0 \cdot C = C^t \cdot J_0 \text{ and } J_0 \cdot B = B^t \cdot J_0\} \quad .$$

$$\mathfrak{u}(2n,2) = \{\begin{pmatrix} A & B \\ -\Lambda_0 \cdot B^* \cdot Q & D \end{pmatrix} \;/\; A^* \cdot \Lambda_0 + \Lambda_0 \cdot A = D^* \cdot Q + Q \cdot D = 0\} \quad .$$

$$\mathfrak{sp}(n,1) = \{\begin{pmatrix} A & B \\ -\Lambda_0 \cdot B^* \cdot Q & -J_0 \cdot A^t \cdot J_0 \end{pmatrix} \;/\; A^* \cdot \Lambda_0 + \Lambda_0 \cdot A = 0 = J_0 \cdot B - B^t \cdot J_0\} \quad .$$

Notice that the canonical embedding $j_* : \mathfrak{su}(n,1) \hookrightarrow \mathfrak{sp}(n,1)$ factorizes through $\mathfrak{u}(n,1)$. The embedding $\mathfrak{u}(n,1) = \{A \;/\; A^* \cdot \Lambda_0 + \Lambda_0 \cdot A = 0\} \longrightarrow \mathfrak{sp}(n,1)$ is the morphism associating to $A \in \mathfrak{u}(n,1)$ the element $\begin{pmatrix} A & 0 \\ 0 & -J_0 \cdot A^t \cdot J_0 \end{pmatrix} \in \mathfrak{sp}(n,1)$.

6.6.2. *Hodge filtration.* The action of the subgroup $S^1$ of $\mathbf{Sp}(n,1)$ with complexified Lie algebra $\mathbb{C}\cdot v$ (where $v = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathfrak{sp}(2n+2,\mathbb{C})$) on the Lie algebra $\mathfrak{sp}(2n+2,\mathbb{C})$ defines a filtration $F^\bullet\mathfrak{sp}(2n+2,\mathbb{C})$ on $\mathfrak{sp}(2n+2,\mathbb{C})$. The non-positively graded pieces are given explicitly as follows (where $J_0$ now denotes the $n\times n$-matrix $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$) :

$$\mathrm{Gr}^0\mathfrak{sp}(2n+2,\mathbb{C}) = \{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & X & R & 0 \\ 0 & S & -J_0\cdot X^t\cdot J_0 & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} / J_0\cdot R = R^t\cdot J_0 \text{ and } J_0\cdot S = S^t\cdot J_0\} \ .$$

$$\mathrm{Gr}^{-1}\mathfrak{sp}(2n+2,\mathbb{C}) = \{ \begin{pmatrix} 0 & u & v & 0 \\ 0 & 0 & 0 & J_0\cdot v^t \\ 0 & 0 & 0 & -J_0\cdot u^t \\ 0 & 0 & 0 & 0 \end{pmatrix} u, v\in\mathbb{C}^n\} \ .$$

$$\mathrm{Gr}^{-2}\mathfrak{sp}(2n+2,\mathbb{C}) = \{ \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} / z\in\mathbb{C}\} \ .$$

This Hodge decomposition restricts to a Hodge decomposition of the complexified Lie algebra $\mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{su}(n,1)\otimes_{\mathbb{R}}\mathbb{C}$.

6.6.3. *Automorphic bundles.* Let $K$ denote the maximal compact subgroup $S(U(n)\times U(1))$ of $SU(n,1)$ and $K_{\mathbb{C}}\simeq GL(n,\mathbb{C})$ its complexification. This is a Levi subgroup of the parabolic subgroup $Q\cap\mathbf{SU}(n,1)(\mathbb{C})$, where $Q$ denotes the parabolic subgroup of $\mathbf{Sp}(2n+2,\mathbb{C})$ with Lie algebra $F^0\mathfrak{sp}(2n+2,\mathbb{C})$. Notice that any $K_{\mathbb{C}}$-module $\mathfrak{m}$ is canonically a $Q\cap\mathbf{SU}(n,1)(\mathbb{C})$-module (realizing $K_{\mathbb{C}}$ as a quotient of $Q\cap\mathbf{SU}(n,1)(\mathbb{C})$). The natural inclusion

$$\mathbf{H}^n_{\mathbb{C}} = SU(n,1)/K \hookrightarrow \mathbf{SU}(n,1)(\mathbb{C})/(Q\cap\mathbf{SU}(n,1)(\mathbb{C})) \simeq \mathbf{P}^n\mathbb{C}$$

is the natural open embedding of the period domain $\mathbf{H}^n_{\mathbb{C}}$ into its dual.

**Definition 6.6.2.** *Given a $K_{\mathbb{C}}$-module $\mathfrak{m}$, we denote by $\mathcal{F}(\mathfrak{m})$ the holomorphic automorphic vector bundle $\Gamma\backslash(\mathbf{SU}(n,1)(\mathbb{C})\times_{Q\cap\mathbf{SU}(n,1)(\mathbb{C})}\mathfrak{m})_{|\mathbf{H}^n_{\mathbb{C}}}$ with fiber $\mathfrak{m}$ on $M$.*

**Definition 6.6.3.** *We denote by $L_M$ the automorphic line bundle $n+1$-th root of $K_M^{-1}$, where $K_M$ denotes the canonical line bundle on $M$.*

6.6.4. *Non-Abelian Hodge theory.* Let $(\mathrm{Gr}\mathcal{P}_f,\theta_f)$ be the system of Hodge bundles associated to the variation of Hodge structures $(\rho,f)$. Thus

$$\mathrm{Gr}\mathcal{P}_f = \mathcal{F}(\mathrm{Gr}^\bullet\mathfrak{sp}(2n+2,\mathbb{C}))\ ,$$

and

$$\theta_f\in H^0(M,\Omega^1_M\otimes\mathrm{Gr}\mathcal{P}_f)\ .$$

As proven in [14] the tangent space $T$ to the space of $\mathbf{Sp}(n,1)$-variations of Hodge structures (equivalently: to the subspace of systems of Hodge bundles in the space of semistable $\mathrm{Sp}(2n+2,\mathbb{C})$-Higgs bundles) on $M$, modulo the trivial deformations in the centralizer $U(n,1)$ of $SU(n,1)$, identifies with the hypercohomology of complex of coherent sheaves:

$$\mathbb{H}^1(M,\mathcal{F}(\frac{\mathrm{Gr}^0\mathfrak{sp}(2n+2,\mathbb{C})}{\mathrm{Gr}^0\mathfrak{gl}(n+1,\mathbb{C})})\xrightarrow{\theta_f}\mathcal{F}(\frac{\mathrm{Gr}^{-1}\mathfrak{sp}(2n+2,\mathbb{C})}{\mathrm{Gr}^{-1}\mathfrak{gl}(n+1,\mathbb{C})})\otimes\Omega^1_M\xrightarrow{\theta_f}\mathcal{F}(\frac{\mathrm{Gr}^{-2}\mathfrak{sp}(2n+2,\mathbb{C})}{\mathrm{Gr}^{-2}\mathfrak{gl}(n+1,\mathbb{C})})\otimes\Omega^2_M)\ .$$

Notice that :

$$
\begin{aligned}
\mathcal{F}(\frac{\mathrm{Gr}^0\mathfrak{sp}(2n+2,\mathbb{C})}{\mathrm{Gr}^0\mathfrak{gl}(n+1,\mathbb{C})}) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S^2\Omega^1_M\otimes L^2_M & 0 \\ 0 & S^2TM\otimes L_M^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{F}(\frac{\mathrm{Gr}^{-1}\mathfrak{sp}(2n+2,\mathbb{C})}{\mathrm{Gr}^{-1}\mathfrak{gl}(n+1,\mathbb{C})}) &= \begin{pmatrix} 0 & 0 & v\in\Omega^1_M\otimes L^2_M & 0 \\ 0 & 0 & 0 & J_0\cdot v^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathcal{F}(\frac{\mathrm{Gr}^{-2}\mathfrak{sp}(2n+2,\mathbb{C})}{\mathrm{Gr}^{-2}\mathfrak{gl}(n+1,\mathbb{C})}) &= \begin{pmatrix} 0 & 0 & 0 & L^2_M \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad .
\end{aligned}
$$

One easily compute $\theta_f$ in these coordinates, where 1 denotes the identity matrix of the adequate size :

$$
T = \mathbb{H}^1(M, \begin{pmatrix} 0 & S^2\Omega^1_M\otimes L^2_M \\ S^2T_M\otimes L_M^{-2} & 0 \end{pmatrix} \xrightarrow{\binom{0}{1}} \begin{pmatrix} (\Omega^1_M\otimes L^2_M)\otimes\Omega^1_M \\ 0 \end{pmatrix} \xrightarrow{\binom{0}{1}} L^2_M\otimes\Omega^2_M) \quad .
$$

Thus:

$$
\begin{aligned}
T &= \mathbb{H}^1(M, \left( \begin{pmatrix} S^2T_M\otimes L_M^{-2} \\ S^2\Omega^1_M\otimes L^2_M \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} S^2\Omega^1_M\otimes L^2_M \\ \Omega^2_M\otimes L^2_M \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \Omega^2_M\otimes L^2_M \right) \\
&= H^1(M, S^2T_M\otimes L_M^{-2}) \quad .
\end{aligned}
$$

Notice that $H^1(M,S^2T_M\otimes L_M^{-2})$ is conjugate to $H^0(M,S^3\Omega^1_M\otimes L^2_M)$ via the natural pairing:

$$
\begin{aligned}
H^0(M,S^3\Omega^1_M\otimes L^2_M)\otimes H^1(M,S^2T_M\otimes L_M^{-2}) &\longrightarrow H^1(M,S^3\Omega^1_M\otimes S^2TM) \\
&\longrightarrow H^1(M,\Omega^1_M) = H^{1,1}(M,\mathbb{C}) \overset{\cdot\wedge\omega_M^{n-1}}{\longrightarrow} \mathbb{C} \quad .
\end{aligned}
$$

This finishes the proof of proposition 6.5.3.

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